Disastrous Defaults*

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Abstract

As the recent financial crisis illustrated, the default of certain entities can have disastrous effects on the economy. This paper presents a framework aimed at analysing the asset pricing and macro implications of the existence of "systemic defaults". This framework is flexible and tractable enough to simultaneously replicate the price fluctuations of various far-out-of-the-money (disaster-exposed) credit and equity derivatives. According to our estimation results, market data imply that the default of a systemic entity is anticipated to be followed by a 4% decrease in consumption. The recessionary influence of systemic defaults implies that financial instruments whose payoffs are exposed to such credit events carry substantial risk premiums.

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1 Introduction

Following the seminal contribution of Rietz (1988), many studies have shown that disaster risk, defined as a sudden and dramatic decrease in output and consumption, helps solve many asset-pricing puzzles [see e.g. Barro (2006), Gabaix (2012), Gourio (2013)]. Disaster risk has notably been proven to successfully account for features of far-out-of-the-money equity option markets [e.g. Du (2011), Wachter (2013), Tsai and Wachter (2015), Siriwardane (2016)], or credit derivatives such as Credit Default Swaps (CDSs) and synthetic Collateralised Debt Obligations (CDOs) [Collin-Dufresne et al. (2012), or Seo and Wachter (2016)].

In these asset-pricing studies, disasters are modelled as exogenous events causing dramatic increases in the default probabilities of bond issuers (or dramatic decreases in the asset values of firms). However, in some cases, this appears to be the default of a systemic entity *per se* that constitutes a disaster. Typically, since its inception, the largest drop in the University of Michigan Consumer Sentiment index took place in September 2008, the month when Lehman Brothers went bankrupt. The existence of systemic entities is at the core of novel regulations on Systemically Important Financial Institutions (SIFIs) [International Monetary Fund (2010), Basel Committee on Banking Supervision (2013), Battiston et al. (2016) or Brownlees and Engle (2017)].

In this paper, we propose a no-arbitrage asset-pricing framework where the defaults of some entities, called systemic entities, may have disastrous economic effects. The underlying credit risk model is that of Gouriéroux et al. (2014).[1] This model considers a finite number of homogeneous credit segments, some of them gathering systemic entities. Because of contagion effects, a systemic default can be the source of default cascades, amplifying the costs of the original bankruptcy [Allen and Gale (2000), Stiglitz (2011)].[2]

In our equilibrium pricing model, the number of systemic defaults affects consumption. Because bankruptcy cascades can be triggered by the default of a systemic entity, each systemic default is likely to eventually result in a sharp decline in consumption. In this context, financial instruments exposed to the default of systemic entities are expected to command substantial risk premiums, the latter being defined as those components of prices that would not exist if agents' were risk-neutral.

We estimate our model by making use of market data on two types of financial instruments

[1]This framework falls in the category of "top-down" models, which focus on default counting (or loss) processes [see e.g. Giesecke et al. (2011), or Azizpour et al. (2011)], contrary to "bottom-up" approaches that consider default processes of individual firms as the model primitives [see e.g. Lando (1998), Duffie and Singleton (1999), Duffie and Gârleanu (2001)]. The "top-down" approach has been shown to satisfactorily capture the existence of default clustering [see Brigo et al. (2007), or Errais et al. (2010)].

[2]Das et al. (2007) find that default clustering cannot be explained only by the firms' joint exposure to observable systematic factors. In a recent paper, Azizpour et al. (2018) provide strong evidence for the fact that contagion, through which the default by one firm has a direct impact on the health of other firms, is a significant clustering source.

directly exposed to systemic risk: senior tranches of synthetic Collateralised Debt Obligations and far-out-of-the-money put options written on market equity indices. In synthetic CDO transactions, the protection buyer receives payments when a pre-specified amount of credit losses in the reference portfolio has been reached. Losses are allocated first to the lowest tranche, known as the equity tranche, and then to successively prioritised tranches (mezzanine tranches, followed by senior tranches). Senior tranches therefore provide non-null payoffs to the protection buyer only once a sufficiently large number of entities in the underlying portfolio have defaulted. Accordingly, the market prices of senior tranches reflect investors' expectations regarding catastrophic events [Longstaff and Rajan (2008), Coval et al. (2007), or Collin-Dufresne et al. (2012)]. The second type of financial instruments, far-out-of-the-money put options, deliver payoffs when the underlying equity index experiences crashes. Typically, if the strike of the option is equal to 70% of the current value of the equity index, this option gives strictly positive payoff if the equity index falls by 30% between the inception date of the contract and the maturity date. Therefore, such equity options should also convey information regarding market perception of systemic risk [e.g. Santa-Clara and Yan (2010), Backus et al. (2011)].

The task of bringing the present equilibrium pricing model to the data is facilitated by the existence of closed-from formulas to price a wide range of equity and credit market derivatives, including synthetic CDOs. Such a degree of tractability is not present in the models employed by Collin-Dufresne et al. (2012), Seo and Wachter (2016), or Christoffersen et al. (2017), who resort to computer-demanding simulations to price tranche products. As a consequence, we can fit the equilibrium model to a wide range of derivative data, covering the pre-crisis, crisis and post-crisis periods.

The empirical application demonstrates the ability of our model to capture a substantial share of the joint fluctuations of stock and credit markets, both in tranquil and stressed periods. The estimation is conducted on euro area data spanning the period from January 2006 to September 2017. We show that two factors allow to jointly account for the main fluctuations of options written on both (i) the EURO STOXX 50 index, one of the main benchmarks of European equity markets, and (ii) the credit portfolio underlying the iTraxx Europe main index, including synthetic CDOs of different maturities and seniority levels. Our estimation procedure recognizes that the 125 constituent entities of the iTraxx indices, that are the most liquid European investment grade credits, are systemic.

Our findings point to the existence of substantial credit risk premiums in the credit derivatives written on systemic entities. In particular, the results suggest that about two thirds of 10-year Credit Default Swaps (CDSs) spreads written on systemic entities correspond to credit risk premiums. In other words, if agents were not risk-averse, these spreads would be three times lower. In line with previous studies [Azizpour et al. (2011), Giesecke and Kim (2011), or Brigo et al. (2009)], we find that an overwhelming share of the prices of the most senior tranches corresponds to risk premiums.

We also analyse the pricing of credit derivatives written on entities that are not systemic. Non-systemic entities are defined as entities whose default does not cause other entities' defaults and that have no macroeconomic impact. These non-systemic entities may however be exposed to systemic defaults – through contagion effects – and/or to other macroeconomic variables. We show that, for a fixed probability of default, the higher the exposure of these entities to systemic defaults, the higher the spreads of CDS written on these (non-systemic) entities.

As a by-product of our calibration exercise, we deduce estimates of the influence of systemic defaults on consumption (in the spirit of Backus et al. (2011)). The calculations suggest that the default of a systemic entity is expected to be followed by a 4% decrease in consumption after two years, taking contagion effects into account. Let us provide some intuition for why this influence can be inferred from our estimation. Our equilibrium model provides some structure regarding risk premiums; specifically, once the relationship between the payoffs of a given asset and consumption have been specified, the model predicts the size of risk premiums asked by investors to carry this asset. Now, the payoffs of a CDS written on a systemic entity critically depend on the default status of this entity. As a result, through the lens of the model, the potential influence of a "systemic default" on consumption can be inferred from indications regarding the size of credit risk premiums included in a CDS spread written on a systemic entity.

We finally exploit our estimated model to derive systemic risk indicators. These indicators are defined as the probabilities of observing a certain number of systemic defaults over specific horizons.[3] The resulting systemic indicators reach their highest levels in late 2008, after the Lehman bankruptcy and in late 2011, when the European sovereign crisis was at its peak. The probabilities of having more than 10% of defaults among iTraxx constituents within two years were of 6% and 4%, respectively, at the time.

The remainder of this paper is organised as follows. Section 2 presents the general framework and derives associated pricing formulas. Section 3 documents the estimation approach. Section 4 explores the asset pricing implications. The data description and the derivation of pricing formulas are gathered in appendices.

[3]Using prices of far-out-of-the-money put options to infer disaster probabilities dates back to Bates (1991). This approach has been applied recently by, among others, Bollerslev and Todorov (2011), Backus et al. (2011), Seo and Wachter (2016), Barro and Liao (2016), or Siriwardane (2016). For a discussion regarding the difficulty in measuring systemic risk, see Hansen (2013).

2 Model

2.1 Credit segments

We consider J homogeneous segments of defaultable entities. For any j, the I_j entities of segment j share the same credit characteristics.

Let $N_{j,t}$ be the number of segment-j entities in default at date t and let N_t be the vector $N_t = [N_{1,t},\dots,N_{J,t}]'$. We denote by $n_{j,t} = N_{j,t} - N_{j,t-1}$ the number of default occurring in segment j on date t. With obvious notations, we also have $n_t = N_t - N_{t-1}$.

The information on current and past values of any process k_t is denoted by $\underline{k_t} = \{k_t, k_{t-1},\dots\}$. Conditional on $N_0 = 0$, the information contained in the information set $\underline{n_{j,t}}$ (respectively $\underline{n_t}$) is equivalent to that in $\underline{N_{j,t}}$ (resp. $\underline{N_t}$).

The first two segments of entities gather large firms supposed to be systemic. We denote by $N_t^s = N_{1,t} + N_{2,t}$ the cumulated number of systemic defaults and by $n_t^s = n_{1,t} + n_{2,t}$ the number of systemic defaults occurring on date t. The only distinction between these first two segments is that the first one contains the constituents of a credit index used as the reference portfolio for traded credit derivatives, whose prices are used to calibrate the model. Having a single segment of systemic entities would be restrictive because it would mean that this specific credit index, namely the iTraxx Europe main, covers all systemic entities in our economy, which may not be realistic. The other segments, $j \geq 3$, gather non-systemic firms, that can be thought of as small firms.

2.2 Default-count processes

We assume that defaults are caused by two exogenous and non-negative factors that we denote by x_t and y_t, which allows to distinguish between long-run and short-run fluctuations of aggregate credit risk. Without loss of generality, we impose $\mathbb{E}(x_t) = \mathbb{E}(y_t) = 1$. Appendix A.1 proposes a specification based on vector auto-regressive gamma processes [see Gouriéroux and Jasiak (2006), or Monfort et al. (2017)]. These processes are Markov processes, with gamma-type transition distributions. In particular, they are such that:

$$\left\{\begin{array}{rcl} x_t - 1 &=& \rho_x(x_{t-1}-1) + \sigma_{x,t}\varepsilon_{x,t} \\ y_t - x_t &=& \rho_y(y_{t-1}-x_{t-1}) + \sigma_{y,t}\varepsilon_{y,t}, \end{array}\right. \tag{1}$$

where $\varepsilon_t = [\varepsilon_{x,t},\varepsilon_{y,t}]'$ is a martingale difference sequence with unit-variance components and where $[\sigma_{x,t}^2,\sigma_{y,t}^2]'$ is affine in $[x_{t-1},y_{t-1}]'$.

Intuitively, if $0 < \rho_y < \rho_x < 1$, the autonomous factor x_t is more persistent than $y_t - x_t$, and x_t can be seen as the low-frequency component of y_t. The residual component $y_t - x_t$, which has a marginal zero expectation, can be interpreted as the higher-frequency component of y_t.

Let us now turn to the conditional distribution of the number of defaults. For any segment j, we assume:

$$n_{j,t+1}|\underline{x_{t+1}},\underline{y_{t+1}},\underline{N_t} \sim \mathscr{P}(\beta_j y_{t+1} + c_j n_t^s), \tag{2}$$

where n_t^s is the total number of systemic defaults taking place on date t. If $c_j > 0$, the occurrence of systemic defaults on date t increases the conditional probability of having defaults in segment j on the next date. In other words, systemic defaults are contagious when $c_j > 0$ for some j.[4,5] By contrast, the defaults of non-systemic segments are not contagious since, for $j > 2$, $n_{j,t}$ does not appear in the parameter of the Poisson distribution in eq. (2).

For parsimony, we consider that entities from the two systemic segments are alike, the only difference being that those from segment 1 are the constituents of traded credit indices. Accordingly, we assume that $c_1 = c_2$ and that $\beta_1 = \beta_2$. That is, assuming also that $I_1 = I_2$, the conditional default probabilities of a systemic entity, be it of segment 1 or 2, are the same.

Note that eq. (2) specifies a default process $N_{j,t}$ that does not necessarily terminate at I_j, the number of entities in segment j. This feature is, however, innocuous because for the relatively large portfolios of interest, the probability of $N_{j,t}$ exceeding I_j during standard contract terms is small for our sample.[6]

2.3 Consumption growth process

We assume that systemic defaults have a negative impact on the log growth rate of per capita consumption, denoted by $\Delta c_t = \log(C_t/C_{t-1})$. To have this, a possibility is to make Δc_t directly depend on the number of systemic defaults n_{t-1}^s. However, this would have the nonrealistic implication that all systemic defaults have the same deterministic effect on consumption growth. Instead, we assume that Δc_t depends on a factor w_t, that depends itself on systemic defaults according to:

$$w_t|\underline{x_t},\underline{y_t},\underline{N_t} \sim \gamma_0(\xi_w n_{t-1}^s, \mu_w), \tag{3}$$

where the zero-gamma distribution γ_0 is a distribution featuring a point mass at zero (Monfort et al., 2017). Specifically, when $n_{t-1}^s > 0$, w_t is drawn from a gamma distribution whose scale parameter is μ_w and whose shape parameter is drawn from $\mathscr{P}(\xi_w n_{t-1}^s)$. When the shape parameter is zero, we have $w_t \equiv 0$. Therefore, in this context, the conditional probability that $w_t = 0$ is $\exp(-\xi_w n_{t-1}^s)$.

[4] In particular, if $c_j > 0$ for $j \in \{1,2\}$ (systemic segments), then systemic defaults are "self-excited" [Aït-Sahalia et al. (2014)].

[5] Systemic defaults are contagious, or "infectious", using Davis and Lo (2001)'s terminology.

[6] Size effects are captured by parameters β_j and c_j. In our empirical study, the segment sizes we use are 50 (EURO STOXX 50, see Appendix D.2) and 125 (iTraxx index, Appendix D.1).

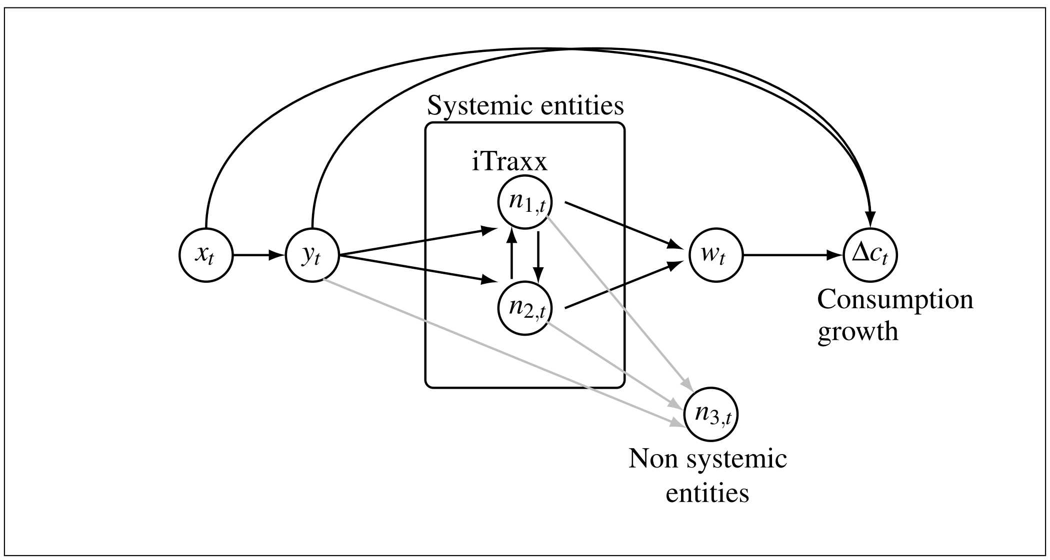
In particular, we have $w_t = 0$ as long as there has been no systemic defaults in the previous period. For identification, we impose $\mathbb{E}(w_t) = 1$, which is obtained by setting $\mu_w = 1/(\xi_w \mathbb{E}(n_t^s))$.

The consumption growth process is then specified as follows:

$$\Delta c_t = \mu_{c,0} + \mu_{c,x} x_t + \mu_{c,y} y_t + \mu_{c,w} w_t. \tag{4}$$

Figure 1 shows the resulting causal scheme. In this model, the defaults of non-systemic segments ($j > 2$) have no causal impact on consumption or on defaults in other segments. As a result, non-systemic segments play no role in the model estimation. We will however use segment 3 in Section 4 to study the implications of the model for the pricing of credit derivatives written on non-systemic entities.

Figure 1: Causal scheme



This figure shows the causal scheme underlying the model. Arrows represent Granger-causality relationships.

2.4 State vector and agents' information sets

On date t, the information set of the representative agent is $\Omega_t = \{\underline{x_t}, \underline{y_t}, \underline{w_t}, \underline{N_t}\}$. It includes the current and past observations of all default counts and underlying factors. In the following, the operator $\mathbb{E}_t$ denotes the expectation conditional on the information available at time t, i.e. $\mathbb{E}_t(\bullet) = \mathbb{E}(\bullet|\Omega_t)$. Note that this information set includes, in particular, the consumption stream $\underline{C_t}$.

In the following, we focus on the state vector $X_t = [x_t, y_t, w_t, N_t', N_{t-1}']'$. As shown below, the payoffs of the financial instruments we consider, as well as their values, are functions of X_t. (As a result, the information contained in the underlying factor values can be recovered from the observed consumption and derivative prices.) The tractability of our approach results from to the fact that X_t is an affine process: the log conditional Laplace transform, denoted by $\psi(v, X_t)$ and defined by:

$$\mathbb{E}_t\left(\exp(v'X_{t+1})\right) = \exp(\psi(v, X_t)),$$

is affine in X_t. Formally, there exist functions ψ_0 and ψ_1 such that:

$$\psi(v, X_t) = \psi_0(v) + \psi_1(v)'X_t, \tag{5}$$

for all values of v. Functions ψ_0 and ψ_1 are made explicit in Appendix A.2 (eqs. a.2 and a.3). As is well-known, the combination of affine processes and an exponential affine stochastic discount factor results in closed-form, or quasi closed-form, expressions for a wide range of financial instruments [e.g. Duffie et al. (2002)].

2.5 Preferences, stochastic discount factor and risk-neutral dynamics

The preferences of the representative agent are of the Epstein and Zin (1989) type, with a unit elasticity of intertemporal substitution (EIS).[7] Specifically, the time-t utility of a consumption stream (C_t) is recursively defined by

$$u_t \quad = \quad (1-\delta)c_t + \frac{\delta}{1-\gamma}\log\left(\mathbb{E}_t \exp\left[(1-\gamma)u_{t+1}\right]\right). \tag{6}$$

where c_t denotes the logarithm of the agent's consumption level C_t, δ the time discount factor and γ the risk aversion parameter. Eq. (6) results from a first-order Taylor expansion around $\rho = 1$ of the general Epstein and Zin (1989) recursive utility defined by

$$u_t = \frac{1}{1-\rho}\log\left((1-\delta)C_t^{1-\rho} + \delta\left(\mathbb{E}_t\left[\exp\{(1-\gamma)u_{t+1}\}\right]\right)^{\frac{1-\rho}{1-\gamma}}\right),$$

where ρ is the inverse of the EIS. Exploiting the affine property of the state vector X_t, we obtain the following solution for u_t in eq. (6):

Proposition 1. *$u_t = u_{t-1} + \mu_{c,0} + \mu_{u,1}'X_t + (\mu_{c,1} - \mu_{u,1})'X_{t-1}$ satisfies eq. (6) for any $[X_t', X_{t-1}']'$ iff $\mu_{u,1}$ satisfies:*

$$\frac{\delta}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) = \mu_{u,1} - \mu_{c,1}. \tag{7}$$

[7] Using a unit EIS facilitates resolution. Piazzesi and Schneider (2007), or Seo and Wachter (2016), among others, also work under this assumption of a unit EIS.

Proof. Appendix B.1. □

The stochastic discount factor (s.d.f.) can then be deduced from Proposition 1:

Proposition 2. *We have:*

$$M_{t,t+1} = \exp[-(\eta_0 + \eta_1' X_t) + \pi' X_{t+1} - \psi(\pi, X_t)], \tag{8}$$

with

$$\begin{cases} \pi &= (1-\gamma)\mu_{u,1} - \mu_{c,1} \\ \eta_0 &= -\log(\delta) + \mu_{c,0} + \psi_0((1-\gamma)\mu_{u,1}) - \psi_0(\pi) \\ \eta_1 &= \psi_1((1-\gamma)\mu_{u,1}) - \psi_1(\pi). \end{cases}$$

Proof. Appendix B.2. □

Because $\mathbb{E}_t(M_{t,t+1}) = \exp[-(\eta_0 + \eta_1' X_t)]$, the short-term risk-free interest rate r_t is affine in X_t and given by:

$$r_t = \eta_0 + \eta_1' X_t. \tag{9}$$

The following term:

$$\frac{M_{t,t+1}}{\mathbb{E}_t(M_{t,t+1})} = \exp[\pi' X_{t+1} - \psi(\pi, X_t)]$$

corresponds to $(d\mathbb{Q}/d\mathbb{P})_{t,t+1}$, that is, it defines the change of probability from the historical to the risk-neutral measure. The associated risk premiums depend on vector π, which reflects the "prices" of the different risk factors.

Let us consider the risk-neutral conditional log Laplace transform $\psi^{\mathbb{Q}}$ of X_t. We have:

$$\begin{aligned} \exp\left(\psi^{\mathbb{Q}}(v, X_t)\right) &= \mathbb{E}_t^{\mathbb{Q}}\left(\exp(v' X_{t+1})\right) = \mathbb{E}_t\left(\exp[\pi' X_{t+1} - \psi(\pi, X_t) + v' X_{t+1}]\right) \\ &= \exp(\psi(v+\pi, X_t) - \psi(\pi, X_t)) = \exp\left(\psi_0^{\mathbb{Q}}(v) + \psi_1^{\mathbb{Q}}(v)' X_t\right), \end{aligned}$$

with

$$\begin{cases} \psi_0^{\mathbb{Q}}(v) &= \psi_0(v+\pi) - \psi_0(\pi) \\ \psi_1^{\mathbb{Q}}(v) &= \psi_1(v+\pi) - \psi_1(\pi). \end{cases} \tag{10}$$

Hence, the state process X_t is also an affine process under the risk-neutral measure $\mathbb{Q}$. This facilitates the pricing of various assets whose payoffs depends on future values of X_t. In particular, Appendix C.1 provides closed-form formulae to compute date-t prices of European derivatives with payoffs of the form $\exp(a' X_{t+h})$, $\exp(a' X_{t+h}) \mathbb{1}_{\{b' X_{t+h} < y\}}$, $a' X_{t+h}$, or $a' X_{t+h} \mathbb{1}_{\{b' X_{t+h} < y\}}$, settled on date $t+h$. These formulas are key building blocks to price specific financial instruments.

2.6 Pricing credit and equity derivatives

2.6.1 Pricing credit index swaps

A credit index swap allows an investor to either buy or sell protection on a credit index, which is a basket of reference entities. There are two main families of credit indices, which serve as reference points for CDS markets, that are the Dow Jones CDX and iTraxx indices. The U.S. Investment-Grade CDX main index and the iTraxx Europe main are each comprised of 125 equally-weighted underlying credits (see Appendix D.1 for more details on the iTraxx Europe main index, the one used in our application).

In a credit index swap transaction, a protection seller agrees to pay all default losses in the index in return for a fixed periodic spread $S_{t,h}^{CI}/q$ paid on the total notional of obligors remaining in the index over a period of h years, where q is the number of time periods per year. Should there be no credit event, the protection buyer pays a regular spread until maturity. Upon default of one of the reference entities, the protection seller provides the buyer with the amount that the latter would have lost if she had held the index bond portfolio. For instance, for a \$100,000 position in a 20-name index, with a recovery rate of 50%, the amount would be \$2,500 ($= 50\% \times 100,000/20$). Following this default, the trade continues with the notional amount reduced by the weight of the defaulted credit. In the previous example, the new notional would be \$95mm; the number of reference entities in the index would be reduced to the remaining (non-defaulted) 19 entities.

In our application, we consider that the names in the credit index coincide with segment 1. The payoffs therefore critically depends on $N_{1,t}$. The spread $S_{t,h}^{CI}$ is determined by equalizing the date-t values of the protection leg and of the premium leg, that is:

$$\underbrace{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}(1-RR)\frac{N_{1,t+k}-N_{1,t+k-1}}{I_1}\right\}}_{\text{Protection leg}} = \underbrace{\frac{S_{t,h}^{CI}}{q}\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}\frac{I_1-N_{1,t+k}}{I_1}\right\}}_{\text{Premium leg}}, \tag{11}$$

where I_1 is the number of entities in segment 1, i.e. the number of names in the index, where RR is the contractual recovery rate, assumed independent of time, and where:

$$\Lambda_{t,t+k} = \exp(-r_t - r_{t+1} - \cdots - r_{t+k-1}), \tag{12}$$

r_t being the riskfree short-term interest rate between periods t and $t+1$.

Hence, credit index swap spreads result from the knowledge of conditional expectations of the form $\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}N_{1,t+k})$ and $\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}N_{1,t+k-1})$, whose computation is addressed by Corollary 1 in Appendix C.1.

Online Appendix O.4 shows that the spread on a CDS written on any entity of segment 1 is also given by eq. (11).

2.6.2 Pricing synthetic Collateralised Debt Obligations (CDOs)

Collateralised Debt Obligations (CDOs), or credit tranches, allow an investor to get a specified exposure to the credit risk of the underlying reference portfolio, or credit index, while in return receiving periodic coupon payments. The credit-tranche market consists of an actively traded segment and an illiquid "buy-and-hold" segment [Scheicher (2008)]. In the actively-traded segment, the underlying credit portfolio is based on the standardised portfolio of a credit index such as the iTraxx or the CDX index. The less-actively-traded segment of the credit-tranche market consists of tailor-made tranches of Collateralised Debt Obligations (CDOs) in which various loans are bundled. Losses due to credit events in the underlying portfolio are allocated first to the lowest tranche, known as the equity tranche, and then to successively prioritised tranches (mezzanine tranches, followed by senior tranches).

The risk of a tranche is determined by attachment and detachment points. The attachment point, denoted by a, determines the subordination of a tranche. The detachment point, denoted by b, $b > a$, determines the point beyond which the tranche has lost its complete notional. The equity tranche takes the first losses on the portfolio, from $a_1 = 0$ up to b_1. When the portfolio has accumulated losses exceeding the fraction b_1 of notional, the next tranche, (a_2, b_2) with $a_2 = b_1$, will incur losses from any additional defaults up to b_2, and so on.

Let us detail the cash-flows induced by an (a,b) credit tranche in the context of the reference portfolio made of segment-1 entities, that are the iTraxx ones. Consider a protection buyer and a protection seller who meet at date t. Their negotiation results in a spread $S_{t,h}^{TDS}(a,b)$, which is the quote associated with this credit tranche at date t, the maturity date of this derivative product being $t+h$. Let us denote by ℓ_t the cumulated loss, that is:

$$\ell_t = (1 - RR)\frac{N_{1,t}}{I_1}.$$

From dates $t+1$ to $t+h$, cash-flows are exchanged between the two parties unless the cumulated losses ℓ_{t+k} (for $k = 1, \dots, h$) have exceeded the detachment point b. Specifically, at date $t+k$, these cash-flows are the following:

- If cumulated losses ℓ_{t+k} have not reached the attachment point a: (i) there is no cash-flow paid by the protection seller and (ii) the protection buyer pays the full premium $S_{t,h}^{TDS}(a,b)/q$.

- If cumulated losses ℓ_{t+k} exceed the attachment point a but remain lower than the detachment point b: (i) the protection seller provides the protection buyer with an amount equal to the fraction of the tranche consumed by new losses between $t+k-1$ and $t+k$, that is $(\ell_{t+k} -$

$\ell_{t+k-1})/(b-a)$, and (ii) the protection buyer pays a premium equal to the multiplication of the full premium $S^{TDS}_{t,h}(a,b)/q$ by the fraction of the tranche that has not been consumed at date $t+k$, that is $(b-\ell_{t+k})/(b-a)$.

The spread $S^{TDS}_{t,h}(a,b)/q$ is such that the two legs have the same value at date t, that is:

$$
\begin{aligned}
& \underbrace{\mathbb{E}^{\mathbb{Q}}_t\left\{\sum_{k=1}^{qh}\frac{1}{b-a}\Lambda_{t,t+k}(\min(\ell_{t+k},b)-\max(\ell_{t+k-1},a))\mathbb{1}_{\{a<\ell_{t+k}\}}\mathbb{1}_{\{\ell_{t+k-1}\leq b\}}\right\}}_{\text{Protection leg}} \\
\approx\; & \underbrace{\mathbb{E}^{\mathbb{Q}}_t\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}\frac{\ell_{t+k}-\ell_{t+k-1}}{b-a}\mathbb{1}_{\{a<\ell_{t+k}\leq b\}}\right\}}_{\text{Protection leg}} \\
=\; & \underbrace{U^{TDS}_{t,h}(a,b)+\mathbb{E}^{\mathbb{Q}}_t\left\{\frac{S^{TDS}_{t,h}(a,b)}{q}\sum_{k=1}^{qh}\Lambda_{t,t+k}\left(\mathbb{1}_{\{\ell_{t+k}\leq a\}}+\frac{b-\ell_{t+k}}{b-a}\mathbb{1}_{\{a<\ell_{t+k}\leq b\}}\right)\right\}}_{\text{Premium leg}}, \qquad (13)
\end{aligned}
$$

where $U^{TDS}_{t,h}(a,b)$ is an upfront payment and where $\Lambda_{t,t+k}$ is defined in eq. (12).[8] Credit tranches are either quoted in terms of spreads $S^{TDS}_{t,h}(a,b)$, or in terms of up-front payments $U^{TDS}_{t,h}(a,b)$. Typically, in the former case, the up-front payment is fixed, and vice versa.

Appendix C.2 shows that by expanding both sides of eq. (13), computing $S^{TDS}_{t,h}(a,b)$ – or, equivalently, $U^{TDS}_{t,h}(a,b)$ – amounts to calculating date-t prices of payoffs of the forms: $\mathbb{1}_{\{N_{1,t+k}<z\}}$, $N_{1,t+k}\mathbb{1}_{\{N_{1,t+k}<z\}}$, and $N_{1,t+k-1}\mathbb{1}_{\{N_{1,t+k}<z\}}$, these payoffs being settled at date $t+k$. The computation of such prices is addressed in Corollaries 2 and 3 (Appendix C.1).

2.6.3 Pricing equity derivatives

The price P_t of a stock at date t can be deduced from the series of future dividends D_{t+h}, $h \geq 1$, as:

$$P_t = \sum_{h=1}^{\infty}\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+h}D_{t+h}),$$

Let us assume that, as consumption growth, the dividend log growth rate $g_{d,t} = \log(D_t/D_{t-1})$ is affine in $[x_t, y_t, w_t]'$:

$$g_{d,t} = \mu_{d,0} + \mu_{d,x}x_t + \mu_{d,y}y_t + \mu_{d,w}w_t. \qquad (14)$$

Proposition 5 (Appendix C.3) provides an approximated solution for the stock returns in the

[8] See e.g. O'Kane and Sen (2003), D'Amato and Gyntelberg (2005), or Morgan Stanley (2011) for an analysis of upfront versus running spread quoting conventions.

general case where the log growth rate of dividends is affine in X_t. As in Bansal and Yaron (2004), this approximated solution is based on the Campbell and Shiller (1988) linearisation of stock returns around the average of the log price-dividend ratio $\tau_t = \log(P_t/D_t)$. In the solution, τ_t is expressed as an affine function of X_t.

The payoffs of equity derivatives depend on P_t. The dynamics of P_t is deduced from that of the ex-dividend return $r^*_{t+1} = \log(P_{t+1}/P_t)$, that we denote by r^*_t. This return is given by:

$$r^*_{t+1} = \log\left(\frac{P_{t+1}}{D_{t+1}} \times \frac{D_t}{P_t} \times \frac{D_{t+1}}{D_t}\right) = \tau_{t+1} - \tau_t + g_{d,t+1}, \tag{15}$$

which is affine in $[X'_{t+1}, X'_t]'$. We therefore have, for any horizon h:

$$\begin{aligned} P_{t+h} &= P_t \exp\left(r^*_{t+1} + \cdots + r^*_{t+h}\right) && (16)\\ &= \exp\left(\tau_{t+h} - \tau_t + g_{d,t+1} + g_{d,t+2} + \cdots + g_{d,t+h}\right). && (17) \end{aligned}$$

Let us consider the price of a European put option of maturity h and strike K. This price is given by $\mathbb{E}^{\mathbb{Q}}_t\left(\Lambda_{t,t+h}(K - P_{t+h})\mathbb{1}_{\{K>P_{t+h}\}}\right)$. Using eq. (16), we obtain:

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}}_t\left(\Lambda_{t,t+h}(K - P_{t+h})\mathbb{1}_{\{K>P_{t+h}\}}\right)\\ = \; & K\mathbb{E}^{\mathbb{Q}}_t\left(\Lambda_{t,t+h}\mathbb{1}_{\{r^*_{t+1}+\cdots+r^*_{t+h}<\log(K)-\log P_t\}}\right)\\ & -P_t\mathbb{E}^{\mathbb{Q}}_t\left(\Lambda_{t,t+h}\exp(r^*_{t+1} + \cdots + r^*_{t+h})\mathbb{1}_{\{r^*_{t+1}+\cdots+r^*_{t+h}<\log(K)-\log P_t\}}\right). \end{aligned} \tag{18}$$

Appendix C.4 provides details about the computation of the two conditional expectations appearing on the right-hand side of eq. (18).

3 Estimation

Bringing the model to the data amounts to determining two types of objects: the model parameters have to be estimated and the latent variables have to be filtered. Some of the model parameters, in particular preference parameters, are calibrated. Thanks to the tractability of our framework, the estimation of remaining parameters and the filtering of unobserved variables are performed jointly by Kalman filter techniques.

3.1 Data

The data cover the period from January 2006 to September 2017 at a bi-monthly frequency. We use credit index swap spreads and prices of tranches associated with the iTraxx Europe main index.

The constituents of this index are 125 large European firms whose credit default swaps are actively traded (see Appendix D.1). For credit index swap spreads, we use maturities of 3, 5, 7 and 10 years. We consider three maturities of synthetic CDOs, 3, 5 and 7 years and, for each maturity, 5 tranches: 0%-3%, 3%-6%, 6%-9%, 9%-12% and 12%-22%. The financial data also include prices of far-out-of-the-money (far-OTM) equity put options written on the EURO STOXX 50, which is one of the most important benchmarks of European equity markets. More precisely, we consider 6-month and 12-month put options protecting against larger-than-30% falls in the equity index (see Appendix D.2).

3.2 Calibrated parameters

The left panel of Table 1 reports the calibrated parameters. Following Seo and Wachter (2016), the risk aversion parameter γ is set to 3 and the annualized rate of time preference to 1.2%. Because our model is at the bi-monthly frequency, this rate of time preference translates into $\delta = (1-1.2\%)^{1/6} \approx 0.998$. As mentioned above (Subsection 2.5), we consider a unit elasticity of intertemporal substitution. Another calibrated moment is the population expectation of consumption growth, that is set to 1.5% (annualized). As in Bansal and Yaron (2004), the log growth rate of dividends is given the same marginal expectation (1.5%, annualized). We take a contractual recovery rate RR of 40%, consistently with standard market practice. We also set the average default rate of the systemic entities to be of 0.3% per year. This is consistent with historical data on investment-grade entities compiled by Moody's.[9]

3.3 State-space model

During the period we consider (2006-2017), there has been no systemic default in the euro area.[10] Accordingly, we have $n_t^s = 0$ and therefore $w_t = 0$ for all dates t in our sample. Then we can focus on the filtering of the other factors x_t and y_t. Let us stress that in spite of the fact that $w_t = 0$ over our sample, the threat of possibly having $w_{t+k} > 0$, $k > 0$, is taken into account by investors on each date t of the sample. Accordingly, the parameters governing the dynamics of w_t, i.e. ξ_w, μ_w, $\mu_{c,w}$ and $\mu_{d,w}$ are, in particular, identifiable through observed derivative prices.

[9] More precisely, this corresponds to the average cumulative issuer-weighted global default rates for Baa-rated firms on the period 1920-2016 [see Moody's (2017), Exhibit 32]. In March 2016, the median rating for the iTraxx index (series 25) is BBB+ at S&P [Société Générale (2016)].

[10] On October 22 2009, CDS contracts written on the French electronics firm Thomson were triggered. This entity was included among the iTraxx constituents. However, we do not consider this credit event to be a systemic event. Indeed, this credit event was not a failure of the firm but a restructuring of its debt. In the U.S., following the so-called "Big Bang" changes in practices on credit events (April 8 2009) restructuring was excluded from the list of credit events triggering American CDSs [see Coudert and Gex (2010)].

Observed variables include credit index swap spreads of different maturities, tranche spreads and equity put prices. Let us denote by Γ_t the vector of observed prices and by θ the vector of model parameters to be estimated. Over our estimation period, our model predicts that these prices are functions of $z_t = [x_t, y_t]'$ (and of $w_t = 0$) and θ. Allowing for measurement errors denoted by ε_t, the set of measurement equations reads:

$$\Gamma_t = F(z_t; \theta) + \varepsilon_t, \tag{19}$$

where the components of ε_t are mutually and serially independent Gaussian shocks, i.e. $\varepsilon_t \sim i.i.d.\mathcal{N}(0, \Sigma_\varepsilon)$, where Σ_ε is a diagonal matrix.

The transition equation describes the dynamics of z_t. Using the formula provided in Appendix A.1, the dynamics of z_t can be expressed as follows:

$$z_{t+1} = \mu_z + \Phi_z z_t + \Sigma_z(z_t)\xi_{t+1}, \tag{20}$$

where ξ_{t+1} is a martingale difference sequence that, conditional on Ω_t, is zero mean and admits an identity covariance matrix.

Eqs. (19) and (20) constitute the state-space form of our model. We employ the extended Kalman filter to approximate the log-likelihood function associated with this state-space model.[11] By maximising this function with respect to θ, we obtain estimates of the parameters that have not been calibrated (Subsection 3.2). A final pass of the Kalman algorithm provides us with filtered values of the latent factors z_t.

4 Results

4.1 Model fit

Table 1 shows calibrated and estimated parameters. It notably appears that c_j parameters ($i, j \in \{1,2\}$) are equal to 0.35, revealing a substantial level of contagion. It implies that an additional default by one systemic firm on date t leads to an increase in the expected number of systemic default on date $t+1$ by 0.70 (2×0.35) on date $t+1$. Responses to systemic defaults will be studied more extensively through impulse response functions in Subsection 4.2. The fact that $\rho_x = 0.977$ and $\rho_y = 0.895$, with associated half-lives of 5 and 1 years, respectively, indicates that the

[11] Derivative of function F with respect to z_t are obtained numerically. In order reduce the number of parameters to estimate, the diagonal entries of Σ_ε (variances of the measurement errors) are calibrated in a preliminary step. We employ the approach of Green and Silverman (1994) and proceed as follows: we apply a smoothing spline to series of observed prices. Next, we compute the sample variances of the differences between the prices and their smoothed counterparts. The variances of the measurement equations are set to these values.

persistence of x_t is larger than that of $y_t - x_t$. This is illustrated by Figure 2, which displays the filtered factors x_t and y_t. This figure shows that y_t is more volatile than x_t. In particular, contrary to x_t, process y_t appears to have been more sensitive to the post-Lehman crisis (late 2008, early 2009) and to the peak of the euro-area sovereign debt crisis (late 2011, early 2012).

Figure 2 displays the estimates of factors x_t and y_t. The long-run factor x_t remained subdued before the euro-area sovereign debt crisis of 2010-2012. Therefore, the peak reached by y_t in late 2008 is mainly due to an increase in the shorter-run component of y_t, i.e. $y_t - x_t$. This suggests that, as regards corporate European credit risk, the post-Lehman crisis was then perceived as a relatively short-lived phenomenon. By contrast, the European sovereign debt crisis triggered an increase in the long-run component of y_t. This indicates that the market considers that this latter crisis will have a longer-lived impact on corporate European credit risk. As of the end of the sample, the level of the long-run factor x_t is higher than before 2008. As a consequence, even though the 2017 level of y_t is below its late 2008 level, conditional medium to long-run expectations of y_t are higher in 2017 than by in late 2008.

Panel (a) of Table 2 reports model-implied population moments. It indicates for instance that the average excess return for our stock index is of 2.00% and that the maximum Sharpe ratio [Hansen and Jagannathan (1991)], evaluated at the average values of the state vector X_t, has a value of 62%, which is for instance comparable to the 70% maximum Sharpe ratio value used by Brennan et al. (2004). Panels (b), (c) and (d) of Table 2 documents the fit resulting from our estimation approach by comparing the sample averages of observed financial data to their model-implied counterparts.

The model fit is also illustrated by Figures 3 to 5. Figure 3 illustrates the fit of the iTraxx index swap spreads of different maturities. Figure 4 compares observed and model-based implied volatilities of far-OTM put options and Figure 5 displays tranche price estimates. These figures suggest that the model is successful in capturing the main joint fluctuations of stock and credit derivatives exposed to systemic risk. In particular, in spite of using a longer sample (2006-2017 versus 2005-2008) –including acute crisis periods– and a larger cross-section of prices than in Collin-Dufresne et al. (2012), Seo and Wachter (2016), or Christoffersen et al. (2017), the fit we obtain is comparable to theirs.

Table 1: Estimated parameters

Panel (a) – Calibrated parameters			Panel (b) – Estimated parameters		
γ		3	$c_i \quad i \in \{1,2\}$		0.35
δ		0.997			
EIS		1.00	$\beta_i \quad i \in \{1,2\}$	$(\times 10^2)$	1.81
			μ_w		118.07
$\mathbb{E}(\Delta c_t)$	$(\times 6)$	1.50%	ξ_w		0.13
$\mathbb{E}(g_{d,t})$	$(\times 6)$	1.50%			
			μ_x	$(\times 10^2)$	1.22
			μ_y	$(\times 10^2)$	5.63
			ρ_x		0.978
			ρ_y		0.858
			$\mu_{c,x}$	$(\times 10^5)$	-0.08
			$\mu_{c,y}$	$(\times 10^5)$	-12.61
			$\mu_{c,w}$	$(\times 10^4)$	-8.17
			$\mu_{d,x}$	$(\times 10^5)$	-0.00
			$\mu_{d,y}$	$(\times 10^5)$	-0.00
			$\mu_{d,w}$	$(\times 10^4)$	-16.15

This table presents the model parameterisation. $\mathbb{E}(\Delta c_t)$ is multiplied by 6 so as to be expressed in annualised terms. The parameterisation is such that $\mathbb{E}(x_t) = \mathbb{E}(y_t) = 1$ (see Appendix A.1). The specification of the consumption growth rate is given by eq. (4). The specification of the dividend growth rate is given by eq. (14). Panel (a) reports calibrated parameters. Panel (b) reports parameters estimated by maximising an approximation of the log-likelihood associated with the state-space model defined by measurement equations (19) and transition equations (20) (see Subsection 3.3).

Table 2: Model fit

Panel (a) Model-implied population moments		
Avg. short-term risk-free rate	1.97%	
St. dev. short-term risk-free rate	0.66%	
Avg. equity excess return	2.00%	
Maximum Sharpe ratio (at $X_t = \bar{X}$, for a one-year investment)	62.2%	

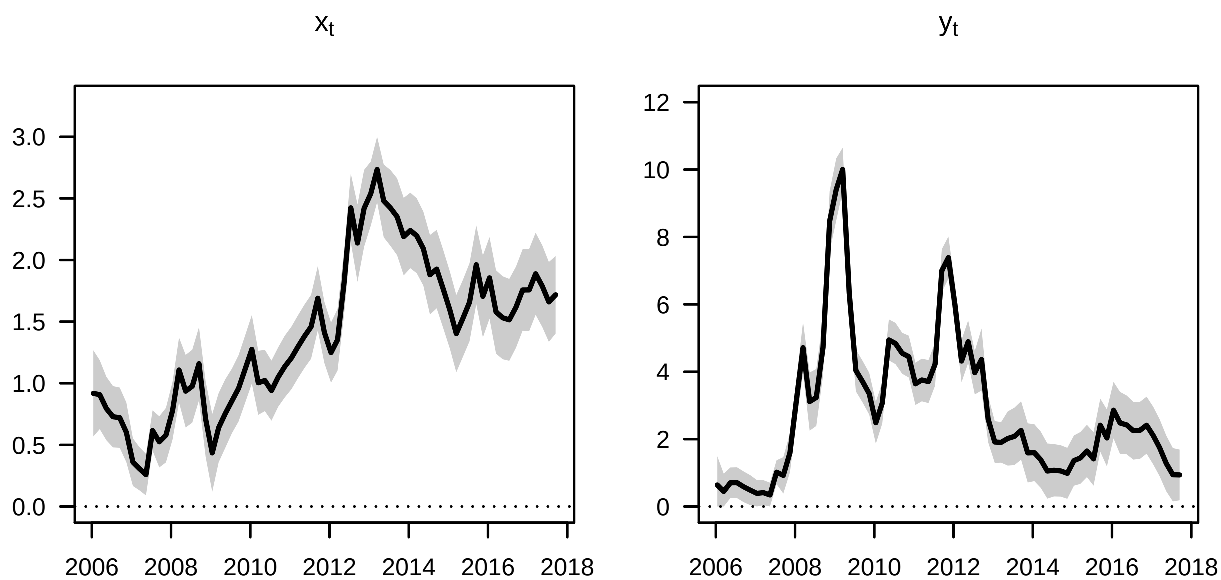
Panel (b) ITRAXX indices (sample averages, in b.p.)	Model	Data
3 years	60	65
5 years	71	88
7 years	83	101
10 years	115	112

Panel (c) ITRAXX tranches (sample averages, in b.p.)	Model	Data
3 years, Tranche: 0-3%	1832	1879
3 years, Tranche: 3-6%	486	772
3 years, Tranche: 6-9%	243	452
3 years, Tranche: 9-12%	145	160
3 years, Tranche: 12-22%	34	113
5 years, Tranche: 0-3%	1497	1444
5 years, Tranche: 3-6%	468	663
5 years, Tranche: 6-9%	260	421
5 years, Tranche: 9-12%	188	151
5 years, Tranche: 12-22%	77	91
7 years, Tranche: 0-3%	1384	1241
7 years, Tranche: 3-6%	471	672
7 years, Tranche: 6-9%	265	439
7 years, Tranche: 9-12%	195	146
7 years, Tranche: 12-22%	90	94

Panel (d) Implied Volatility (sample averages, in p.p.)	Model	Data
Maturity: 6 months	30%	33%
Maturity: 12 months	31%	30%

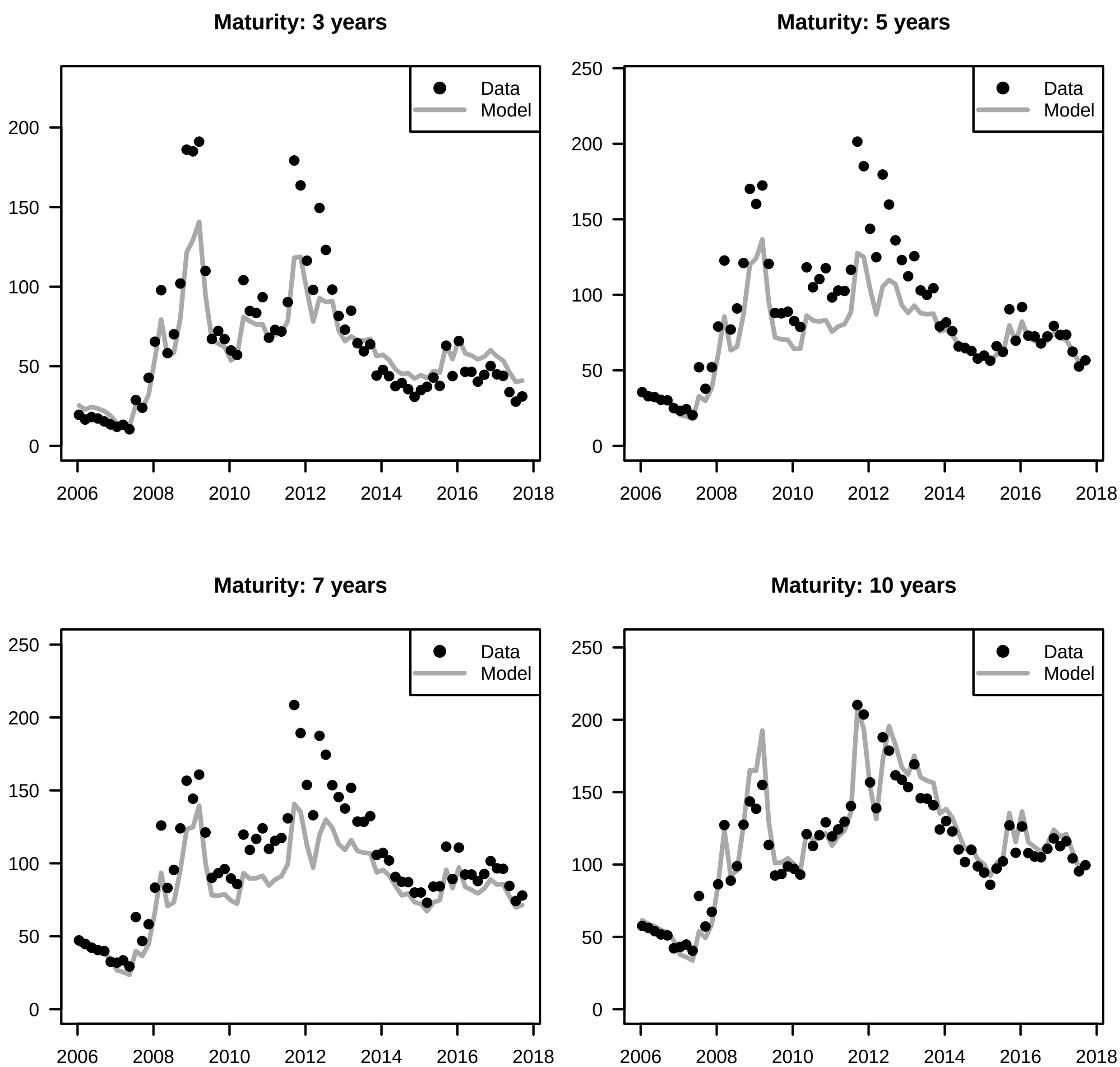
This table documents the fit of the model. Model-implied prices are evaluated by setting factors x_t and y_t to their filtered values derived from the extended Kalman filter applied to the state-space model defined by measurement equations (19) and transition equations (20) (see Subsection 3.3). The reported maximum Sharpe ratio (see the Online Appendix O.5 for its computation) is evaluated at the population mean of the state vector, i.e. for $X_t = \bar{X}$.

Figure 2: Estimated factors x_t and y_t



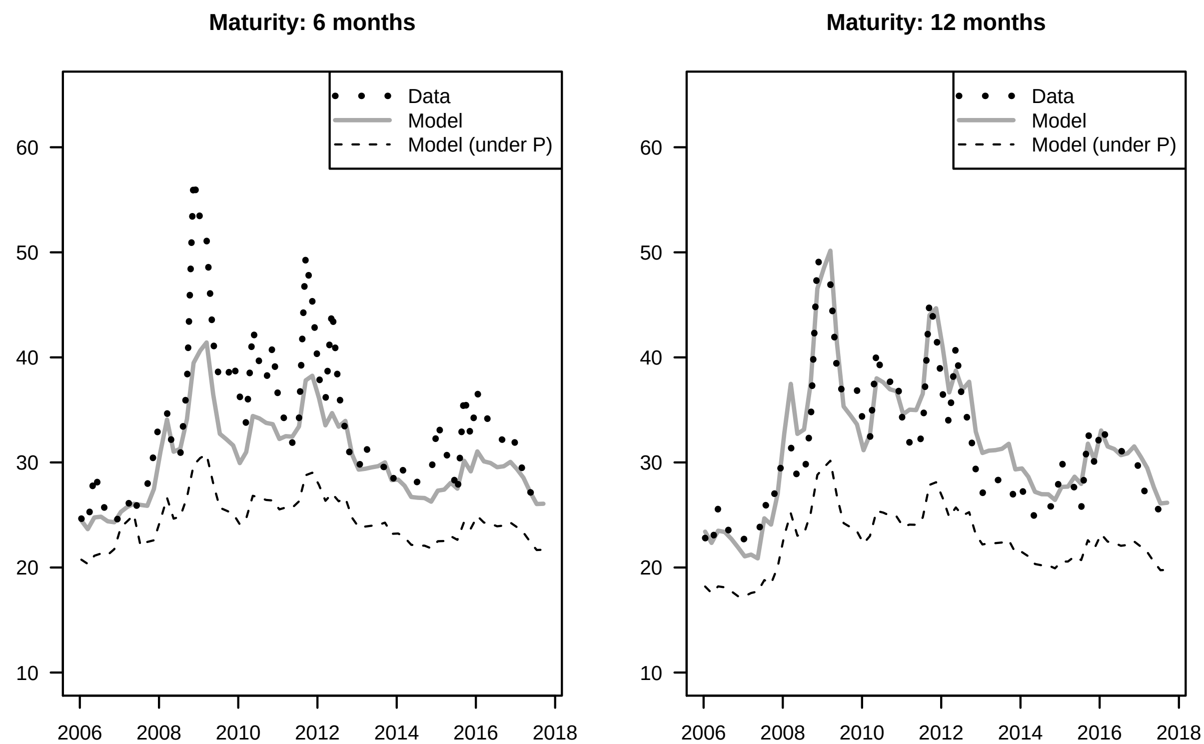
This figure displays the filtered values of x_t and y_t. These values stem from the extended Kalman filter applied on the state-space model whose measurement and transition are eqs. (19) and (20), respectively. Grey-shaded areas are 95% (prediction) confidence intervals.

Figure 3: Fit of iTraxx index swap spreads



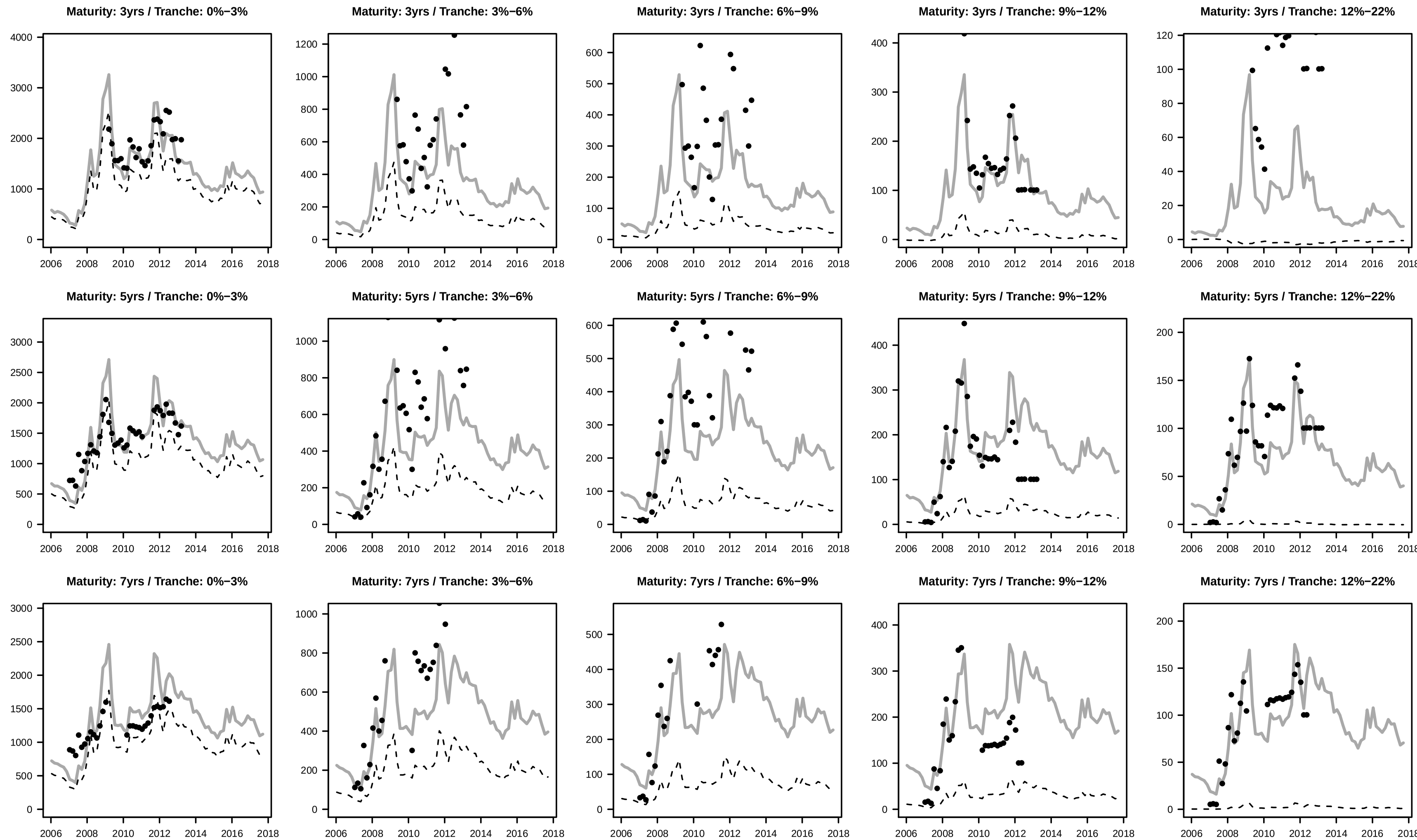
This figure displays index swap spreads (iTraxx Europe main index, solid lines) and their model-implied counterparts (symbols). The data cover the period from January 2006 to September 2017 at the bi-monthly frequency. Spreads are expressed in basis points.

Figure 4: Equity options



This figure displays implied volatilities of put options written on the EURO STOXX 50 index (black dots) and their model-implied counterparts (grey lines). The dashed black lines represent (model-based) implied volatilities that would prevail if agents were risk-neutral; they correspond to the implied volatilities computed under the physical measure $\mathbb{P}$.

Figure 5: Fit of tranche values



This figure displays model-implied iTraxx tranche values (in grey). Black dots represent observed market prices. The dashed black lines correspond to counterfactual tranche prices that would be observed if agents were risk-neutral (i.e. “under $\mathbb{P}$”). The differences between the grey and black lines are credit risk premiums. All prices are expressed in basis points. For each date, maturity and tranche, we convert all quotes into an equivalent running spread with no upfront payment by using the risky duration approach [see e.g. O’Kane and Sen (2003), D’Amato and Gyntelberg (2005), or Morgan Stanley (2011)].

4.2 Dynamic effects of systemic defaults

This subsection examines the dynamic implications of a systemic default. We focus on consumption and stock returns; the implications on credit derivative prices will be considered in the next subsection. The dynamic analysis relies on impulse response functions (IRFs) where the initial shock consists of an unexpected additional default by a systemic entity. Figure 6 displays the results.

The left-hand panel of Figure 6 shows the dynamic responses of the number of systemic defaults following an unexpected systemic default on date $t = 0$. Because of contagion phenomena, the initial default increases the expected number of subsequent systemic defaults. More precisely, it appears that a systemic default triggers two additional systemic defaults in the subsequent two years, on average. The middle panel shows, in black, the response of consumption following the systemic default. This response is gradual, going from 0 to -4% in the two years following the shock. The economic impact of a systemic default is therefore substantial. For the sake of comparison, Laeven and Valencia (2012) find that a systemic banking crisis is, on average, followed by a 23% decrease in output, which would correspond to about 6 defaults of systemic entities (assuming that consumption and GDP move in tandem). Interestingly, in our model, a systemic default has not only an impact on conditional expectations, but also on conditional variances: upon arrival of a systemic default, we observe a jump of the volatility of consumption growth, i.e. a dramatic increase in economic uncertainty (right-hand plot of Figure 6).

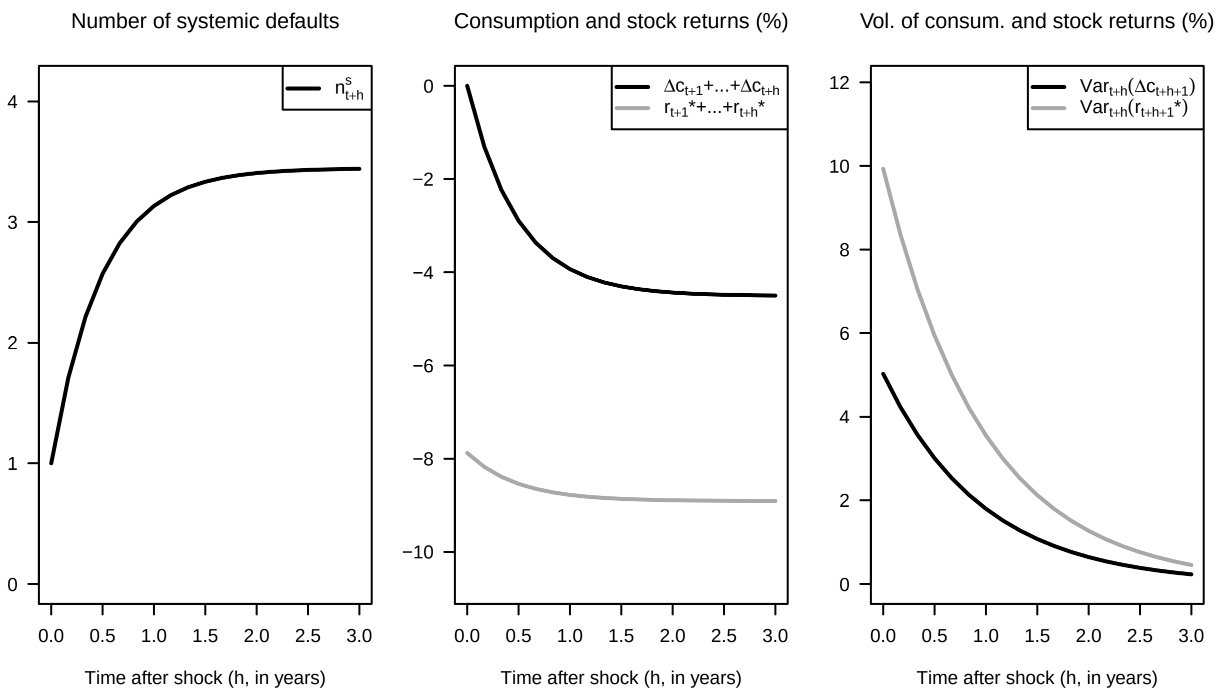
The middle and right-hand panels of Figure 6 further display the respective responses of stock returns r_t^* and their volatility. Following a systemic default, the conditional level and volatility of the stock index undergo the same types of effects as consumption does, except that the stock price response (in level, central plot) is immediate. This is consistent with the forward-looking nature of stock returns.

4.3 Credit risk premiums

Let us now turn to the study of credit risk premiums, defined as the differences between model-implied prices and those prices that would be observed if agents were not risk averse. The latter prices are computed by replacing $\mathbb{E}^{\mathbb{Q}}$ by $\mathbb{E}^{\mathbb{P}} \equiv \mathbb{E}$ in the pricing formulae. Such counterfactual prices are said to be computed under the physical measure $\mathbb{P}$; standard model-implied prices are said to be computed under the risk-neutral measure $\mathbb{Q}$.

Let us first consider the decompositions of Credit Default Swap (CDS) spreads. Figure 7 displays the $\mathbb{P}$ (grey) and $\mathbb{Q}$ (black) CDS spreads associated with maturities of 5 and 10 years. The differences between the two types of spreads are credit risk premiums. The solid lines correspond to spreads of CDSs written on systemic entities. In late 2011, CDS premiums accounted for al-

Figure 6: Responses to an unexpected default of a systemic entity



This figure displays response functions of different variables to an additional default of a systemic entity at date $t = 0$. That is, the initial shock is $n^s_{t=0} = \mathbb{E}(n^s_t) + 1$. The left-hand side panel displays the effect on the number of systemic defaults. The middle panel displays changes in expectations of future consumption and of future stock index. The right-hand panel shows the effect on the expectations of future conditional variances of consumption growth and of stock returns. To facilitate the reading, we plot the square roots of the expected conditional variance.

most 80% of the 10-year CDS spread. Such high risk premiums reflect the fact that the default of a systemic entity is a particularly bad state of the world, i.e. a state of high marginal utility: when it happens, agents dramatically revise their future consumption path downward (consistently with the IRF plotted on the middle panel of Figure 6). For a CDS written on a systemic entity, the protection seller expects to face large losses in bad states of the world. As a result, she is willing to provide this protection only if the compensation is high enough, i.e. if the CDS spread is sufficiently above her expected loss, which translates into high credit risk premiums.

The triangles in Figure 7 correspond to $\mathbb{P}$ (grey) and $\mathbb{Q}$ (black) CDS spreads associated with non-systemic entities. Note that, at this stage, we have not discussed the parameterisation of the number of non-systemic defaults ($n_{3,t}$). Because this number does not cause any other variable in the model (see Figure 1), it does not affect the prices we have considered until then. In particular, it was not necessary to parameterise the conditional distribution of $n_{3,t}$ to estimate the model. Thus we now are free to choose the exposure of non-systemic entities. The triangles in Figure 7 are obtained for the following exposures: $\beta_3 = \mathbb{E}(n_t^s)$ and $c_3 = 0$. Assuming arbitrarily that $I_3 = I_1$, these exposures (β_3, c_3) are such that segment-3 entities feature the same average default probability than the systemic entities (segments 1 and 2). However, Figure 7 shows that the spreads of CDS written on these entities are far lower than those for systemic entities. This figure also shows that the "$\mathbb{P}$ parts" of the CDS spreads of systemic entities and segment-3 entities are close. This was expected as $\mathbb{P}$ CDS spreads essentially reflect default probabilities and segment-3 entities have, on average, the same default probability as systemic entities. The reason why credit risk premiums are far lower for segment-3 entities is that the defaults of such entities tend to occur in *relatively* better states of the world than is the case for systemic entities. Though defaults of segment-3 entities are more likely to happen when y_t is high, the decline in consumption may then remain subdued as long as such a high level of y_t has not triggered (recessionary) defaults of systemic entities.

Again, the exposure (β_3, c_3) chosen for the segment-3 entities was arbitrary. Another exposure (β_3, c_3) would have resulted in different dotted lines in Figure 7. In particular, we could have chosen $\beta_3 < \beta_1$ and $c_3 > c_1$ (say), still keeping the average default probability constant. In this case, compared to systemic entities, a larger fraction of defaults of segment-3 entities would take place in particularly bad states of the world. Accordingly, we would expect higher CDS spreads for this new type of entities than for the systemic ones. Note that they would remain "non systemic" because their default would still not cause consumption growth or other defaults.

Let us define the $\mathbb{Q}$-$\mathbb{P}$ ratio as the ratio between model-implied CDS spreads and the counterfactual $\mathbb{P}$ CDS spreads. Figure 8 explores in a systematic way the relationship between the exposures to the risk factors (β_3, c_3) on the one hand, and the 10-year-maturity $\mathbb{Q}$-$\mathbb{P}$ ratio on the other hand. On Figure 8, we connect, with black lines, those pairs of exposures resulting in the same average $\mathbb{Q}$-$\mathbb{P}$ ratio. We also connect, with dashed grey lines, pairs of exposures resulting in the same average one-year probability of default. While the black square represents the segment-3 entities we considered in Figure 7, the triangle indicates an entity that features the same exposures as our systemic entities. While the average default probabilities of these two types of entities are close, their $\mathbb{Q}$-$\mathbb{P}$ ratios differ substantially (3 and 1, respectively). The figure also shows that, for each average probability of default, there exists a maximum $\mathbb{Q}$-$\mathbb{P}$ ratio. Typically, for a one-year probability of default of 0.4%, the maximum $\mathbb{Q}$-$\mathbb{P}$ ratio is about 3.75.

Credit risk premiums are also present in iTraxx tranche spreads. On Figure 5, these risk premiums are the differences between the grey lines and the dashed black lines: while the grey lines are the model-implied tranche prices, the dotted lines are their $\mathbb{P}$ counterparts, i.e. the (model-implied) prices that would prevail if agents were not risk averse. The more senior the tranche, the higher the relative importance of credit risk premiums. This is consistent with the fact that more senior tranches are more exposed to catastrophic events.

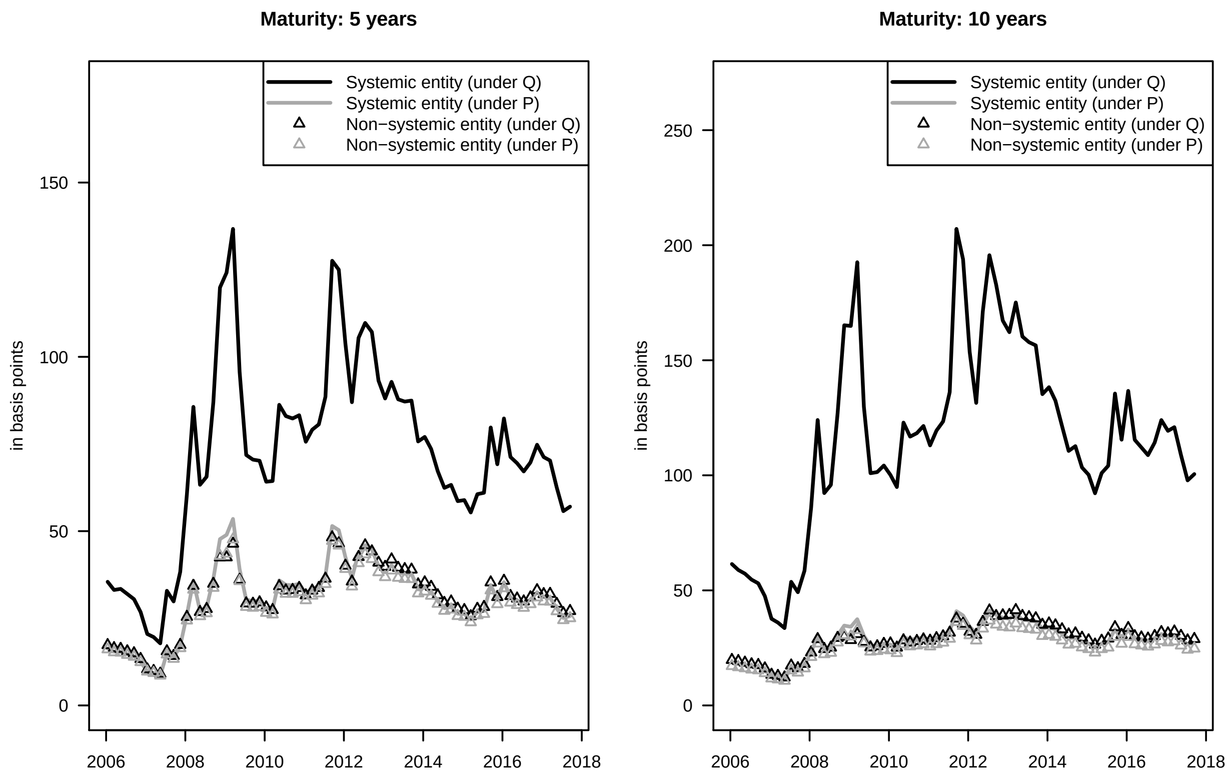
4.4 Measuring systemic risk

Our approach provides us with natural measures of systemic risk, by considering the probabilities of having at least ω systemic defaults (say) at any given horizon h.[12]

As an illustration, Figure 9 plots the probability to observe at least 10 defaults of iTraxx constituents in the next 12 months (dotted line) and 24 months (solid line). We also report vertical lines indicating significant dates of the financial crisis. Our systemic indicators reached their maximum levels in late 2008, after the Lehman bankruptcy and in late 2011, when the European sovereign crisis was at its peak. For these two dates, the conditional probabilities to have more than 10% of defaults among iTraxx constituents within two years were then of 6% and 4%, respectively.

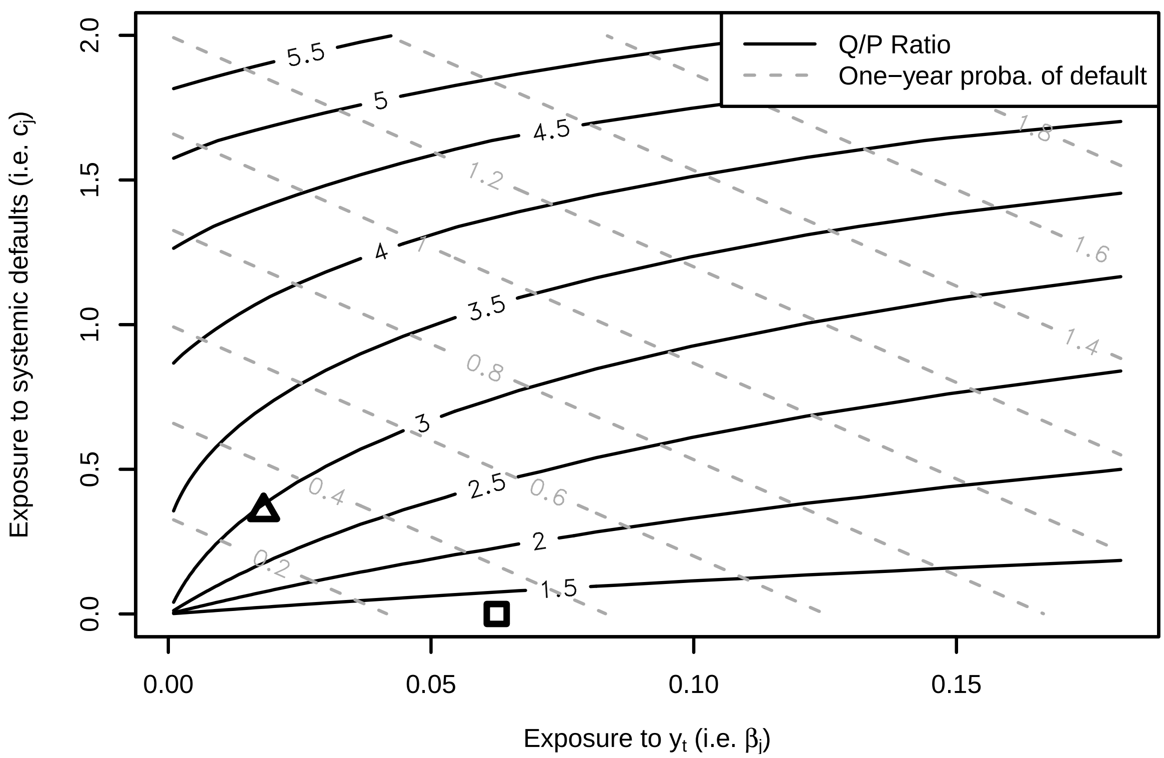
[12] Closed-form formulae can be deduced from a straightforward adaptation of Corollary 2.

Figure 7: Credit risk premiums in iTraxx Europe main indices



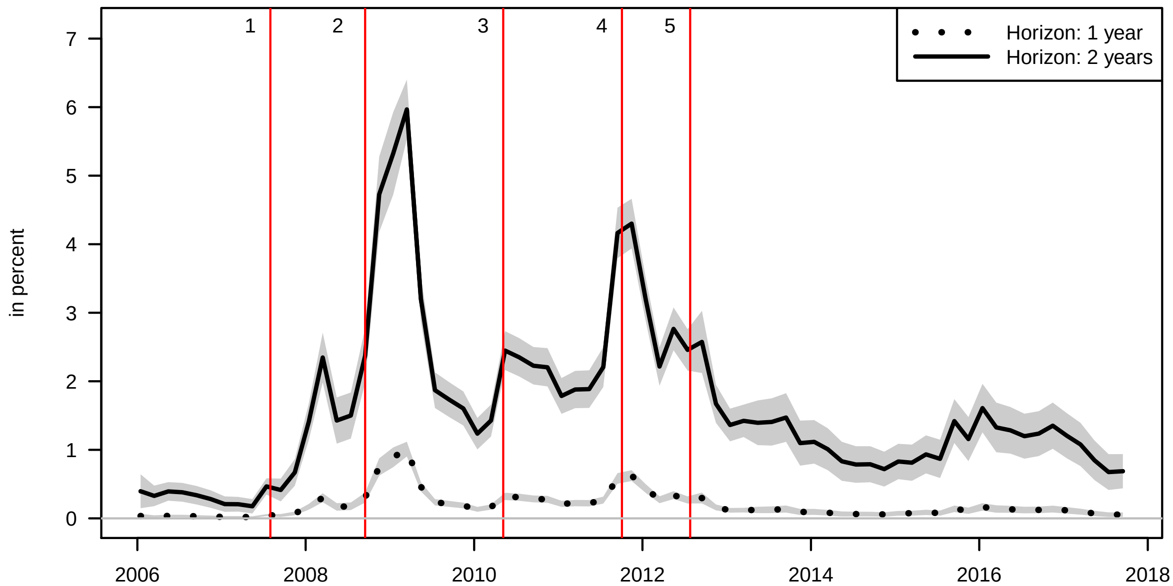
This figure illustrates the importance of credit risk premiums in iTraxx Europe main indices. The black solid line is the model-implied iTraxx index. The grey solid line is the (counterfactual) iTraxx index that would prevail if agents were not risk averse (said to be the iTraxx index "under the physical measure $\mathbb{P}$"). The difference between the black and grey solid lines reflects credit risk premiums. The dotted lines correspond to ($\mathbb{P}$ and $\mathbb{Q}$) CDS spreads associated with a firm from the third segment. See Subsection 4.2 for more details.

Figure 8: Impact of exposures to the exogenous factor y_t (measured by β_j) and to the number of systemic defaults n_t^s (measured by c_j) on the average size of credit risk premiums



This figure illustrates the influence of the exposure to the risk factors – that are the exogenous variable y_t and the number of systemic defaults n_t^s – on the relative importance of risk premiums in CDS spreads. The coordinates of each point correspond to the exposure of a given non-systemic entity to factor y_t (abscissa) and to the number of systemic defaults, i.e. n_t^s (ordinate). The black lines connect those pairs of exposures implying the same Q-P ratio, defined as the ratio between the (model-implied) CDS spread and the counterfactual CDS spread that would be observed if agents were risk-neutral. (The former is the one computed under the pricing, or risk-neutral, measure $\mathbb{Q}$; the latter is computed under the physical measure $\mathbb{P}$, hence the denomination "Q-P ratio".) We consider the 10-year maturity. The grey dashed lines connect pairs of exposures implying the same average probability of default. Figures reported in grey are probabilities of default expressed in annualized percentage points. The triangle indicates a pair of exposures corresponding to the systemic entities. The square indicates the pair of exposures of those non-systemic entities whose CDS indices are displayed on Figure 7.

Figure 9: Probability that at least 10% of iTraxx constituents default in the next two years



This figure displays the (model-implied) probabilities that at least 10% of the iTraxx constituents – considered to be systemic entities – default in the coming 12 months (grey line) and 24 months (black line). Grey-shaded areas are 95% confidence bands; they reflect the uncertainty surrounding filtered x_t and y_t. The vertical bars correspond to important dates of the financial crisis (see Bruegel, http://bruegel.org/2015/09/euro-crisis/): (1) August 2007: European interbank markets seize-up; (2) 15 September 2008: Collapse of Lehman Brothers; (3) 7 May 2010: Emergency measures to safeguard financial stability; (4) October 2011: Spain and Italy are hit by a wave of rating downgrades by the three main rating agencies; (5) 26 July 2012: ECB President Mario Draghi says that the ECB will do “whatever it takes to preserve the euro”.

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A State-vector dynamics

A.1 The dynamics of $z_t = [x_t, y_t]'$

We assume that, conditional on $\{\underline{x_t}, \underline{y_t}\}$, x_{t+1} and y_{t+1} are independently drawn from non-centered Gamma distributions:[13]

$$
\begin{aligned}
x_{t+1}|\underline{x_t}, \underline{y_t} &\sim \gamma_{\nu_x}(\zeta_x x_t, \mu_x) \\
y_{t+1}|\underline{x_t}, \underline{y_t} &\sim \gamma_{\nu_y}(\zeta_{y,x} x_t + \zeta_{y,y} y_t, \mu_y).
\end{aligned}
$$

In this case, we have that (see Monfort et al. (2017)):

$$
\begin{aligned}
x_t &= \mu_x \nu_x + \mu_x \zeta_x x_{t-1} + \sigma_{x,t}\varepsilon_{x,t} \\
y_t &= \mu_y \nu_y + \zeta_{y,x} x_{t-1} + \mu_y \zeta_{y,y} y_{t-1} + \tilde{\sigma}_{y,t}\tilde{\varepsilon}_{y,t},
\end{aligned}
$$

where $\tilde{\varepsilon}_t = [\varepsilon_{x,t}, \tilde{\varepsilon}_{y,t}]'$ is a martingale difference sequence with identity covariance matrix and where:

$$
\begin{aligned}
\sigma_{x,t} &= \mu_x \sqrt{\nu_x + 2\zeta_x x_{t-1}} \\
\tilde{\sigma}_{y,t} &= \mu_y \sqrt{\nu_y + 2\zeta_{y,y} y_{t-1} + 2\zeta_{y,x} x_{t-1}}.
\end{aligned}
$$

Let us use the notations $\rho_x = \mu_x \zeta_x$ and $\rho_y = \mu_y \zeta_{y,y}$ and let us assume that (i) $1 - \rho_x = \mu_x \nu_x = \mu_y \nu_y$ and that (ii) $\rho_x - \rho_y = \mu_y \zeta_{y,x}$. We get:

$$
\left\{
\begin{aligned}
x_t - 1 &= \rho_x (x_{t-1} - 1) + \sigma_{x,t}\varepsilon_{x,t} \\
y_t &= 1 - \rho_x + \rho_x x_{t-1} + \rho_y (y_{t-1} - x_{t-1}) + \tilde{\sigma}_{y,t}\tilde{\varepsilon}_{y,t}.
\end{aligned}
\right.
$$

Defining $\varepsilon_{y,t} = \dfrac{\tilde{\sigma}_{y,t}\tilde{\varepsilon}_{y,t} - \sigma_{x,t}\varepsilon_{x,t}}{\sqrt{\tilde{\sigma}_{y,t}^2 + \sigma_{x,t}^2}}$ and $\sigma_{y,t} = \sqrt{\tilde{\sigma}_{y,t}^2 + \sigma_{x,t}^2}$ leads to System (1).

[13] The random variable W is drawn from a non-centered Gamma distribution $\gamma_\nu(\varphi, \mu)$, iif there exists a $\mathscr{P}(\varphi)$-distributed variable Z such that $W|Z \sim \gamma(\nu + Z, \mu)$ where Z and μ are, respectively, the shape and scale parameters of the Gamma distribution [see e.g. Gouriéroux and Jasiak (2006)]. When $Z = 0$ and $\nu = 0$, then $W = 0$. When $\nu = 0$, this distribution is called Gamma$_0$ distribution; this case is introduced and studied by Monfort et al. (2017).

A.2 The conditional log-Laplace transform of $X_t = [x_t, y_t, w_t, N_t', N_{t-1}']'$

The dynamics followed by $X_t = [x_t, y_t, w_t, N_t', N_{t-1}']'$ is a special case of the general case treated in the Online Appendix (see O.1) with:

$$\zeta_F = \begin{bmatrix} \zeta_x & \zeta_{y,x} & 0 \\ 0 & \zeta_{y,y} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \zeta_n = \begin{bmatrix} 0 & 0 & \xi_w \\ 0 & 0 & \xi_w \\ 0 & 0 & 0 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 & 0 & 0 \\ \beta_1 & \beta_2 & \beta_3 \\ 0 & 0 & 0 \end{bmatrix}, \quad c = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_1 & c_2 & c_3 \\ 0 & 0 & 0 \end{bmatrix},$$

and $\mu = [\mu_x, \mu_y, \mu_w]'$, $\nu = [\nu_x, \nu_y, 0]'$.

As shown in the Online Appendix (O.1), in this case, the conditional log Laplace transform of X_t is given by:

$$\mathbb{E}_t\left(\exp(v'X_{t+1})\right) = \exp(\psi(v, X_t)) = \exp(\psi_0(\nu) + \psi_1(\nu)'X_t), \tag{a.1}$$

where

$$\begin{cases} \psi_0(v) &= d\left(\sum_{j=1}^J(\exp(v_{B,j})-1)\beta_j + v_A\right) \\ \psi_1(v) &= \quad [\psi_{1,A}(v)', \psi_{1,B}(v)', \psi_{1,C}(v)']', \end{cases} \tag{a.2}$$

with $d(w) = -\nu' \log(1 - w \odot \mu)$, where $\odot$ is the element-by-element (Hadamard) product (and where, by abuse of notations, the log operator is applied element-by-element wise) and where:

$$\begin{cases} \psi_{1,A}(v) &= a(\sum_{j=1}^J(\exp(v_{B,j})-1)\beta_j + v_A) \\ \psi_{1,B}(v) &= b(\sum_{j=1}^J(\exp(v_{B,j})-1)\beta_j + v_A) + \sum_{j=1}^J(\exp(v_{B,j})-1)c^j + v_B + v_C \\ \psi_{1,C}(v) &= c(\sum_{j=1}^J(\exp(v_{B,j})-1)\beta_j + v_A) - \sum_{j=1}^J(\exp(v_{B,j})-1)c^j, \end{cases} \tag{a.3}$$

where β^j and c^j respectively denote the j^{th} columns of β and of c, and where $v = [v_A', v_B', v_C']'$, v_A being a n_F-dimensional vector and v_B and v_C being J-dimensional vectors, and with:

$$a(w) \;=\; \zeta_F\left(\frac{w \odot \mu}{1 - w \odot \mu}\right), \quad b(w) = \zeta_n\left(\frac{w \odot \mu}{1 - w \odot \mu}\right), \quad c(w) = -b(w),$$

where, again, the log and division operator are applied element-by-element wise, by abuse of notations.

B S.d.f. derivation

B.1 Proof of Prop. 1

Let us consider the following specification for Δu_t:

$$\Delta u_t = \mu_{u,0} + \mu_{u,1}' X_t + \mu_{u,2}' X_{t-1}.$$

Then, for a given $[X_t', X_{t-1}']'$, we have:

$$\begin{aligned} \text{eq. (6)} \quad \Leftrightarrow \quad & \mu_{u,0} + \mu_{u,1}' X_t + \mu_{u,2}' X_{t-1} \\ & = \mu_{c,0} + \mu_{c,1}' X_t + \frac{\delta}{1-\delta} \frac{1}{1-\gamma} \left\{ [\psi_1((1-\gamma)\mu_{u,1}) + (1-\gamma)\mu_{u,2}]'(X_t - X_{t-1}) \right\}. \end{aligned}$$

Therefore eq. (6) is satisfied for any $[X_t', X_{t-1}']'$ iff

$$\left\{ \begin{array}{l} \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) + \frac{1}{1-\delta}\mu_{u,2} = 0 \\ \mu_{u,1} - \mu_{c,1} - \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) - \frac{\delta}{1-\delta}\mu_{u,2} = 0 \\ \mu_{u,0} = \mu_{c,0}, \end{array} \right.$$

or

$$\left\{ \begin{array}{l} \frac{\delta}{1-\delta}\frac{1}{1-\gamma}\psi_1((1-\gamma)\mu_{u,1}) + \frac{1}{1-\delta}\mu_{u,2} = 0 \\ \mu_{u,1} + \mu_{u,2} - \mu_{c,1} = 0 \\ \mu_{u,0} = \mu_{c,0}, \end{array} \right. \tag{a.4}$$

which leads to the result.

B.2 Proof of Prop. 2

Epstein and Zin (1989) have shown that when agent's preferences are as in eq. (6), the s.d.f. is given by:

$$M_{t,t+1} = \delta \left(\frac{C_{t+1}}{C_t} \right)^{-1} \frac{\exp[(1-\gamma)u_{t+1}]}{\mathbb{E}_t(\exp[(1-\gamma)u_{t+1}])}.$$

Therefore, we have:

$$\begin{aligned}
\log M_{t,t+1} &= \log\delta - \Delta c_{t+1} + (1-\gamma)u_{t+1} - \log\mathbb{E}_t(\exp[(1-\gamma)u_{t+1}]) \\
&= \log(\delta) - \mu_{c,0} - \mu_{c,1}' X_{t+1} + (1-\gamma)(\mu_{u,0} + \mu_{u,1}' X_{t+1} + \mu_{u,2}' X_t) \\
&\quad - \log\mathbb{E}_t(\exp[(1-\gamma)(\mu_{u,0} + \mu_{u,1}' X_{t+1} + \mu_{u,2}' X_t)]) \\
&= \log(\delta) - \mu_{c,0} - \psi_0((1-\gamma)\mu_{u,1}) + [(1-\gamma)\mu_{u,1} - \mu_{c,1}]' X_{t+1} - \psi_1((1-\gamma)\mu_{u,1})' X_t,
\end{aligned}$$

which leads to the result.

C Pricing formulae

C.1 Generic pricing formulae

C.1.1 Pricing of $\exp(u'X_{t+h})$ and $v'X_{t+h}$, settled at date $t+h$

Proposition 3. *The date-t price $p(u,h,X_t)$ of the payoff $\exp(u'X_{t+h})$, that is settled at date $t+h$ is given by $\exp(\Gamma_{0,h}(u) + \Gamma_{1,h}'(u)X_t)$, where:*

$$\begin{cases}
\Gamma_{1,h+1}(u) &= \psi_1^{\mathbb{Q}}(\Gamma_{1,h}(u)) - \eta_1 \\
\Gamma_{0,h+1}(u) &= \psi_0^{\mathbb{Q}}(\Gamma_{1,h}(u)) - \eta_0 + \Gamma_{0,h}(u)
\end{cases}$$

with $\Gamma_{1,0}(u) = u$ and $\Gamma_{0,0}(u) = 0$.

Proof. This proposition is clearly satisfied for $h=0$. Assume that, for a given $h \geq 0$ and for all admissible u and X_t, we have $p(u,h,X_t) = \exp(\Gamma_{0,h}(u) + \Gamma_{1,h}(u)'X_t)$, then

$$\begin{aligned}
p(u,h+1,X_t) &= \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(p(u,h,X_{t+1})) \\
&= \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(\Gamma_{0,h}(u) + \Gamma_{1,h}(u)'X_{t+1})) \\
&= \exp(-\eta_0 + \Gamma_{0,h}(u) - \eta_1' X_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(\Gamma_{1,h}(u)'X_{t+1})) \\
&= \exp(-\eta_0 + \Gamma_{0,h}(u) - \eta_1' X_t)\exp\left(\psi_0^{\mathbb{Q}}(\Gamma_{1,h}(u)) + \psi_1^{\mathbb{Q}}(\Gamma_{1,h}(u))'X_t\right),
\end{aligned}$$

which leads to the result. □

Corollary 1. *The date-t price of the payoff $v'X_{t+h}$, conditional on $X_t = x$, with payoff settlement at date $t+h$, is given by:*

$$\Pi(v,h,x) = v'\nabla_u p(u,h,x)\big|_{u=0}, \tag{a.5}$$

where $p(u,h,x)$ is defined in Proposition 3 and where ∇_u denotes the Jacobian operator with respect to the first argument of the function.

Let us denote by $\mathbf{0}_{r\times c}$ and $\mathbf{1}_{r\times c}$ the matrices of dimensions $r\times c$ filled with 0 and 1, respectively. In addition, let e_j denote the j^{th} row vector of the identify matrix of dimension $J\times J$. Using the previous corollary with $v' = [\mathbf{0}_{1\times 3}, e_j, \mathbf{0}_{1\times J}]$ and $v' = [\mathbf{0}_{1\times 3}, \mathbf{0}_{1\times J}, e_j]$ respectively results in the prices of the payoffs $N_{j,t+h}$ and $N_{j,t+h-1}$, settled at date $t+h$.

Corollary 2. *The date-t price of the payoff* $\exp(a'X_{t+h})\mathbb{1}_{\{b'X_{t+h}<y\}}$*, conditional on $X_t = x$, with payoff settlement at date $t+h$, is given by:*

$$\begin{aligned} g(a,b,y,h,x) &= \mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp(a'X_{t+h})\mathbb{1}_{\{b'X_{t+h}<y\}}|X_t = x\right) \\ &= \frac{p(a,h,x)}{2} - \frac{1}{\pi}\int_0^{\infty}\frac{Im\left[p(a+ivb,h,x)\exp(-ivy)\right]}{v}dv, \end{aligned} \tag{a.6}$$

where $Im(z)$ denotes the imaginary part of the complex number z and where $\lambda_{t,t+h}$ is defined in eq. (12).

This result is proved in Duffie et al. (2000). Note that the formula for $g(a,b,y,h,x)$ is quasi explicit since it only involves a simple (one-dimensional) integration.

Corollary 3. *The date-t price of the payoff $a'X_{t+h}\mathbb{1}_{\{b'X_{t+h}<y\}}$, conditional on $X_t = x$, with payoff settlement at date $t+h$, is given by:*

$$\Gamma(a,b,y,h,x) = a'\nabla_u g(u,b,y,h,x)\big|_{u=0}. \tag{a.7}$$

Let us consider the date-t prices of the following payoffs, settled at date $t+k$: (i) $\mathbb{1}_{\{N_{1,t+k}<z\}}$, (ii) $N_{1,t+k}\mathbb{1}_{\{N_{1,t+k}<z\}}$ and (iii) $N_{1,t+k-1}\mathbb{1}_{\{N_{1,t+k}<z\}}$. Using the notations introduced in Corollaries 2 and 3, these prices respectively write: (i) $g(0,\omega_0,z,h,X_t)$, (ii) $\Gamma(\omega_0,\omega_0,z,h,X_t)$ and (iii) $\Gamma(\omega_1,\omega_0,z,h,X_t)$, with $\omega_0 = [\mathbf{0}_{1\times 3}, e_1, \mathbf{0}_{1\times J}]'$ and $\omega_1 = [\mathbf{0}_{1\times 3}, \mathbf{0}_{1\times J}, e_1]'$, where e_1 is a J-dimensional vector whose entries are 0, except the first one that is equal to 1.

C.1.2 Pricing of $\exp(u_1'X_{t+1}+\cdots+u_1'X_{t+h-1}+u_2'X_{t+h})$, settled at date $t+h$

Proposition 4. *Using the notation $\mathbf{u} = \{u_1,u_2\}$, the date-t price $\tilde{p}(\mathbf{u},h,X_t)$ of the payoff*

$$\exp(u_1'X_{t+1}+\cdots+u_1'X_{t+h-1}+u_2'X_{t+h}), \quad \textit{for } h>1$$

and of $\exp(u_2' X_{t+1})$ *for* $h = 1$*, settled at date* $t+h$*, is given by* $\exp\left(\tilde{\Gamma}_{0,h}(u) + \tilde{\Gamma}_{1,h}(u)' X_t\right)$*, where:*

$$\begin{cases} \tilde{\Gamma}_{1,h+1}(u) & = \psi_1^{\mathbb{Q}}(\tilde{\Gamma}_{1,h}(u) + u_1) - \eta_1 \\ \tilde{\Gamma}_{0,h+1}(u) & = \psi_0^{\mathbb{Q}}(\tilde{\Gamma}_{1,h}(u) + u_1) - \eta_0 + \tilde{\Gamma}_{0,h}(u) \end{cases}$$

with $\tilde{\Gamma}_{1,1}(u) = \Gamma_{1,1}(u)$ *and* $\tilde{\Gamma}_{0,1}(u) = \Gamma_{0,1}(u)$*.*

Proof. This proposition is clearly satisfied for $h = 1$. Assume that, for a given $h \geq 1$ and for all admissible $\mathbf{u}$ and X_t, we have $\exp\left(\tilde{\Gamma}_{0,h}(u) + \tilde{\Gamma}_{1,h}(u)' X_t\right)$, then

$$\begin{aligned} \tilde{p}(u, h+1, X_t) &= \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(u_1' X_{t+1})\tilde{p}(u, h, X_{t+1})) \\ &= \exp(-r_t)\mathbb{E}_t^{\mathbb{Q}}(\exp(\tilde{\Gamma}_{0,h}(u) + \tilde{\Gamma}_{1,h}(u)' X_{t+1} + u_1' X_{t+1})) \\ &= \exp(-\eta_0 + \tilde{\Gamma}_{0,h}(u) - \eta_1' X_t)\mathbb{E}_t^{\mathbb{Q}}(\exp([\tilde{\Gamma}_{1,h}(u) + u_1]' X_{t+1})) \\ &= \exp(-\eta_0 + \tilde{\Gamma}_{0,h}(u) - \eta_1' X_t)\exp\left(\psi_0^{\mathbb{Q}}(\tilde{\Gamma}_{1,h}(u) + u_1) + \psi_1^{\mathbb{Q}}(\tilde{\Gamma}_{1,h}(u) + u_1)' X_t\right), \end{aligned}$$

which leads to the result. □

C.2 Tranche products formula

Let's rewrite eq. (13):

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh} \Lambda_{t,t+k} \frac{\tilde{N}_{t+k} - \tilde{N}_{t+k-1}}{\bar{b} - \bar{a}} \mathbb{1}_{\{\bar{a} < \tilde{N}_{t+k} \leq \bar{b}\}}\right\} \\ = \; & U_{t,h}^{TDS}(a,b) + \mathbb{E}_t^{\mathbb{Q}}\left\{\frac{S_{t,h}^{TDS}(a,b)}{q} \sum_{k=1}^{qh} \Lambda_{t,t+k}\left(\mathbb{1}_{\{\tilde{N}_{t+k} \leq \bar{a}\}} + \frac{\bar{b} - \tilde{N}_{t+k}}{\bar{b} - \bar{a}} \mathbb{1}_{\{\bar{a} < \tilde{N}_{t+k} \leq \bar{b}\}}\right)\right\}, \end{aligned}$$

where $\bar{a} = a\frac{\tilde{I}}{1-RR}$ and $\bar{b} = b\frac{\tilde{I}}{1-RR}$. We obtain:

$$S_{t,h}^{TDS}(a,b) = \frac{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh} \Lambda_{t,t+k}(\tilde{N}_{t+k} - \tilde{N}_{t+k-1})\left(\mathbb{1}_{\{\tilde{N}_{t+k} \leq \bar{b}\}} - \mathbb{1}_{\{\tilde{N}_{t+k} \leq \bar{a}\}}\right)\right\} - (\bar{b} - \bar{a})U_{t,h}^{TDS}(a,b)}{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh} \Lambda_{t,t+k}\left((\bar{b} - \bar{a})\mathbb{1}_{\{\tilde{N}_{t+k} \leq \bar{a}\}} + (\bar{b} - \tilde{N}_{t+k})\left(\mathbb{1}_{\{\tilde{N}_{t+k} \leq \bar{b}\}} - \mathbb{1}_{\{\tilde{N}_{t+k} \leq \bar{a}\}}\right)\right)\right\}}.$$

C.3 Approximated stock returns

Proposition 5. *If the log growth rate of dividends is affine in* X_t*, i.e. if:*

$$g_{d,t} = \mu_{d,0} + \mu_{d,1}' X_t, \tag{a.8}$$

then stock returns are approximately given by:

$$r^s_{t+1} = \kappa_0 + A_0(\kappa_1 - 1) + \mu_{d,0} + (\kappa_1 A_1 + \mu_{d,1})' X_{t+1} - A_1' X_t, \tag{a.9}$$

where κ_0 and κ_1 are given by

$$\begin{cases} \kappa_1 &= \dfrac{\exp(\bar{\tau})}{1+\exp(\bar{\tau})} \\ \kappa_0 &= \log(1+\exp(\bar{\tau})) - \kappa_1 \bar{\tau}, \end{cases} \tag{a.10}$$

where A_1 satisfies

$$\psi_1(\kappa_1 A_1 + \mu_{d,1} + \pi) = A_1 + \eta_1 + \psi_1(\pi),$$

and where

$$A_0 = \frac{-\kappa_0 - \mu_{d,0} + \eta_0 + \psi_0(\pi) - \psi_0(\kappa_1 A_1 + \mu_{d,1} + \pi)}{\kappa_1 - 1}.$$

Proof. Let us introduce the log price-dividend ratio defined by $\tau_t = \log(P_t/D_t)$ and let us denote by $\bar{\tau}$ its marginal expectation. The following lemma is based on the log-linearisation proposed by Campbell and Shiller (1988).

Lemma 1. *if $\tau_t - \bar{\tau}$ is relatively small, then the stock return r^s_{t+1} can be approximated by*

$$r^s_{t+1} = \log\left(\frac{P_{t+1} + D_{t+1}}{D_t}\right) \approx \kappa_0 + \kappa_1 \tau_{t+1} - \tau_t + g_{d,t+1}. \tag{a.11}$$

Proof. See Online Appendix O.3 □

Assume that τ_t is affine in X_t, i.e.:

$$\tau_t = A_0 + A_1' X_t. \tag{a.12}$$

Substituting for τ_t in eq. (a.11) leads to eq. (a.9). Let us now determine the constraints that should be satisfied by A_0 and A_1. The returns of stocks have to satisfy the Euler equation:

$$0 = \log \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+1} \exp(r^s_{t+1})). \tag{a.13}$$

Using eqs. (8) and (a.9), we obtain:

$$M_{t,t+1} \exp(r^s_{t+1})) = \tag{a.14}$$
$$\exp(\kappa_0 + A_0(\kappa_1 - 1) + \mu_{d,0} - \eta_0 - \psi_0(\pi) + (\kappa_1 A_1 + \mu_{d,1} + \pi)' X_{t+1} - (A_1 + \eta_1 + \psi_1(\pi))' X_t).$$

eqs. (a.14) and (a.13) are satisfied if:

$$\left\{\begin{array}{rcl} \kappa_0 + A_0(\kappa_1 - 1) + \mu_{d,0} - \eta_0 - \psi_0(\pi) + \psi_0(\kappa_1 A_1 + \mu_{d,1} + \pi) & = & 0 \\ \psi_1(\kappa_1 A_1 + \mu_{d,1} + \pi) - (A_1 + \eta_1 + \psi_1(\pi)) & = & 0, \end{array}\right.$$

which proves Prop. 5. □

C.4 Equity option pricing

If τ_t and $g_{d,t}$ are affine in X_t (as in eqs. (a.8) and (a.12)), then eq. (17) implies that P_{t+h}/P_t is exponential affine in X_t.

Let us introduce function φ defined by:

$$(u,h,X_t) \to \varphi(u,h,X_t) = \mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp\left(u\log\left(\frac{P_{t+h}}{P_t}\right)\right)\right).$$

Using eq. (17) and Prop. 4 (see Subsection C.1.2), one can find functions $a_h^s(\bullet)$ and $b_h^s(\bullet)$ that are such that:

$$\varphi(u,h,X_t) = \exp(a_h^s(u) + b_h^s(u)'X_t).$$

Replacing $p(a,h,x)$ by $\varphi(a,h,x)$ in Corollary 2 of Subsection C.1.1 provides formulae to compute

$$\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp\left[a\log\left(\frac{P_{t+h}}{P_t}\right)\right]\mathbb{1}_{\left\{b\log\left(\frac{P_{t+h}}{P_t}\right)<y\right\}}\middle|X_t = x\right).$$

Let us denote by $g^*(a,b,y,h,x)$ the previous expression. With this notation, the price of a put option (eq. 18) reads:

$$\begin{array}{rl} & \mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}(K-P_{t+h})\mathbb{1}_{\{K>P_{t+h}\}}\right) \\ = & K\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\mathbb{1}_{\{r^*_{t+1}+\dots+r^*_{t+h}<\log(K)-\log P_t\}}\right) \\ & -P_t\mathbb{E}_t^{\mathbb{Q}}\left(\Lambda_{t,t+h}\exp(r^*_{t+1}+\dots+r^*_{t+h})\mathbb{1}_{\{r^*_{t+1}+\dots+r^*_{t+h}<\log(K)-\log P_t\}}\right) \\ = & Kg^*(0,1,\log(K)-\log P_t,h,X_t) - P_t g^*(1,1,\log(K)-\log P_t,h,X_t). \end{array}$$

D Data

D.1 Credit index and tranche prices (iTraxx)

D.1.1 The iTraxx credit index and constituents

To estimate the model, we employ financial data based on the iTraxx Europe main index, an index involving 125 large European firms. iTraxx indices roll every six month. That is, every six months, a new series of the index is created with updated constituents. Derivatives written on previous series continues trading, although liquidity is concentrated on options written on the on-the-run series [see Markit (2014)].

The roll consists of a series of steps which are administered by Markit, a financial services information company that owns and compiles CDX and iTraxx indices. For the Markit iTraxx Europe indices, liquidity lists are formed from the trading volumes from the DTCC Trade Information Warehouse.[14] Markit then applies index rules to determine the index constituents among the most liquid names [see e.g. Markit (2016)]. For iTraxx Europe main (the index used in this study), the final Index comprises 30 Autos & Industrials, 30 Consumers, 20 Energy, 20 TMT, 25 Financials.

Constituents of the iTraxx Europe main index must have an investment grade rating. That is, to be included in the list of constituents, entities have to be rated BBB-/Baa3/BBB- (Fitch/-Moody's/S&P) or higher. In March 2016, the median rating for the iTraxx index (series 25) is BBB+ at S&P [Société Générale (2016)].

D.1.2 Data sources and preliminary transformations

We extract spreads of iTraxx indices from Thomson Datastream. These spreads correspond to maturities of 3, 5, 7 and 10 years. iTraxx tranche prices come from the Markit website.[15] For each maturity, we use prices associated with the following tranches: 0%-3%, 3%-6%, 6%-9%, 9%-12% and 12%-22%. We do not use prices associated with the super-senior tranche (22%-100%) as well as prices associated with the 10-year maturity given the very low liquidity of these contracts. Note also that, for liquidity reasons, our Markit data do not cover all dates in our sample. In particular, we do not have tranche prices before January 2007 and after March 2013.

Because each index roll features fixed maturity dates, market prices are not of the "constant-maturity" type. To deal with this, for each considered maturity and for (i) each date and (ii) each pair of attachment/detachment points, we look for the tranche price whose residual maturity is the

[14] http://www.dtcc.com/derivatives-services/trade-information-warehouse.

[15] http://www.creditfixings.com/CreditEventAuctions/itraxx.jsp. For each date, maturity and tranche, we convert all quotes into an equivalent running spread with no upfront payment by using the risky duration approach [see e.g. O'Kane and Sen (2003), D'Amato and Gyntelberg (2005), or Morgan Stanley (2011)].

closest to the considered one. If the residual maturity of the resulting tranche is not in a ± 1 year window around the targeted maturity, no price is reported.

D.2 Equity options (EURO STOXX 50)

Equity put options are far out-of-the-money options written on the EURO STOXX 50 index. We consider two maturities, 6 and 12 months, and strikes equal to 70% of the current value of the index. That is, the payoffs of these options become strictly positive in case of a fall of the index by more than 30%. Note that such option prices are not directly available on Thomson Datastream; option prices reported on those database are for contracts with standardized maturity dates and strikes. We compute the prices of our out-of-money options by applying interpolation splines on available data, in both the time and strike dimensions. Following market convention, we convert our put option prices into implied volatilities using the Black-Scholes formula.

– Online Appendix –

Disastrous Defaults

Christian GOURIÉROUX, Alain MONFORT, Sarah MOUABBI and Jean-Paul RENNE

O.1 A general model of X_t's dynamics

We consider the state vector $X_t = [F_t', N_t', N_{t-1}']'$, with $F_t = [x_t, y_t, w_t]'$. Vectors F_t and N_t are, respectively, n_F-dimensional and J-dimensional. Conditional on $\underline{X_t} = \{X_t, X_{t-1}, \dots\}$, the different components of F_{t+1} are independent and drawn from non-centered Gamma distributions:[16]

$$F_{i,t+1}|\underline{X_t} \sim \gamma_{\nu_i}(\zeta_{i,0} + \zeta_{i,F}' F_t + \zeta_{i,n}' n_t, \mu_i),$$

where ν_i, $\zeta_{i,0}$ and μ_i are scalar, $\zeta_{i,F}$ is a n_F-dimensional vector and $\zeta_{i,n}$ is a J-dimensional vector.

In this case, we have (see Monfort et al. (2017)):

$$\mathbb{E}_t\left(\exp(w'F_{t+1})\right) = \exp\left(a(w)'F_t + b(w)'N_t + c(w)'N_{t-1} + d(w)\right), \text{ for any } w \in V. \tag{a.15}$$

with

$$\begin{aligned} a(w) &= \zeta_F\left(\frac{w\odot\mu}{1-w\odot\mu}\right), \quad b(w) = \zeta_n\left(\frac{w\odot\mu}{1-w\odot\mu}\right), \quad c(w) = -b(w), \\ d(w) &= \zeta_0'\left(\frac{w\odot\mu}{1-w\odot\mu}\right) - \nu'\log(1-w\odot\mu), \end{aligned} \tag{a.16}$$

and where V is the set of vector w whose components w_i are in $]-\infty, 1/\mu_i[$. with $\zeta_F = [\zeta_{1,F}, \dots, \zeta_{n_F,F}]$, $\zeta_n = [\zeta_{1,n}, \dots, \zeta_{J,n}]$, $\zeta_0 = [\zeta_{1,0}, \dots, \zeta_{n_F,0}]'$, $\mu = [\mu_1, \dots, \mu_{n_F}]'$, $\nu = [\nu_1, \dots, \nu_{n_F}]'$, where $\odot$ is the element-by-element (Hadamard) product and where, by abuse of notations, the log and division operator are applied element-by-element wise.

[16] The random variable W is drawn from a non-centered Gamma distribution $\gamma_\nu(\varphi, \mu)$, iif there exists a $\mathscr{P}(\varphi)$-distributed variable Z such that $W|Z \sim \gamma(\nu + Z, \mu)$ where Z and μ are, respectively, the shape and scale parameters of the Gamma distribution [see e.g. Gouriéroux and Jasiak (2006)]. When $Z = 0$ and $\nu = 0$, then $W = 0$. When $\nu = 0$, this distribution is called Gamma$_0$ distribution; this case is introduced and studied by Monfort et al. (2017).

Besides, conditional on $\Omega_t^* = \{F_{t+1}, \Omega_t\}$, we assume that $n_{j,t} = \Delta N_{j,t},\ j = 1, \ldots, J$ are independent with Poisson distributions:

$$n_{j,t+1}|\Omega_t^* \sim \mathscr{P}(\beta_j' F_{t+1} + c_j' n_t + \gamma_j). \tag{a.17}$$

Proposition O.1. *The log conditional Laplace transform of process (X_t), denoted by $\psi(v, X_t)$ and defined by:*

$$\mathbb{E}_t\left(\exp(v' X_{t+1})\right) = \exp(\psi(v, X_t)), \tag{a.18}$$

is affine in X_t. That is, we have:

$$\psi(v, X_t) = \psi_0(v) + \psi_1(v)' X_t, \tag{a.19}$$

with

$$\psi_0(v) = d\left(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A\right) + \sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\gamma_j,$$

and where $\psi_1(v) = [\psi_{1,A}(v)', \psi_{1,B}(v)', \psi_{1,C}(v)']'$, where:

$$\begin{cases} \psi_{1,A}(v) & = a(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A) \\ \psi_{1,B}(v) & = b(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A) + \sum_{j=1}^{J}(\exp(v_{B,j}) - 1)c_j + v_B + v_C \\ \psi_{1,C}(v) & = c(\sum_{j=1}^{J}(\exp(v_{B,j}) - 1)\beta_j + v_A) - \sum_{j=1}^{J}(\exp(v_{B,j}) - 1)c_j, \end{cases} \tag{a.20}$$

with $v = [v_A', v_B', v_C']'$, v_A being a n_F-dimensional vector and v_B and v_C being J-dimensional vectors.

Proof. We have:

$$\begin{aligned} & \mathbb{E}_t\left(\exp((v_A', v_B', v_C')X_{t+1})\right) \\ = & \mathbb{E}_t\left(\exp(v_A' F_{t+1} + v_B' N_{t+1} + v_C' N_t)\right) = \mathbb{E}_t\left(\mathbb{E}\left(\exp(v_A' F_{t+1} + v_B' N_{t+1} + v_C' N_t)\middle| \Omega_t, F_{t+1}\right)\right) \\ = & \mathbb{E}_t\left(\exp(v_A' F_{t+1} + (v_B + v_C)' N_t)\mathbb{E}\left(\exp(v_B'(N_{t+1} - N_t))\middle| \Omega_t^*\right)\right) \\ = & \mathbb{E}_t\left(\exp(v_A' F_{t+1} + (v_B + v_C)' N_t)\mathbb{E}\left(\exp\left[\sum_{j=1}^{J} v_{B,j}(N_{j,t+1} - N_{j,t})\right]\middle| \Omega_t^*\right)\right) \\ = & \mathbb{E}_t\left(\exp\left(v_A' F_{t+1} + (v_B + v_C)' N_t + \sum_{j=1}^{J}\left(\beta_j' F_{t+1} + c_j'(N_t - N_{t-1}) + \gamma_j\right)(e^{v_{B,j}} - 1)\right)\right), \end{aligned}$$

the last equality resulting from the fact that the $n_{j,t+1}$s, are independent conditional on Ω_t^*. Therefore:

$$
\begin{aligned}
& \mathbb{E}_t\left(\exp((v_A', v_B', v_C')X_{t+1})\right) \\
= \; & \exp\left(\left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)c_j + v_B + v_C\right\}' N_t - \left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)c_j\right\}' N_{t-1} + \sum_{j=1}^{J}\gamma_j(e^{v_{B,j}}-1)\right) \times \\
& \mathbb{E}_t\left(\exp\left(\left\{\sum_{j=1}^{J}(e^{v_{B,j}}-1)\beta_j + v_A\right\}' F_{t+1}\right)\right),
\end{aligned}
$$

and the result follows. □

O.2 Conditional and unconditional moments of $[F_t', n_t']'$

We have (see Monfort et al. (2017)):

$$
\begin{aligned}
\mathbb{E}(F_{t+1}|\underline{X_t}) &= \mu \odot (\zeta_0 + \nu) + \mu \odot (\zeta_F' F_t + \zeta_n' n_t) \qquad (a.21) \\
&=: \mu_F + \Phi_{FF} F_t + \Phi_{Fn} n_t \\
\mathbb{V}ar(F_{t+1}|\underline{X_t}) &= \text{diag}\left[\mu \odot \mu \odot (2\zeta_0 + \nu) + 2(\{\mu \odot \mu\}\mathbf{1}') \odot (\zeta_F' F_t + \zeta_n' n_t)\right] \qquad (a.22) \\
&=: \text{diag}\left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t\right),
\end{aligned}
$$

where $\mathbf{1}$ is a n_F-dimensional vector of ones. Using further eq. (a.17), we obtain:

$$
\begin{aligned}
\mathbb{E}_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right) &= \begin{bmatrix} \mu_F \\ \gamma + \beta'\mu_F \end{bmatrix} + \begin{bmatrix} \Phi_{FF} & \Phi_{Fn} \\ \beta'\Phi_{FF} & \beta'\Phi_{Fn} + c' \end{bmatrix} \begin{bmatrix} F_t \\ n_t \end{bmatrix} \\
\mathbb{V}ar_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right) &= \mathbb{E}_t\left(\mathbb{V}ar\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\middle| \underline{X_t}^*\right)\right) + \mathbb{V}ar_t\left(\mathbb{E}\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\middle| \underline{X_t}^*\right)\right) \\
&= \mathbb{E}_t\left(\begin{bmatrix} 0 & 0 \\ 0 & \text{diag}(\beta' F_{t+1} + c' n_t + \gamma) \end{bmatrix}\right) + \mathbb{V}ar_t\left(\begin{bmatrix} F_{t+1} \\ \beta' F_{t+1} + c' n_t + \gamma \end{bmatrix}\right) \\
&= \begin{bmatrix} 0_{n_F \times n_F} & 0_{n_F \times J} \\ 0_{J \times n_F} & \text{diag}(\beta'(\mu_F + \Phi_{FF} F_t + \Phi_{Fn} n_t) + c' n_t + \gamma) \end{bmatrix} + \\
& \quad \begin{bmatrix} I_{n_F} \\ \beta' \end{bmatrix} \text{diag}\left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t\right) \begin{bmatrix} I_{n_F} & \beta \end{bmatrix} \\
&= \begin{bmatrix} 0_{n_F \times J} \\ I_J \end{bmatrix} \text{diag}(\beta'(\mu_F + \Phi_{FF} F_t + \Phi_{Fn} n_t) + c' n_t + \gamma) \begin{bmatrix} 0_{J \times n_F} & I_J \end{bmatrix} \\
& \quad \begin{bmatrix} I_{n_F} \\ \beta' \end{bmatrix} \text{diag}\left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t\right) \begin{bmatrix} I_{n_F} & \beta \end{bmatrix}.
\end{aligned}
$$

Using the fact that, for any n-dimensional vector a:

$$vec(diag(a)) = \sum_{i=1}^{n} vec(diag[e_i e_i' a]) = \sum_{i=1}^{n} vec(e_i e_i') a_i = \underbrace{\left(\sum_{i=1}^{n} vec(e_i e_i') e_i'\right)}_{=:S_n} a,$$

we obtain:

$$\begin{aligned}
& vec\left(\mathbb{V}ar_t\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\right)\right) \\
= & \left(\left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array}\right] \otimes \left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array}\right]\right) S_J(\beta'(\mu_F + \Phi_{FF} F_t + \Phi_{Fn} n_t) + c' n_t + \gamma) + \\
& \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right] \otimes \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\right) S_{n_F}\left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t\right) \\
= & \left(\left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array}\right] \otimes \left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array}\right]\right) S_J(\beta' \mu_F + \gamma) + \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right] \otimes \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\right) S_{n_F} \mu_F^{var} + \\
& \left\{\left(\left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array}\right] \otimes \left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array}\right]\right) S_J \beta' \Phi_{FF} + \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right] \otimes \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\right) S_{n_F} \Phi_{FF}^{var}\right\} F_t + \\
& \left\{\left(\left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array}\right] \otimes \left[\begin{array}{c} 0_{n_F \times J} \\ I_J \end{array}\right]\right) S_J \beta' \Phi_{Fn} + \left(\left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right] \otimes \left[\begin{array}{c} I_{n_F} \\ \beta' \end{array}\right]\right) S_{n_F} \Phi_{Fn}^{var}\right\} n_t.
\end{aligned}$$

Therefore:

$$\begin{aligned}
& vec\left(\mathbb{V}ar_t\left(\left[\begin{array}{c} F_{t+1} \\ n_{t+1} \end{array}\right]\right)\right) \\
= & \left[\begin{array}{c}
S_{n_F}\left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t\right) \\
(I_{n_F} \otimes \beta') S_{n_F}\left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t\right) \\
(\beta' \otimes I_{n_F}) S_{n_F}\left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t\right) \\
S_J(\beta'(\mu_F + \Phi_{FF} F_t + \Phi_{Fn} n_t) + c' n_t + \gamma) + (\beta' \otimes \beta') S_{n_F}\left(\mu_F^{var} + \Phi_{FF}^{var} F_t + \Phi_{Fn}^{var} n_t\right)
\end{array}\right]
\end{aligned}$$

$$
= \begin{bmatrix} S_{n_F}\mu_F^{var} \\ (I_{n_F}\otimes\beta')S_{n_F}\mu_F^{var} \\ (\beta'\otimes I_{n_F})S_{n_F}\mu_F^{var} \\ S_J(\beta'\mu_F+\gamma)+(\beta'\otimes\beta')S_{n_F}\mu_F^{var} \end{bmatrix} +
\begin{bmatrix} S_{n_F}\Phi_{FF}^{var} \\ (I_{n_F}\otimes\beta')S_{n_F}\Phi_{FF}^{var} \\ (\beta'\otimes I_{n_F})S_{n_F}\Phi_{FF}^{var} \\ S_J\beta'\Phi_{FF}+(\beta'\otimes\beta')S_{n_F}\Phi_{FF}^{var} \end{bmatrix} F_t +
\begin{bmatrix} S_{n_F}\Phi_{Fn}^{var} \\ (I_{n_F}\otimes\beta')S_{n_F}\Phi_{Fn}^{var} \\ (\beta'\otimes I_{n_F})S_{n_F}\Phi_{Fn}^{var} \\ S_J\beta'\Phi_{Fn}+(\beta'\otimes\beta')S_{n_F}\Phi_{Fn}^{var} \end{bmatrix} n_t.
$$

Hence, both $vec\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right)\right)$ and $\mathbb{E}_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right)$ are affine functions of $\begin{bmatrix} F_t \\ n_t \end{bmatrix}$. Let us introduce the obvious notations:

$$
\begin{aligned}
\mathbb{E}_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right) &= \mu_E+\Phi_E\begin{bmatrix} F_t \\ n_t \end{bmatrix} \\
vec\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right)\right) &= \mu_V+\Phi_V\begin{bmatrix} F_t \\ n_t \end{bmatrix}.
\end{aligned}
$$

Assuming that $\begin{bmatrix} F_t \\ n_t \end{bmatrix}$ is covariance-stationary, we have:

$$
\mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)=\mathbb{E}\left(\begin{bmatrix} F_{t+1} \\ n_{t+1} \end{bmatrix}\right)=\mu_E+\Phi_E\mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right),
$$

which leads to

$$
\mathbb{E}\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)=(I-\Phi_E)^{-1}\mu_E.
$$

We also have:

$$
\mathbb{V}ar\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) = \mathbb{E}\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right)+\mathbb{V}ar\left(\mathbb{E}_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right).
$$

Using that

$$
\begin{cases}
\mathbb{E}\left(\mathbb{V}ar_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) = \mathbb{E}\left(\mu_V+\Phi_V\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right) \\
\mathbb{V}ar\left(\mathbb{E}_t\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\right) = \mathbb{V}ar\left(\mu_E+\Phi_E\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)=\Phi_E\mathbb{V}ar\left(\begin{bmatrix} F_t \\ n_t \end{bmatrix}\right)\Phi_E',
\end{cases}
$$

we get:

$$vec\left(\mathbb{V}ar\left(\left[\begin{array}{c} F_t \\ n_t \end{array}\right]\right)\right) = (I - \Phi_E \otimes \Phi_E)^{-1}\left[\mu_V + \Phi_V \mathbb{E}\left(\left[\begin{array}{c} F_t \\ n_t \end{array}\right]\right)\right].$$

O.3 Proof of Lemma 1

We have:

$$\begin{aligned} r^s_{t+1} &= \log\left(\frac{P_{t+1}+D_{t+1}}{P_t}\right) \\ &= \log(P_{t+1}+D_{t+1}) - \log(P_t) + \log(P_{t+1}) - \log(P_{t+1}) + \\ &\quad \log(D_{t+1}) - \log(D_{t+1}) + \log(D_t) - \log(D_t) \\ &= \tau_{t+1} - \tau_t + g_{d,t+1} + \log\left(1 + \frac{D_{t+1}}{P_{t+1}}\right). \end{aligned} \tag{a.23}$$

Besides:

$$\begin{aligned} \log[1 + D_{t+1}/P_{t+1}] &= \log[1+\exp(-\tau_{t+1})] \\ &\approx \log[1+\exp(-\bar{\tau})\{1-(\tau_{t+1}-\bar{\tau})\}] \\ &\approx \log[1+\exp(-\bar{\tau}) - \exp(-\bar{\tau})(\tau_{t+1}-\bar{\tau})] \\ &\approx \log[1+\exp(-\bar{\tau})] + \log\left[1 - \frac{\exp(-\bar{\tau})(\tau_{t+1}-\bar{\tau})}{1+\exp(-\bar{\tau})}\right] \\ &\approx \log[1+\exp(-\bar{\tau})] - \frac{\tau_{t+1}-\bar{\tau}}{1+\exp(\bar{\tau})}. \end{aligned}$$

Therefore:

$$\begin{aligned} r^s_{t+1} &\approx \tau_{t+1} - \tau_t + g_{d,t+1} + \log[1+\exp(-\bar{\tau})] - \frac{\tau_{t+1}-\bar{\tau}}{1+\exp(\bar{\tau})} \\ &\approx \log[1+\exp(-\bar{\tau})] + \frac{\bar{\tau}}{1+\exp(\bar{\tau})} + \frac{\exp(\bar{\tau})}{1+\exp(\bar{\tau})}\tau_{t+1} - \tau_t + g_{d,t+1} \\ &\approx \log[1+\exp(\bar{\tau})] - \frac{\exp(\bar{\tau})}{1+\exp(\bar{\tau})}\bar{\tau} + \frac{\exp(\bar{\tau})}{1+\exp(\bar{\tau})}\tau_{t+1} - \tau_t + g_{d,t+1}, \end{aligned}$$

which leads to eq. (a.11).

O.4 Credit Default Swap

The Credit Default Swap (CDS) is the most common credit derivative. It is an agreement between a protection buyer and a protection seller, whereby the buyer pays a periodic fee in return for a contingent payment by the seller upon a credit event, such as bankruptcy or failure to pay, of a reference entity. The contingent payment usually replicates the loss incurred by a creditor of the reference entity in the event of its default [See e.g. Duffie (1999)].

More specifically, a CDS works as follows: the protection buyer pays a regular (annual, semi-annual or quarterly) premium to the so-called protection seller. These payments end either after a given period of time (the maturity of the CDS) or at default of the reference entity i from segment j. In the case of the default of this debtor, the protection seller compensates the protection buyer for the loss the latter would incur upon default of the reference entity (assuming that the latter effectively holds a bond issued by the reference entity). The CDS spread, also called CDS premium, is the regular payment paid by the protection buyer (expressed in percentage of the notional and in annualized terms). Since, in our model, the segments of credit are homogeneous, the CDS spreads are the same for all entities belonging to the same segment. Let us denote by $S^{CDS}_{j,t,h}$ the maturity-h CDS spread of segment-j entities, by q the number of premium payments made per year and by RR the recovery rate.[17]

Let $d_{j,i,t}$ be the indicator of default of entity i belonging to segment j: $d_{j,i,t} = 1$ if entity i is in default at time t (or before) and $d_{j,i,t} = 0$ otherwise.[18] Note that we have $N_{j,t} = \Sigma_{i=1}^{I_j} d_{j,i,t}$.

At inception of the CDS contract, there is no cash-flow exchanged between both parties: Indeed, the CDS spread $S^{CDS}_{j,t,h}$ is determined so as to equalize the present discounted values of the payments promised by each of them. If the maturity h is expressed in years, we have:

$$\underbrace{\mathbb{E}^{\mathbb{Q}}_t \left\{ \sum_{k=1}^{qh} \Lambda_{t,t+k}(1-RR)(d_{j,i,t+k} - d_{j,i,t+k-1}) \right\}}_{\text{Protection leg}} = \underbrace{\mathbb{E}^{\mathbb{Q}}_t \left\{ \frac{S^{CDS}_{j,t,h}}{q} \sum_{k=1}^{qh} \Lambda_{t,t+k}(1 - d_{j,i,t+k}) \right\}}_{\text{Premium leg}}. \quad \text{(a.24)}$$

By expanding the latter equality, it is clear that the CDS spread $S_{j,t,h}$ is easily derived if one can compute $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k})$, $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k} d_{j,i,t+k})$ and $\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k} d_{j,i,t+k-1})$ for all $k > 0$. By symmetry arguments, assuming that $d_{j,i,t} = 0$, we have:

$$\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k} d_{j,i,t+k}) = \mathbb{E}^{\mathbb{Q}}_t \left(\Lambda_{t,t+k} \frac{\overline{N_{j,t+k}} - \overline{N_{j,t}}}{I_j - \overline{N_{j,t}}} \right) = \frac{1}{I_j - \overline{N_{j,t}}} \left(\mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k} \overline{N_{j,t+k}}) - \overline{N_{j,t}} \mathbb{E}^{\mathbb{Q}}_t(\Lambda_{t,t+k}) \right).$$

where $\overline{N_{j,t}} = \min(N_{j,t}, I_j)$. While exact formulae are available to compute these quantities, we will proceed under the assumption that the probability of having $N_{j,t} > I_j$ is so small that we have, in

[17] While the model is extensible to the case of stochastic recovery rates, we restrict our attention here to that of deterministic recovery rates as is common practice in pricing exotic credit derivatives.

[18] In what follows, we augment the filtration Ω_t with $\underline{d_t}$.

particular, $\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}\overline{N_{j,t+k}}) \approx \mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}N_{j,t+k})$. In this context, we obtain:

$$\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}d_{j,i,t+k}) \approx \frac{1}{I_j - N_{j,t}} \left(\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}N_{j,t+k}) - N_{j,t}\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}) \right). \tag{a.25}$$

Similarly, we obtain:

$$\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}d_{j,i,t+k-1}) \approx \frac{1}{I_j - N_{j,t}} \left(\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}N_{j,t+k-1}) - N_{j,t}\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+k}) \right). \tag{a.26}$$

Using eqs. (a.25) and (a.26) in eq. (a.24), we get:

$$S_{j,t,h}^{CDS} \approx q(1-RR)\frac{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}[N_{j,t+k} - N_{j,t+k-1}]\right\}}{\mathbb{E}_t^{\mathbb{Q}}\left\{\sum_{k=1}^{qh}\Lambda_{t,t+k}(I_j - N_{j,t+k})\right\}}.$$

It can be seen that this expression is the same as that for spreads of credit indices (eq. 11).

O.5 Maximum Sharpe ratio between dates t and $t+h$

The maximum Sharpe ratio of an investment realized between dates t to $t+h$ is given by [see Hansen and Jagannathan (1991)]:

$$\mathscr{M}_{t,t+h} = \frac{\sqrt{\mathbb{V}ar_t(M_{t,t+h})}}{\mathbb{E}_t^{\mathbb{Q}}(\Lambda_{t,t+h})}.$$

Using the notation $M_{t,t+1} = \exp(\mu_{0,m} + \mu_{1,m}'X_{t+1} + \mu_{2,m}'X_t)$, we have:

$$\begin{aligned} M_{t,t+h} &= \exp\left(h\mu_{0,m} + \mu_{2,m}'X_t\right) \times \\ &\quad \exp\left([\mu_{1,m}+\mu_{2,m}]'X_{t+1} + \dots + [\mu_{1,m}+\mu_{2,m}]'X_{t+h-1} + \mu_{1,m}'X_{t+h}\right) \end{aligned}$$

Therefore:

$$\mathscr{M}_{t,t+h} = \frac{\sqrt{\theta_{t,h}(2[\mu_{1,m}+\mu_{2,m}], 2\mu_{1,m}) - \theta_{t,h}(\mu_{1,m}+\mu_{2,m}, \mu_{1,m})^2}}{\theta_{t,h}(\mu_{1,m}+\mu_{2,m}, \mu_{1,m})},$$

where

$$\theta_{t,h}(u,v) = \mathbb{E}_t(\exp(u'X_{t+1} + \dots + u'X_{t+h-1} + v'X_{t+h})).$$

When X_t is an affine process, $\theta_{t,h}(u,v)$ can be computed in closed-form by using recursive formulae as in Prop. 4 using the physical measure instead of the risk-neutral one and setting the risk-free short-term rate to 0, i.e. using $\eta_0 = 0$ and $\eta_1 = 0$.