A Preferred-Habitat Model of Term Premia and Currency Risk

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Abstract

We propose an integrated preferred-habitat model of term premia and exchange rates, building on Vayanos and Vila (2019). Our model generates deviations from UIP and also that the term structure of currency risk premia is decreasing. Using our framework we explore the transmission of monetary policy to domestic and currency markets, as well as the spillovers to the foreign term premia; the effect of non-conventional monetary policy on the domestic and foreign economies; and the effect of shifts in the 'specialness' of one country's bonds or currency.

* We thank ...

1 Introduction

The literature on exchange rate determination has had limited success so far in explaining how exchange rate movements connect with other financial prices and returns, as well as with macroeconomics determinants. On the financial side, it is well-known that the uncovered interest parity condition (UIP), linking expected movements in exchange rates to the short-term interest rate differential, is strongly rejected by the data (Fama, 1984). On the macroeconomic side, it is equally well-known that exchange rate movements appear disconnected from traditional macroeconomic fundamentals such as growth, external imbalances, monetary policy etc... (see Meese and Rogoff (1983) and the literature on the 'exchange rate disconnect puzzle').

In standard models, the determination of the exchange rate results from the confrontation of two equilibrium conditions: one is the standard asset pricing/Euler equation condition that characterizes the intertemporal saving and portfolio decisions of agents. The other is an intertemporal budget constraint that requires that consumption/saving/portfolio choices be consistent with the present value of available resources.

To simplify, the latter condition pins down the long-run ''level' of equilibrium exchange rates that ensures choices remain within the relevant budget sets. The former determines the 'slope' of the exchange rate, i.e. how they respond to changes in the economic environment by equating the marginal expected utility across available investment strategies. In the leading representative no-arbitrage models of international finance, this equilibrium condition imposes a tremendous amount of structure. These models typically have a difficult time reproducing observed empirical patterns. For instance, Lustig, Stathopoulos, and Verdelhan (2019) observe that no-arbitrage models cannot replicate both the strong evidence of deviations from UIP and the evidence that the term structure of currency risk-premia is decreasing, i.e. that the expected fixed-horizon return on a generalized carry trade strategy that invests in domestic and foreign bond of maturity τ decreases as τ increases. Similarly, Engel (2016) observes that standard representative agent models cannot explain simultaneously the UIP puzzle -which through the lens of these models implies that the high interest rate currency is more risky- and the fact that high interest rate currency tend to have a stronger currency -which through the lens of these models suggests that the high interest rate currency is less risky.

A recent promising avenue of research consists in introducing some level of international market segmentation. At the theoretical level, this relaxes the arbitrage condition by focusing instead on the risk-return tradeoff of the relevant global investors. Gabaix and Maggiori (2015) present a stylized model along those lines, reviving an important older literature on portfolio balance models (Kouri, 1982). These models naturally generate deviations from UIP as financial arbitrageurs need to be compensated for their external exposure. Itskhoki and Mukhin (2017) present such a model where financial arbitrageurs also need to absorb liquidity demand arising from noise traders, as in Jeanne and Rose (2002). These liquidity demand shocks translate, in equilibrium, into 'UIP shocks', i.e. deviations from the UIP condition. Quantitatively, Itskhoki and Mukhin (2017) conclude that these UIP shocks account for more than 90% of the fluctuations in the nominal and real exchange rate, but very little of the fluctuations in output (thus explaining the disconnect).

At the institutional level, market segmentation seems a very plausible assumption: the marginal investor in currency markets is much more likely to be a specialized investor such as a large macro global hedge fund, the trading desk of a multinational corporation, a sovereign wealth fund, or the fixed-income desk of a global broker-dealer, rather than the representative household trying to diversify the risks to the marginal utility of its consumption stream.

In these models, the segmentation hypothesis is extreme: currency market are segmented, but domestic rate markets are not. This is too extreme. In particular, it implies that, while deviations from UIP may occur, the rational expectation hypothesis (EH) would still be a good guide to understanding the term structure. Yet, a body of evidence indicates that risk premia on currency markets and term premia on bond markets are related. One such piece of evidence was already mentioned: we know from Lustig, Stathopoulos, and Verdelhan (2019) that the term structure of currency risk premia, which one can understand as a combination of UIP deviations and term premia on domestic and foreign bond markets, is downward sloping, declining to zero. This strongly suggests that segmentation matters both for bond and currency markets.

In this paper, we propose such an integrated analysis of global rates markets. Our approach builds on the work of Vayanos and Vila (2019) as well as Ray (2019), on preferred-habitat models. These earlier papers focused on a closed economy model and analyzed segmentation along the term structure. In these models, there are two types of investors: local investors specialized in specific maturity segments, and term-structure arbitrageurs. Because the arbitrageurs are risk-averse and have finite resources, deviations from the expectation hypothesis of the term structure persist. These models are particularly useful to investigate how 'local shocks' to the supply of or demand for specific maturities can propagate along the term structure. Ray (2019) embeds such a segmented asset market structure into a New Keynesian model and explores how non-conventional policies, such as QE or forward guidance can be deployed effectively.

The current paper considers an extension of this framework to two countries. In each country, a monetary authority sets short term policy rate exogenously. Further, local investors are situated along the domestic and foreign term structure. These investors are specialized in a given currency and maturity segment. In addition, as in Itskhoki and Mukhin (2017), there are specialized (noise) investors in the currency market. Lastly 'global rates market' risk averse arbitrageurs can invest limited capital in all fixed income instruments, foreign and domestic. Because these global arbitrageurs operate both on the term structure in each country, and in currency markets, term premia and currency risk premia will be linked in equilibria.

Our framework allows us to answer a number of specific questions. First, we can characterize the time series behavior of term-premia and currency risk-premia, given the underlying policy and demand shocks. Our model recovers deviations from UIP and also very naturally the Lustig, Stathopoulos, and Verdelhan (2019) term structure of currency risk premia: In our model, as the maturity of the bond increases, the short term excess return decreases to zero. The reason is precisely that long term bond and currency risk premia are linked: as arbitrageurs become more exposed to domestic policy shocks, domestic long term bonds and foreign currency are equally undesirable: their premia increase by similar amounts, which account for the decline in the term structure of currency risk premia. Second, our framework allows us to explore how shocks to the policy rate in one country transmit to the domestic term structure, the currency, and the foreign term structure. Under UIP and the EH, a change in the domestic policy rate would leave the domestic and foreign term structures unchanged: all the adjustment would be in the expected rate of depreciation of the exchange rate. This is no longer the case when global rates market investors can arbitrage across these markets. A domestic policy shock, for instance a decrease in the domestic policy rate, will transmit to the domestic term structure: as long bond become more desirable, global rates investor increase their exposure. Because their exposure increases, they need to be compensated, hence the expected return from holding these long term bonds needs to increase. By the same token, however, foreign bonds also become more desirable and global rates investors shift their portfolios towards foreign bonds. This leads to a depreciation of the domestic currency. However, the increased exposure to foreign bonds also needs to be compensated, hence a deviation from UIP arises. Lastly, as this also increases the global rates investors for foreign long bonds, a positive term premium arises also in the foreign term structure. This suggests that the transmission of domestic monetary policy to the domestic economy is impaired, as in Ray (2019), and that domestic monetary policy has spillovers to foreign long real rates, even under a regime of flexible exchange rates. To the extent that long rates matter for economic activity, this is another instance where the Friedman-Obstfeld-Taylor trilemma fails.

Third, the model allows us to investigate how non-conventional policies such as Quantitative Easing or Forward Guidance transmit, both domestically and abroad.

Fourth, if we interpret the Home country as the United States, the model also lets us investigate how shifts in the demand for US Treasuries (i.e. a generalized shift in the demand for domestic bonds) differs from a shift in the demand for dollars (i.e. a shift in the demand on the currency markets). This allows us to better understand whether the current environment is one characterized by the specialness of the U.S. dollar, or the specialness of U.S. Treasuries (Jiang, Krishnamurthy, and Lustig, 2018, 2019).

Greenwood, Hanson, Stein, and Sunderam (2019) develop independently a model similar to ours, with arbitrageurs trading bonds and currency across two countries. They find, as we do, that bond and currency carry trades are profitable, and that an increase in bond demand in one country causes the currency of that country to depreciate and bond prices in both countries to rise. They also introduce segmented arbitrage, e.g., some arbitrageurs can only trade bonds in one country, and some can trade only currency. Their model is set up in discrete time and assumes only a short and a long bond. By contrast, ours is set up in continuous time and derives the entire term structure of interest rates in each country. This allows us to compare the predictability of bond and currency movements across different horizons, and to perform a quantitative exercise in which we can compare model-generated moments to those in the data.

2 Model

Time is continuous and goes from zero to infinity. There are two countries, Home (H) and Foreign (F). We define the exchange rate as the units of home currency that one unit of foreign currency can buy, and denote it by e_t at time t. An increase in e_t corresponds to a home currency depreciation.

In each country $j = H, F$, a continuum of zero-coupon government bonds can be traded. The bonds' maturities lie in the interval $(0, T)$, where T can be finite or infinite. The country-j bond with maturity τ at time t pays off one unit of country j's currency at time $t+\tau$. We denote by $P_{jt}^{(\tau)}$ the time-t price of that bond, expressed in units of country j's currency, and by $y_{jt}^{(\tau)}$ the bond's yield. The yield is the spot rate for maturity τ, and is related to the price through

$$y_{jt}^{(\tau)} = -\frac{\log\left(P_{jt}^{(\tau)}\right)}{\tau}. \tag{2.1}$$

The country-j and time-t short rate r_{jt} is the limit of the yield $y_{jt}^{(\tau)}$ when τ goes to zero. We take r_{jt} as exogenous, and describe its dynamics later in this section (Equation 2.9). An exogenous r_{jt} could result from actions that the central bank in country j takes in response to exogenous shocks.

There are three types of agents: arbitrageurs, bond investors and currency traders. Arbitrageurs are competitive and maximize a mean-variance objective over instantaneous changes in wealth. We express their wealth in units of the home currency, thus assuming that the home currency is the riskless asset for them. We allow arbitrage to be global or segmented. When arbitrage is global, arbitrageurs can invest in the currencies and bonds of both countries. When instead arbitrage is segmented, arbitrageurs can invest in the currency of the home country (the riskless asset), and in a single additional asset class: foreign currency for some arbitrageurs, home bonds for others, and foreign bonds for the remainder. We assume that the arbitrageurs investing in foreign bonds have a zero net position in foreign-currency instruments: they hedge their bond position with an equally sized position in the foreign short rate. Segmented arbitrage is a useful benchmark, as the interactions between bond and currency markets that global arbitrage generates are not present.

In the case of global arbitrage, we denote by W_t the arbitrageurs' time-t wealth, by W_{Ht} and W_{Ft} their net position in home and foreign-currency instruments, respectively, and by $X_{Ht}^{(\tau)}$ and $X_{Ft}^{(\tau)}$ their position in the home and foreign bond with maturity τ, respectively, all expressed in units of the home currency. The arbitrageurs' budget constraint is

$$W_{t+dt} = \left(W_{Ht} - \int_0^T X_{Ht}^{(\tau)} d\tau\right)(1 + r_{Ht}dt) + \int_0^T X_{Ht}^{(\tau)} \frac{P_{H,t+dt}^{(\tau-dt)}}{P_{Ht}^{(\tau)}} d\tau$$

$$+ \left(W_{Ft} - \int_0^T X_{Ft}^{(\tau)} d\tau\right)(1 + r_{Ft}dt)\frac{e_{t+dt}}{e_t} + \int_0^T X_{Ft}^{(\tau)} \frac{P_{F,t+dt}^{(\tau-dt)} e_{t+dt}}{P_{Ft}^{(\tau)} e_t} d\tau. \tag{2.2}$$

The first term in the right-hand side of (2.2) corresponds to a position in the home short rate, the second term to a position in home bonds, the third term to a position in the foreign short rate, and the fourth term to a position in foreign bonds. In the third term, $W_{Ft} - \int_0^T X_{Ft}^{(\tau)} d\tau$ units of the home currency are converted at time t to units of the foreign currency by dividing by e_t. They earn the foreign short rate between time t and $t+dt$, and are converted back at time $t+dt$ to units of the home currency by multiplying by e_{t+dt}. In the fourth term, $X_{Ft}^{(\tau)}$ units of the home currency are converted at time t to units of the foreign currency by dividing by e_t, and then to units of the foreign bond with maturity τ by dividing by $P_{Ft}^{(\tau)}$, the price of the bond in foreign currency. They are converted back at time $t+dt$ to units of the home currency by multiplying by $P_{F,t+dt}^{(\tau-dt)} e_{t+dt}$.

Subtracting $W_t = W_{Ht} + W_{Ft}$ from both sides of (2.2) and rearranging, we find

$$dW_t = W_t r_{Ht} dt + W_{Ft}\left(\frac{de_t}{e_t} + (r_{Ft} - r_{Ht})dt\right)$$

$$+ \int_0^T X_{Ht}^{(\tau)} \left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - r_{Ht}dt\right) d\tau + \int_0^T X_{Ft}^{(\tau)} \left(\frac{d(P_{Ft}^{(\tau)} e_t)}{P_{Ft}^{(\tau)} e_t} - \frac{de_t}{e_t} - r_{Ft}dt\right) d\tau. \tag{2.3}$$

If arbitrageurs invest all their wealth in the home short rate, then the instantaneous change dW_t in their wealth is $W_t r_{Ht} dt$, the first term in the right-hand side of (2.3). Relative to that case, arbitrageurs can earn an additional return from investing in three sets of assets: foreign currency, home bonds, and foreign bonds. The returns from these investments correspond to the second, third and fourth term, respectively, in the right-hand side of (2.3). The optimization problem of a global arbitrageur is

$$\max_{W_{Ft}, \{X_{jt}^{(\tau)}\}_{\tau \in (0,T), j=H,F}} \left[\mathbb{E}_t(dW_t) - \frac{a}{2}\mathbb{V}\text{ar}_t(dW_t)\right], \tag{2.4}$$

where $a \geq 0$ is a risk-aversion coefficient that characterizes the trade-off between mean and variance. By possibly redefining a, we assume that global arbitrageurs are in measure one. Arbitrageurs with the objective (2.4) can be interpreted as overlapping generations living over infinitesimal periods.

In the case of segmented arbitrage, the budget constraint of any given arbitrageur is derived from (2.3) by setting two of the terms to zero. For an arbitrageur who can invest in foreign currency, the third and fourth terms are zero ($X_{Ht}^{(\tau)} = X_{Ft}^{(\tau)} = 0$); for an arbitrageur who can invest in home bonds, the second and fourth terms are zero ($W_{Ft} = X_{Ft}^{(\tau)} = 0$); and for an arbitrageur who can invest in foreign bonds, with a zero net position in foreign-currency instruments, the second and third terms are zero ($W_{Ft} = X_{Ht}^{(\tau)} = 0$). The optimization problem is derived from (2.4) by restricting the choice variables accordingly. We denote by a_e, a_H and a_F, respectively, the risk-aversion coefficient of an arbitrageur who can invest in foreign currency, home bonds and foreign bonds. By possibly redefining (a_e, a_H, a_F), we assume that each type of arbitrageur is in measure one.

Bond investors have preferences ("habitats") for specific countries and maturities. For example, pension funds in the home country prefer long-maturity home bonds because these match their pension liabilities, which are long-term and denominated in home currency. For tractability, we assume that preferences take an extreme form, whereby investors demand only the bond closest to their preferred characteristics. That is, investors with preferences for country j and maturity τ at time t hold a position $Z_{jt}^{(\tau)}$ in the country-j bond with maturity τ and hold no other bond. We express the position $Z_{jt}^{(\tau)}$ in units of the home currency, and assume that it is linear and decreasing in the logarithm of the bond price:

$$Z_{jt}^{(\tau)} = -\alpha_j(\tau) \log\left(P_{jt}^{(\tau)}\right) - \beta_{jt}^{(\tau)}. \tag{2.5}$$

The slope coefficient $\alpha_j(\tau) \geq 0$ is constant over time but can depend on maturity τ and country j. The intercept coefficient $\beta_{jt}^{(\tau)}$ can depend on t, τ and j. For simplicity, we refer to $\alpha_j(\tau)$ and $\beta_{jt}^{(\tau)}$ as demand slope and demand intercept, respectively. The actual intercept is $-\beta_{jt}^{(\tau)}$.

The demand intercept $\beta_{jt}^{(\tau)}$ takes the form

$$\beta_{jt}^{(\tau)} = \zeta_j(\tau) + \theta_j(\tau)\beta_{jt}, \tag{2.6}$$

where $\{\zeta_j(\tau), \theta_j(\tau)\}_{j=H,F}$ are constant over time but can depend on country j and maturity τ, and $\{\beta_{jt}\}_{j=H,F}$ are independent of τ but can depend on j and t. We refer to $\{\beta_{jt}\}_{j=H,F}$ as demand risk factors, and describe their dynamics later in this section (Equation 2.9). Vayanos and Vila (2019) provide an optimizing foundation for the demand specification (2.5)-(2.6) in a setting where investors form overlapping generations consuming at the end of their life, are infinitely risk-averse, and can invest in bonds and in a private opportunity with exogenous return.

Currency traders generate a downward-sloping demand for foreign currency as a function of the exchange rate e_t. These agents can be interpreted as exporters and importers. Suppose, following Gabaix and Maggiori (2015), that e_t is low, i.e., the foreign currency is cheap relative to the home currency. Then, agents in the home country export less in value terms than what they import, and hence there is an excess demand of foreign currency. For tractability, we assume that the demand of currency traders, expressed in units of the home currency, is linear and decreasing in the logarithm of the exchange rate:

$$Z_{et} = -\alpha_e \log(e_t) - (\zeta_e + \theta_e \gamma_t), \tag{2.7}$$

where $\alpha_e \geq 0$ is a slope coefficient, (ζ_e, θ_e) are constants, and γ_t is a demand risk factor, whose dynamics we describe later in this section (Equation 2.9).

The demand (2.7) for foreign currency is expressed in the spot market. We could alternatively assume that some of the demand is expressed in the forward market. Indeed, according to BIS (2019), spot transactions accounted for only one-third of total trading volume in the currency market over recent years, with forward and swap transactions accounting for most of the remainder. We assume that currency traders' demand, expressed in units of the home currency, for the foreign-currency forward contract with maturity τ is

$$Z_{et}^{(\tau)} = -(\zeta_e(\tau) + \theta_e(\tau)\gamma_t), \tag{2.8}$$

where $(\zeta_e(\tau), \theta_e(\tau))$ are functions of τ, and γ_t is a demand risk factor, whose dynamics we describe later in this section (Equation 2.9).

Under covered interest parity (CIP), the demand $Z_{et}^{(\tau)}$ for the foreign-currency forward contract with maturity τ is equivalent to the combination of (i) a demand $Z_{et}^{(\tau)}$ for foreign currency in the spot market, (ii) a demand $Z_{et}^{(\tau)}$ for the foreign bond with maturity τ, and (iii) a demand $-Z_{et}^{(\tau)}$ for the home bond with maturity τ. Hence, the equilibrium with the forward market is identical to one without it but with the demands (i)-(iii) added to (2.5) and (2.7). We derive the equilibrium in the latter form. CIP holds in our model when arbitrage is global, as the same agents can trade all the instruments involved in CIP arbitrage. We introduce currency demand in the forward market only when arbitrage is global, leaving CIP violations for future work.

The 5×1 vector $q_t \equiv (r_{Ht}, r_{Ft}, \beta_{Ht}, \beta_{Ft}, \gamma_t)^\top$ follows the process

$$dq_t = -\Gamma(q_t - \bar{q})dt + \Sigma dB_t, \tag{2.9}$$

where $\bar{q}$ is a constant 5×1 vector, (Γ, Σ) are constant 5×5 matrices, dB_t is a 5×1 vector $(dB_{rHt}, dB_{rFt}, dB_{\beta Ht}, dB_{\beta Ft}, dB_{\gamma t})^\top$ of independent Brownian motions, and $\top$ denotes transpose. Equation (2.9) nests the case where the factors $(r_{Ht}, r_{Ft}, \beta_{Ht}, \beta_{Ft}, \gamma_t)$ are mutually independent, and the case where they are correlated. Independence arises when the matrices (Γ, Σ) are diagonal. When instead Σ is non-diagonal, shocks to the factors are correlated, and when Γ is non-diagonal, the drift (instantaneous expected change) of each factor depends on all other factors. We assume that Σ has full rank so that each factor is not perfectly correlated with the others, and that the eigenvalues of Γ have negative real parts so that q_t is stationary. Since q_t is stationary, (2.9) implies that the long-run mean of q_t is $\bar{q}$. We set the means of the demand factors to zero ($\bar{q}_3 = \bar{q}_4 = \bar{q}_5 = 0$). This is without loss of generality since we can redefine $\{\zeta_j(\tau)\}_{j=H,F}$ and ζ_e to include a non-zero long-run mean. Using the same redefinitions, we can set the supply of each bond and of foreign currency to zero.

The combination of mean-variance preferences for arbitrageurs and log-linear demand functions for the remaining agents yields a tractable equilibrium in which bond prices and the exchange rate are log-linear functions of the risk factors. Key to the tractability is that all demand functions are expressed in terms of the same numeraire, which is also the riskless asset for arbitrageurs. The numeraire can be the currency in one of the two countries, and we take it to be the home currency. Note that by expressing demand for foreign bonds in the home currency, we preclude that a foreign-currency depreciation, holding bond yields constant, lowers foreign-bond demand in home-currency terms. We also preclude that the foreign currency is the riskless asset for some arbitrageurs (e.g., foreign ones).

3 Segmented Arbitrage

In this section we study the case of segmented arbitrage, whereby foreign currency, home bonds, and foreign bonds are traded by three disjoint sets of arbitrageurs. For simplicity, we assume that there is no demand risk for bonds and foreign currency ($\beta_{Ht} = \beta_{Ft} = \gamma_t = 0$) and that the home and foreign short rates (r_{Ht}, r_{Ft}) are independent. This amounts to taking the matrices (Γ, Σ) in (2.9) to be diagonal and setting $\Sigma_{3,3} = \Sigma_{4,4} = \Sigma_{5,5} = 0$. Setting $(\Gamma_{1,1}, \Gamma_{2,2}, \bar{q}_1, \bar{q}_2, \Sigma_{1,1}, \Sigma_{2,2}) \equiv$

$(\kappa_{rH}, \kappa_{rF}, \overline{r}_H, \overline{r}_F, \sigma_{rH}, \sigma_{rF})$, we can write the dynamics of the country-j short rate as

$$dr_{jt} = \kappa_{rj}(\overline{r}_j - r_{jt})dt + \sigma_{rj}dB_{rjt}. \tag{3.1}$$

3.1 Equilibrium

We conjecture that the equilibrium exchange rate is a log-linear function of the home and the foreign short rate, and that equilibrium bond yields in country $j = H, F$ are affine functions of that country's short rate. That is, there exist three scalars $(\{A_{rje}\}_{j=H,F}, C_e)$ and four functions $\{A_{rj}(\tau), C_j(\tau)\}_{j=H,F}$ that depend only on τ, such that

$$e_t = e^{-[A_{rHe}r_{Ht} - A_{rFe}r_{Ft} + C_e]}, \tag{3.2}$$

$$P_{jt}^{(\tau)} = e^{-[A_{rj}(\tau)r_{jt} + C_j(\tau)]}. \tag{3.3}$$

When arbitrage is segmented, the exchange rate, the yields of home bonds, and the yields of foreign bonds are determined independently, and they reflect the risk aversion of the corresponding arbitrageurs.

3.1.1 Exchange Rate

We determine the exchange rate by deriving the arbitrageurs' first-order condition and combining it with market clearing. Applying Ito's Lemma to (3.2), and using the dynamics (3.1) of r_{jt}, we find that the instantaneous return on foreign currency is

$$\frac{de_t}{e_t} = \mu_{et}dt - A_{rHe}\sigma_{rH}dB_{rHt} + A_{rFe}\sigma_{rF}dB_{rFt}, \tag{3.4}$$

where

$$\mu_{et} \equiv -A_{rHe}\kappa_{rH}(\overline{r}_H - r_{Ht}) + A_{rFe}\kappa_{rF}(\overline{r}_F - r_{Ft}) + \frac{1}{2}A_{rHe}^2\sigma_{rH}^2 + \frac{1}{2}A_{rFe}^2\sigma_{rF}^2 \tag{3.5}$$

is the expected return. Substituting the return (3.4) into the budget constraint of the subset of arbitrageurs who can invest in foreign currency (and whose budget constraint is derived from (2.3) by setting $X_{Ht}^{(\tau)} = X_{Ft}^{(\tau)} = 0$), we find

$$dW_t = [W_t r_{Ht} + W_{Ft}(\mu_{et} + r_{Ft} - r_{Ht})]\,dt - W_{Ft}(A_{rHe}\sigma_{rH}dB_{rHt} - A_{rFe}\sigma_{rF}dB_{rFt}).$$

The optimization problem of these arbitrageurs is

$$\max_{W_{Ft}} \left[W_{Ft} \left(\mu_{et} + r_{Ft} - r_{Ht} \right) - \frac{a_e}{2} W_{Ft}^2 \left(A_{rHe}^2 \sigma_{rH}^2 + A_{rFe}^2 \sigma_{rF}^2 \right) \right],$$

and their first-order condition is

$$\mu_{et} + r_{Ft} - r_{Ht} = a_e W_{Ft} \left(A_{rHe}^2 \sigma_{rH}^2 + A_{rFe}^2 \sigma_{rF}^2 \right). \tag{3.6}$$

Equation (3.6) describes the arbitrageurs' risk-return trade-off when investing in the *currency carry trade* (CCT). We term CCT the trade of borrowing short-term in the home country, exchanging the borrowed amount in the foreign currency, investing it short-term in the foreign country, and exchanging it back in the home currency.[1] The CCT's return is $\frac{de_t}{e_t} + (r_{Ft} - r_{Ht})dt$, equal to the return on foreign currency plus that on the foreign-home short-rate differential.

If arbitrageurs invest an extra unit of home currency in the CCT, then their expected return increases by the CCT's expected return $\mu_{et} + r_{Ft} - r_{Ht}$. This is the left-hand side of (3.6). The right-hand side is the increase in the the arbitrageurs' portfolio risk, times their risk-aversion coefficient a_e. The increase in portfolio risk is equal to the variance of the CCT's return, times the arbitrageurs' wealth W_{Ft} invested in foreign currency.

We next combine the arbitrageurs' first-order condition (3.6) with market clearing in foreign currency. Market clearing requires that the time-t positions of arbitrageurs and currency traders sum to zero:

$$W_{Ft} + Z_{et} = 0. \tag{3.7}$$

Using (3.7), we can write (3.6) as

$$\begin{aligned}
\mu_{et} + r_{Ft} - r_{Ht} &= -a_e Z_{et} \left(A_{rHe}^2 \sigma_{rH}^2 + A_{rFe}^2 \sigma_{rF}^2 \right) \\
&= a_e \left[\alpha_e \log(e_t) + \zeta_e \right] \left(A_{rHe}^2 \sigma_{rH}^2 + A_{rFe}^2 \sigma_{rF}^2 \right) \\
&= a_e \left[\zeta_e - \alpha_e \left(A_{rHe} r_{Ht} - A_{rFe} r_{Ft} + C_e \right) \right] \left(A_{rHe}^2 \sigma_{rH}^2 + A_{rFe}^2 \sigma_{rF}^2 \right),
\end{aligned} \tag{3.8}$$

where the second step follows from (2.7) and $\gamma_t = 0$, and the third step follows from (3.2). Substituting μ_{et} from (3.5) into (3.8), we can write the latter equation as

$$\begin{aligned}
&- A_{rHe} \kappa_{rH} (\overline{r}_H - r_{Ht}) + A_{rFe} \kappa_{rF} (\overline{r}_F - r_{Ft}) + \frac{1}{2} A_{rHe}^2 \sigma_{rH}^2 + \frac{1}{2} A_{rFe}^2 \sigma_{rF}^2 + r_{Ft} - r_{Ht} \\
&= a_e \left[\zeta_e - \alpha_e \left(A_{rHe} r_{Ht} - A_{rFe} r_{Ft} + C_e \right) \right] \left(A_{rHe}^2 \sigma_{rH}^2 + A_{rFe}^2 \sigma_{rF}^2 \right).
\end{aligned} \tag{3.9}$$

[1] For simplicity, we deviate from market terminology, according to which the CCT borrows in the currency with the low interest rate.

Equation (3.9) is affine in (r_{Ht}, r_{Ft}). Identifying the linear terms in (r_{Ht}, r_{Ft}) and the constant terms yields three equations for the three scalars $(\{A_{rje}\}_{j=H,F}, C_e)$.

Proposition 3.1. *When arbitrage is segmented, the exchange rate e_t is given by (3.2), with $(\{A_{rje}\}_{j=H,F}, C_e)$ solving*

$$\kappa_{rj} A_{rje} - 1 = -a_e \alpha_e A_{rje} \left(\sigma_{rH}^2 A_{rHe}^2 + \sigma_{rF}^2 A_{rFe}^2\right), \tag{3.10}$$

$$-\kappa_{rH}\overline{r}_H A_{rHe} + \kappa_{rF}\overline{r}_F A_{rFe} + \frac{1}{2}\sigma_{rH}^2 A_{rHe}^2 + \frac{1}{2}\sigma_{rF}^2 A_{rFe}^2 = a_e\left(\zeta_e - \alpha_e C_e\right)\left(\sigma_{rH}^2 A_{rHe}^2 + \sigma_{rF}^2 A_{rFe}^2\right). \tag{3.11}$$

Equation (3.10) yields a system of two non-linear equations in (A_{rHe}, A_{rFe}). Given a solution to this system, (3.11) determines C_e.

3.1.2 Bond Yields

The determination of bond yields parallels that of the exchange rate. Applying Ito's Lemma to (3.3) for $j = H$, using the dynamics (3.1) of r_{jt} for $j = H$, and noting that $t + \tau$ stays constant when taking the derivative, we find that the time-t instantaneous return on the home bond with maturity τ is

$$\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \mu_{Ht}^{(\tau)} dt - A_{rH}(\tau)\sigma_{rH} dB_{rHt}, \tag{3.12}$$

where

$$\mu_{Ht}^{(\tau)} \equiv A_{rH}'(\tau) r_{Ht} + C_H'(\tau) - A_{rH}(\tau)\kappa_{rH}(\overline{r}_H - r_{Ht}) + \frac{1}{2}A_{rH}(\tau)^2 \sigma_{rH}^2 \tag{3.13}$$

is the expected return. Likewise, (3.1) and (3.3) for $j = F$, combined with (3.2), imply that the time-t instantaneous return on the foreign bond with maturity τ, expressed in home-currency terms, minus the instantaneous return on foreign currency, is

$$\frac{d(P_{Ft}^{(\tau)} e_t)}{P_{Ft}^{(\tau)} e_t} - \frac{de_t}{e_t} = \mu_{Ft}^{(\tau)} dt - A_{rF}(\tau)\sigma_{rF} dB_{rFt}, \tag{3.14}$$

where

$$\mu_{Ft}^{(\tau)} \equiv A_{rF}'(\tau) r_{Ft} + C_F'(\tau) - A_{rF}(\tau)\kappa_{rF}(\overline{r}_F - r_{Ft}) + \frac{1}{2}A_{rF}(\tau)\left(A_{rF}(\tau) - 2A_{rFe}\right)\sigma_{rF}^2. \tag{3.15}$$

We next substitute the return (3.12) into the budget constraint of the subset of arbitrageurs who can invest in home bonds (and whose budget constraint is derived from (2.3) by setting $W_{Ft} = X_{Ft}^{(\tau)} = 0$). We do the same for (3.14) and the subset of arbitrageurs who can invest in foreign bonds and have a zero net exposure in foreign-currency instruments ($W_{Ft} = X_{Ht}^{(\tau)} = 0$). For the arbitrageurs investing in the bonds of country $j = H, F$, we find

$$dW_t = \left[W_t r_{Ht} + \int_0^T X_{jt}^{(\tau)} \left(\mu_{jt}^{(\tau)} - r_{jt}\right) d\tau\right] dt - \int_0^T X_{jt}^{(\tau)} A_{rj}(\tau)\sigma_{rj} dB_{rjt}.$$

The optimization problem of these arbitrageurs is

$$\max_{\{X_{jt}^{(\tau)}\}_{\tau\in(0,T)}} \left[\int_0^T X_{jt}^{(\tau)} \left(\mu_{jt}^{(\tau)} - r_{jt}\right) d\tau - \frac{a_j}{2}\left(\int_0^T X_{jt}^{(\tau)} A_{rj}(\tau) d\tau\right)^2 \sigma_{rj}^2\right],$$

and their first-order condition, which follows from point-wise differentiation, is

$$\mu_{jt}^{(\tau)} - r_{jt} = a_j \sigma_{rj}^2 A_{rj}(\tau)\left(\int_0^T X_{jt}^{(\tau)} A_{rj}(\tau) d\tau\right). \tag{3.16}$$

Equation (3.16) describes the arbitrageurs' risk-return trade-off when investing in the *bond carry trade* (BCT) in country j. We term BCT in country j the trade of borrowing short-term in that country and investing the borrowed amount in that country's bonds.[2] The return on the BCT in the home country and for maturity τ is $\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - r_{Ht}dt$, equal to the return on the home bond with maturity τ minus that on the home short rate. The return on the BCT in the foreign country, expressed in home-currency terms, is $\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - r_{Ft}dt$. This is equal to the return on the foreign bond with maturity τ, expressed in home-currency terms, minus that on foreign currency, minus that on the foreign short rate.

If arbitrageurs invest an extra unit of home currency in the BCT for country j and maturity τ, then their expected return increases by the BCT's expected return $\mu_{jt}^{(\tau)} - r_{jt}$. This is the left-hand side of (3.16). The right-hand side is the increase in the arbitrageurs' portfolio risk, times their risk-aversion coefficient a_j. The increase in portfolio risk is equal to the covariance between the return on the BCT in country j and for maturity τ, and the return on the BCT portfolio of arbitrageurs

[2] For simplicity, we deviate from market terminology, according to which the BCT borrows at maturities with a low interest rate.

in country j and across all maturities. Since these returns depend only on the country j short rate r_{jt}, their covariance is the product of their sensitivities to r_{jt} times the instantaneous variance σ_{rj}^2 of r_{jt}. Equations (3.12) and (3.14) imply that the return sensitivities to r_{jt} are $-A_{rj}(\tau)$ and $-\int_0^T X_{jt}^{(\tau)} A_{rj}(\tau)$, respectively.

We next combine the arbitrageurs' first-order condition (3.16) with market clearing for country j bonds. Market clearing requires that the time-t positions of arbitrageurs and bond investors sum to zero:

$$X_{jt}^{(\tau)} + Z_{jt}^{(\tau)} = 0. \tag{3.17}$$

Using (3.17), we can write (3.16) as

$$\begin{aligned}
\mu_{jt}^{(\tau)} - r_{jt} &= -a_j \sigma_{rj}^2 A_{rj}(\tau) \left(\int_0^T Z_{jt}^{(\tau)} A_{rj}(\tau) d\tau \right) \\
&= a_j \sigma_{rj}^2 A_{rj}(\tau) \left(\int_0^T \left[\alpha_j(\tau) \log\left(P_{jt}^{(\tau)} \right) + \zeta_j(\tau) \right] A_{rj}(\tau) d\tau \right) \\
&= a_j \sigma_{rj}^2 A_{rj}(\tau) \left(\int_0^T \left[\zeta_j(\tau) - \alpha_j(\tau) \left(A_{rj}(\tau) r_{jt} + C_j(\tau) \right) \right] A_{rj}(\tau) d\tau \right)
\end{aligned} \tag{3.18}$$

where the second step follows from (2.7) and $\beta_{jt} = 0$, and the third step follows from (3.3). Substituting μ_{Ht} from (3.13) into (3.18) for $j = H$, we find an equation affine in r_{Ht}. Identifying the linear terms in r_{Ht} and the constant terms yields two ordinary differential equations (ODEs) for the two functions $(A_{rH}(\tau), C_{rH}(\tau))$. Repeating this process for the foreign bond, yields two ODEs for $(A_{rF}(\tau), C_{rF}(\tau))$.

Proposition 3.2. *When arbitrage is segmented, bond prices $P_{jt}^{(\tau)}$ in country $j = H, F$ are given by (3.3), with $(A_{rj}(\tau), C_{rj}(\tau))$ solving*

$$A'_{rj}(\tau) + \kappa_{rj} A_{rj}(\tau) - 1 = -a_j \sigma_{rj}^2 A_{rj}(\tau) \int_0^T \alpha_j(\tau) A_{rj}(\tau)^2 d\tau, \tag{3.19}$$

$$C'_j(\tau) + \frac{1}{2}\sigma_{rj}^2 A_{rj}(\tau) \left(A_{rj}(\tau) - 2A_{rFe} 1_{\{j=F\}} \right) = a_j \sigma_{rj}^2 A_{rj}(\tau) \int_0^T \alpha_j(\tau) \left[\zeta_j(\tau) - \alpha_j(\tau) C_j(\tau) \right] A_{rj}(\tau) d\tau, \tag{3.20}$$

with the initial conditions $A_{rj}(0) = C_j(0) = 0$.

Equation (3.19) is a linear ODE in $A_{rj}(\tau)$, with the complication that the coefficient of $A_{rj}(\tau)$ depends on the integral $\int_0^T \alpha_j(\tau) A_{rj}(\tau)^2 d\tau$. To compute $A_{rj}(\tau)$, we solve the ODE taking the integral as given, and then substitute back into the integral to derive one equation in one unknown. Given $A_{rH}(\tau)$, we determine $C_{rH}(\tau)$ from (3.20) following the same approach. A full exposition of the solution method is in Vayanos and Vila (2019).

3.2 Short-Rate Shocks, Carry Trades and Risk Premia

We next determine how bond yields and the exchange rate respond to short-rate shocks, and what the implications are for the profitability of carry trades.

3.2.1 Bonds

Proposition 3.3. *Suppose that arbitrage is segmented. Following a drop in the short rate in country j, bond yields drop in that country ($A_{rj}(\tau) > 0$) and do not change in the other country. When additionally bond arbitrageurs in country j are risk-averse ($a_j > 0$) and the demand of bond investors in that country is price-elastic ($\alpha_j(\tau) > 0$ in a positive-measure set of $(0, T)$):*

- *Bond yields do not drop all the way to the value implied by the Expectations Hypothesis (EH) ($A_{rj}(\tau) < \frac{1-e^{-\kappa_{rj}\tau}}{\kappa_{rj}}$).*
- *The expected return of the BCT rises ($\frac{\partial\left(\mu_{jt}^{(\tau)}-r_{jt}\right)}{\partial r_{jt}} < 0$).*

When the short rate in country j drops, bond prices in that country rise (and bond yields drop) because of a standard discounting effect. Prices do not rise all the way to the value implied by the EH, however. Indeed, if prices remain the same as before the shock, then the drop in the short rate renders the BCT in country j more profitable, raising its expected return $\mu_{jt}^{(\tau)} - r_{jt}$. Hence, bond arbitrageurs in country j seek to invest in the BCT, increasing their bond holdings $X_{jt}^{(\tau)}$. When the demand by bond investors in country j is price-elastic, both bond prices $P_{jt}^{(\tau)}$ and arbitrageurs' bond holdings $X_{jt}^{(\tau)}$ increase in equilibrium. Risk-averse arbitrageurs, however, do not trade all the way to the point where $P_{jt}^{(\tau)}$ reaches its EH value. Instead, as shown in Vayanos and Vila (2019), the BCT's expected return $\mu_{jt}^{(\tau)} - r_{jt}$ remains higher than before the shock to compensate arbitrageurs for the risk generated by their larger bond position. Prices adjust all the way to their EH value when bond arbitrageurs in country j are risk neutral, since they do not require such compensation. They also adjust to their EH value when the demand by bond investors in country

j is price-elastic, because arbitrageurs' activity causes prices to rise to rise without any change in $X_{jt}^{(\tau)}$.

Proposition 3.3 implies a positive relationship between the expected return of the BCT in country j and the slope of the term structure in that country. Indeed, when the short rate is country j is low, both BCT expected return and term-structure slope are high. A positive relationship between the two variables has been documented by Fama and Bliss (FB 1987). We explore quantitatively the link between our model and the empirical findings in FB and other papers mentioned in this section and Section 4, in Section **??**.

3.2.2 Foreign Currency

Proposition 3.4. *Suppose that arbitrage is segmented. Following a drop in the home short rate or a rise in the foreign short rate, the foreign currency appreciates ($A_{rHe} > 0$, $A_{rFe} > 0$). When additionally currency arbitrageurs are risk-averse ($a_e > 0$) and the demand of currency traders is price-elastic ($\alpha_e > 0$),*

- *The foreign currency does not appreciate all the way to the level implied by Uncovered Interest Parity (UIP) ($A_{rHe} < \frac{1}{\kappa_{rH}}$, $A_{rFe} < \frac{1}{\kappa_{rF}}$).*
- *The expected return of the CCT rises ($\frac{\partial(\mu_{et}+r_{Ft}-r_{Ht})}{\partial r_{Ht}} < 0$ and $\frac{\partial(\mu_{et}+r_{Ft}-r_{Ht})}{\partial r_{Ft}} > 0$).*

When the home short rate drops or the foreign short rate rises, the foreign currency appreciates. These movements are in the direction implied by UIP. The foreign currency does not appreciate all the way to the value implied by UIP, however. Indeed, if the exchange rate remains the same as before the shock, then the drop in r_{Ht} or rise in r_{Ft} render the CCT more profitable, raising its expected return $\mu_{et} + r_{Ft} - r_{Ht}$. Hence, currency arbitrageurs seek to increase their holdings W_{Ft} of the foreign currency. When the demand by currency traders is price-elastic, both the exchange rate e_t and arbitrageurs' foreign-currency holdings W_{Ft} increase in equilibrium. Risk-averse arbitrageurs, however, do not trade all the way to the point where e_t reaches its UIP value. Instead, in a spirit similar to Gabaix and Maggiori (2015), the CCT's expected return $\mu_{et}+r_{Ft}-r_{Ht}$ remains higher than before the shock to compensate arbitrageurs for the risk generated by their larger foreign-currency position. The exchange rate adjusts all the way to its UIP value when currency arbitrageurs are risk-neutral or when the demand by currency traders is price-inelastic.

Proposition 3.3 implies a positive relationship between the expected return of the CCT and the differential between the foreign and the home short rate. Such a relationship holds in the data. Bilson (1981) and Fama (1984) document that following an increase in the foreign-minus-home short-rate differential, the expected return on the foreign currency typically increases. Moreover, even in samples where it decreases, it does so less than implied by UIP. Hence, the CCT becomes more profitable.

3.3 Demand Shocks

We next determine how bond yields and the exchange rate respond to changes in the demand for bonds and foreign currency. Since we assume no demand risk in this section, we take the demand changes to be unanticipated and one-off. Demand changes by bond investors in country j correspond to shocks to the demand factor β_{jt}. Demand changes by currency traders correspond to shocks to the demand factor γ_t. Following the shocks, the demand factors revert deterministically to their mean of zero.

Without loss of generality, we take θ_e to be positive, which means that an increase in γ_e corresponds to a drop in demand for foreign currency. We take $\theta_j(\tau)$ to be positive for all τ, which means that an increase in β_{jt} corresponds to a drop in demand for the bonds of country j.

Proposition 3.5. *Suppose that arbitrage is segmented, $\theta_e > 0$ and $\theta_j(\tau) > 0$ for all τ.*

- *A drop in investor demand for the bonds of country j (increase in β_{jt}) raises bond yields in country j if bond arbitrageurs in that country are risk-averse ($a_j > 0$). It has no effect on bond yields in the other country and on the exchange rate.*
- *A drop in currency traders' demand for foreign currency (increase in γ_t) causes the foreign currency to depreciate if currency traders are risk-averse ($a_e > 0$). It has no effect on bond yields.*

When arbitrage is segmented, changes to the demand for an asset class—foreign currency, home bonds, foreign bonds—affect that asset class only. When, for example, the demand for bonds in country j drops, these bonds become cheaper and their yields increase, while foreign currency and bonds in the other country are unaffected.

4 Global Arbitrage

In the remainder of this paper we study the case of global arbitrage. In this section we maintain the other assumptions of Section 3, i.e., no demand risk for bonds and foreign currency, and independent short rates. We relax these assumptions in later sections.

4.1 Equilibrium

We conjecture that the equilibrium exchange rate takes the same form (3.2) as in Section 3. In contrast to Section 3, we allow bond yields in each country $j = H, F$ to also depend on the other country's short rate because this is the case in equilibrium. Thus, we replace (3.3) by

$$P_{jt}^{(\tau)} = e^{-\left[A_{rjj}(\tau)r_{jt} + A_{rjj'}(\tau)r_{j't} + C_j(\tau)\right]} \tag{4.1}$$

for $j' \neq j$ and six functions $(\{A_{rjj'}(\tau)\}_{j,j'=H,F}, \{C_j(\tau)\}_{j=H,F})$ that depend only on τ.

When arbitrage is global, the exchange rate and bond yields are determined jointly. The first step in determining them is to derive the first-order condition of arbitrageurs. Applying Ito's Lemma to (4.1) for $j = H$, we find the following counterpart of (3.12):

$$\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \mu_{Ht}^{(\tau)} dt - A_{rHH}(\tau)\sigma_{rH} dB_{rHt} - A_{rHF}(\tau)\sigma_{rF} dB_{rFt}, \tag{4.2}$$

where

$$\begin{aligned}\mu_{Ht}^{(\tau)} \equiv & A'_{rHH}(\tau)r_{Ht} + A'_{rHF}(\tau)r_{Ft} + C'_H(\tau) - A_{rHH}(\tau)\kappa_{rH}(\overline{r}_H - r_{Ht}) - A_{rHF}(\tau)\kappa_{rF}(\overline{r}_F - r_{Ft}) \\ & + \frac{1}{2}A_{rHH}(\tau)^2\sigma_{rH}^2 + \frac{1}{2}A_{rHF}(\tau)^2\sigma_{rF}^2 \end{aligned} \tag{4.3}$$

Likewise, (4.1) for $j = F$ and (3.2) yield the following counterpart of (3.14):

$$\frac{d(P_{Ft}^{(\tau)} e_t)}{P_{Ft}^{(\tau)} e_t} - \frac{de_t}{e_t} = \mu_{Ft}^{(\tau)} dt - A_{rFH}(\tau)\sigma_{rH} dB_{rHt} - A_{rFF}(\tau)\sigma_{rF} dB_{rFt}, \tag{4.4}$$

where

$$\begin{aligned}\mu_{Ft}^{(\tau)} \equiv & A'_{rFH}(\tau)r_{Ht} + A'_{rFF}(\tau)r_{Ft} + C'_F(\tau) - A_{rFH}(\tau)\kappa_{rH}(\overline{r}_H - r_{Ht}) - A_{rFF}(\tau)\kappa_{rF}(\overline{r}_F - r_{Ft}) \\ & + \frac{1}{2}A_{rFH}(\tau)\left(A_{rFH}(\tau) + 2A_{rHe}\right)\sigma_{rH}^2 + \frac{1}{2}A_{rFF}(\tau)\left(A_{rFF}(\tau) - 2A_{rFe}\right)\sigma_{rF}^2. \end{aligned} \tag{4.5}$$

Substituting the returns (3.4), (4.2) and (4.4) into the arbitrageurs' budget constraint (2.3), we can write their optimization problem (2.4) as

$$\max_{W_{Ft},\{X_{jt}^{(\tau)}\}_{\tau\in(0,T),j=H,F}} \Bigg\{ W_{Ft}\left(\mu_{et}+r_{Ft}-r_{Ht}\right)+\sum_{j=H,F}\int_0^T X_{jt}^{(\tau)}\left(\mu_{jt}^{(\tau)}-r_{jt}\right)d\tau$$
$$-\frac{a}{2}\sum_{j,j'=H,F,j\neq j'}\left[W_{Ft}A_{rej}(-1)^{1_{\{j=F\}}}+\int_0^T X_{jt}^{(\tau)}A_{rjj}(\tau)d\tau+\int_0^T X_{j't}^{(\tau)}A_{rj'j}(\tau)d\tau\right]^2\sigma_{rj}^2\Bigg\}. \tag{4.6}$$

The first-order conditions are

$$\mu_{et}+r_{Ft}-r_{Ht}=A_{rHe}\lambda_{rHt}-A_{rFe}\lambda_{rFt}, \tag{4.7}$$

$$\mu_{jt}^{(\tau)}-r_{jt}=A_{rjj}(\tau)\lambda_{rjt}+A_{rjj'}(\tau)\lambda_{rj't}, \tag{4.8}$$

where $j,j'=H,F$, $j\neq j'$ and

$$\lambda_{rjt}\equiv a\sigma_{rj}^2\left[W_{Ft}A_{rej}(-1)^{1_{\{j=F\}}}+\int_0^T X_{jt}^{(\tau)}A_{rjj}(\tau)d\tau+\int_0^T X_{j't}^{(\tau)}A_{rj'j}(\tau)d\tau\right]. \tag{4.9}$$

The left-hand side of (4.7) and (4.8) is the increase in the arbitrageurs' expected return if they invest one unit of home currency in the CCT and in the country j BCT, respectively. The right-hand side is the increase in the arbitrageurs' portfolio risk, times their risk-aversion coefficient a. Portfolio risk increases by the covariance between the corresponding trade (CCT or country j BCT) and the arbitrageurs' portfolio. To compute the covariance, we multiply the sensitivity of the trade's return to the short rate in country j, times the sensitivity λ_{rjt} of the arbitrageurs' portfolio return to the same factor, times the factor's variance σ_{rj}^2. We then sum over $j=H,F$. In the terminology of no-arbitrage models, the sensitivity λ_{rjt} is the price of the risk factor r_{jt}. Global arbitrage connects bond and currency markets by equalizing factor prices λ_{rjt} across all trades (CCT, home BCT, foreign BCT).

We next combine the arbitrageurs' first-order conditions (4.7) and (4.8) with the market-clearing equations (3.7) and (3.17). Proceeding as in Section 3, we characterize the exchange rate and bond prices by a system of scalar equations and ODEs.

Proposition 4.1. *When arbitrage is global, the exchange rate e_t is given by (3.2) and bond prices*

$P_{jt}^{(\tau)}$ *in country* $j = H, F$ *are given by (4.1), with* $(\{A_{rje}\}_{j=H,F}, C_e)$ *solving*

$$\kappa_{rj} A_{rje} - 1 = a\sigma_{rj}^2 \bar{\lambda}_{rjj} A_{rje} - a\sigma_{rj'}^2 \bar{\lambda}_{rjj'} A_{rj'e}, \tag{4.10}$$

$$- \kappa_{rH} \overline{r}_H A_{rHe} + \kappa_{rF} \overline{r}_F A_{rFe} + \frac{1}{2}\sigma_{rH}^2 A_{rHe}^2 + \frac{1}{2}\sigma_{rF}^2 A_{rFe}^2 = A_{rHe}\lambda_{rHC} - A_{rFe}\lambda_{rFC}, \tag{4.11}$$

and $(A_{rjj}(\tau), A_{rjj'}(\tau), C_j(\tau))$ *solving*

$$A'_{rjj}(\tau) + \kappa_{rj} A_{rjj}(\tau) - 1 = a\sigma_{rj}^2 \bar{\lambda}_{rjj} A_{rjj}(\tau) + a\sigma_{rj'}^2 \bar{\lambda}_{rjj'} A_{rjj'}(\tau), \tag{4.12}$$

$$A'_{rjj'}(\tau) + \kappa_{rj'} A_{rjj'}(\tau) = a\sigma_{rj}^2 \bar{\lambda}_{rj'j} A_{rjj}(\tau) + a\sigma_{rj'}^2 \bar{\lambda}_{rj'j'} A_{rjj'}(\tau), \tag{4.13}$$

$$\begin{aligned} &C'_j(\tau) - \kappa_{rj}\overline{r}_j A_{rjj}(\tau) - \kappa_{rj'}\overline{r}_{j'} A_{rjj'}(\tau) + \frac{1}{2}\sigma_{rj}^2 A_{rjj}(\tau)\left(A_{rjj}(\tau) - 2A_{rFe}1_{\{j=F\}}\right) \\ &+ \frac{1}{2}\sigma_{rj'}^2 A_{rjj'}(\tau)\left(A_{rjj'}(\tau) + 2A_{rHe}1_{\{j=F\}}\right) = a\sigma_{rj}^2 \bar{\lambda}_{rjC} A_{rjj}(\tau) + a\sigma_{rj'}^2 \bar{\lambda}_{rj'C} A_{rjj'}(\tau), \end{aligned} \tag{4.14}$$

with the initial conditions $A_{rjj}(\tau) = A_{rjj'}(0) = C_j(0) = 0$*, where* $j' \neq j$ *and*

$$\bar{\lambda}_{rjj} \equiv -\left[\int_0^T \alpha_j(\tau) A_{rjj}(\tau)^2 d\tau + \int_0^T \alpha_{j'}(\tau) A_{rj'j}(\tau)^2 d\tau + \alpha_e A_{rje}^2\right],$$

$$\bar{\lambda}_{rjj'} \equiv -\left[\int_0^T \alpha_j(\tau) A_{rjj}(\tau) A_{rjj'}(\tau) d\tau + \int_0^T \alpha_{j'}(\tau) A_{rj'j'}(\tau) A_{rj'j}(\tau) d\tau - \alpha_e A_{rje} A_{rj'e}\right],$$

$$\bar{\lambda}_{rjC} \equiv \sum_{k=H,F} \int_0^T \left(\zeta_k(\tau) - \alpha_k(\tau) C_k(\tau)\right) A_{rkj}(\tau) d\tau + \left(\zeta_e - \alpha_e C_e\right) A_{rje} (-1)^{1_{\{j=F\}}}.$$

Equations (4.12) and (4.13) form a system of two linear ODEs in $(A_{rjj}(\tau), A_{rjj'}(\tau))$, with the complication that the coefficients of $(A_{rjj}(\tau), A_{rjj'}(\tau))$ depend on integrals involving these functions as well as the functions obtained by inverting j and $j' \neq j$. We solve the system taking $\bar{\lambda}_{rjj}$, $\bar{\lambda}_{rjj'} = \bar{\lambda}_{rj'j}$ and $\bar{\lambda}_{rj'j'}$ as given. We do the same for the system obtained by inverting j and j', and for the scalar system (4.10) in the unknowns (A_{rHe}, A_{rFe}). We then substitute back into the definitions of $\bar{\lambda}_{rjj}$, $\bar{\lambda}_{rjj'} = \bar{\lambda}_{rj'j}$ and $\bar{\lambda}_{rj'j'}$ to derive a non-linear system of three equations in these three unknowns. The properties that we show in the remainder of this section hold for any solution of this system.

4.2 Short-Rate Shocks, Carry Trades and Risk Premia

Proposition 4.2. *Suppose that arbitrage is global.*

- *The effects of short-rate shocks on the exchange rate and on the CCT's expected return have the same properties as in Proposition 3.4.*
- *The effects of shocks to the country-j short-rate r_{jt} on bond yields in country j and on the BCT's expected return have the same properties as in Proposition 3.1, except that the price-elasticity condition can hold for currency traders rather than bond investors.*
- *When arbitrageurs are risk-averse ($a > 0$) and the demand by currency traders is price-elastic ($\alpha_e > 0$), a drop in r_{jt} causes bond yields in country $j' \neq j$ to rise ($A_{j'j}(\tau) > 0$) and the BCT's expected return to drop ($\frac{\partial\left(\mu_{j't}^{(\tau)} - r_{j't}\right)}{\partial r_{jt}} > 0$).*
- *The effect of r_{jt} on bond yields is smaller in country j' than in country j ($A_{jj}(\tau) > A_{j'j}(\tau)$).*

The response of the exchange rate to short-rate shocks is similar under global and segmented arbitrage: the exchange rate moves in the direction implied by UIP, and there is under-reaction when arbitrageurs are risk-averse ($a > 0$) and the demand by currency traders is price-elastic ($\alpha_e > 0$). Global and segmented arbitrage differ in how bond yields respond to shocks. Under segmented arbitrage, a shock to the short rate r_{jt} in country j affects bond yields in that country only. By contrast, under global arbitrage, and provided that $a\alpha_e > 0$, the shock affects bond yields in both countries, even though the short rate $r_{j't}$ in country $j' \neq j$ does not change. When r_{jt} drops, bond yields in both countries drop.

Since short-rate shocks are transmitted across countries, monetary policy in one country has a direct effect on the other country's interest rates. When the central bank in country j lowers the short rate r_{jt}, interest rates for longer maturities in country j' drop. This is so even though the central bank in country j' leaves the short rate $r_{j't}$ unchanged. Hence, each country is not insulated from monetary-policy shocks in the other country, despite the exchange rate being fully flexible.

Short-rate shocks are transmitted across countries because global arbitrageurs engage in the CCT and use the bond market to hedge. Recall that under both segmented and global arbitrage, a drop in the home short rate r_{Ht} raises the profitability of the CCT, making it more attractive to arbitrageurs. When the demand by currency traders is price-elastic, the arbitrageurs' equilibrium investment in the CCT increases. Because arbitrageurs hold more foreign-currency instruments (higher W_{Ft}), they become more exposed to the risk that the foreign short rate r_{Ft} drops and the foreign currency depreciates. Global arbitrageurs hedge that risk by buying foreign bonds because their price rises when r_{Ft} drops. The arbitrageurs' activity pushes the prices of foreign bonds up and their yields down.

An additional consequence of hedging by global arbitrageurs is greater under-reaction of home bonds to the home short rate. When r_{Ht} drops, arbitrageurs invest more in the CCT, and hence become more exposed to a rise in r_{Ht}. Investing in home bonds, whose prices drop when r_{Ht} rises, adds to that risk. Hence, global arbitrageurs are less eager than segmented arbitrageurs to buy home bonds following a drop in r_{Ht}, and the expected return of the home BCT increases more than under segmented arbitrage. In particular, when the demand by home bond investors is price-inelastic (and that by currency traders is elastic), a drop in r_{Ht} raises the home BCT's expected return under global arbitrage but leaves it unaffected under segmented arbitrage.

We next turn to variants of the CCT studied in the empirical literature. We show that these trades can be viewed as combinations of the BCT and the (basic) CCT, and that Proposition 4.2 can shed light on empirical findings concerning these trades.

One variant is a hybrid CCT in which the trading horizon is short but the trading instruments are long-term. Borrowing in the home country and investing in the foreign country is done with the respective τ-year bonds, and the positions are held for a short horizon dt. The return of the hybrid CCT in home-currency units is

$$\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} = \left(\frac{de_t}{e_t} + r_{Ft} - r_{Ht}\right) + \left(\frac{d(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{de_t}{e_t} - r_{Ft}\right) - \left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - r_{Ht}\right). \quad (4.15)$$

Hence, the hybrid CCT can be viewed as a combination of (i) the basic CCT, (ii) a long position in the foreign BCT, and (iii) a short position in the home BCT.

A second variant is a long-horizon CCT, in which borrowing in the home country and investing in the foreign country is done with the respective τ-year bonds, and the positions are held until the bonds' maturity. The return of the long-horizon CCT in home-currency units and log terms is

$$\log\left(\frac{e_{t+\tau}}{P_{Ft}^{(\tau)}e_t}\right) - \log\left(\frac{1}{P_{Ht}^{(\tau)}}\right) = \int_t^{t+\tau}\left(\log\left(\frac{e_{s+ds}}{e_s}\right) + r_{Fs}ds - r_{Hs}ds\right)$$
$$+ \left(\tau y_{Ft}^{(\tau)} - \int_t^{t+\tau} r_{Fs}ds\right) - \left(\tau y_{Ht}^{(\tau)} - \int_t^{t+\tau} r_{Hs}ds\right), \quad (4.16)$$

where the equality follows from (2.1). Hence, the long-horizon CCT can be viewed as the combination of (i) a sequence of basic CCTs, (ii) a long position in a long-horizon foreign BCT, and (iii) a short position in a long-horizon home BCT. The long-horizon BCT in country j involves buying bonds in country j and financing that position by borrowing short-term and rolling over.

Proposition 4.3. *Suppose that arbitrage is global.*

- *The expected returns of the hybrid CCT and the long-horizon CCT rise following a drop in the home short rate r_{Ht} or a rise in the foreign short rate r_{Ft}.*
- *When the maturity τ of the bonds involved in the hybrid CCT and the long-horizon CCT goes to infinity, these trades' expected returns and their sensitivity to (r_{Ht}, r_{Ft}) go to zero.*

Short rate shocks move the expected return of the hybrid and the long-horizon CCT in the same direction as for the basic CCT. The effect goes to zero, however, when the maturity τ of the bonds in these trades goes to infinity. Our results are consistent with Lustig, Stathopoulos, and Verdelhan (2019), who document that short rates lose their predictive power for the return of the hybrid CCT, while they predict strongly the return of the basic CCT. They are also consistent with Chinn and Meredith (2004), who document that UIP cannot be rejected over long horizons.

Short rate shocks lose their predictive power for the hybrid and the long-horizon CCT because the risk of these trades arises from long-horizon exchange-rate movements, which are unrelated to current short-rate shocks. Indeed, an arbitrageur entering in the long-horizon CCT at time t receives a fixed amount of foreign currency and pays a fixed amount of domestic currency at time $t + \tau$. Mean-reverting short-rate shocks do not affect the risk borne by the arbitrageur when τ is large. The same is true for the hybrid CCT because that trade is identical to the long-horizon CCT except that it is unwound at time $t + dt$.

Under segmented arbitrage, a weaker version of Proposition 4.2 holds. Short-rate shocks have a smaller effect on the expected return of the hybrid CCT than of the basic CCT. Likewise, the effect is smaller for the long-horizon CCT than for the sequence of basic CCTs. This is because the shocks' effect through the BCTs work in the opposite direction. Consider, for example, a drop in the home short rate. Propositions 3.3 and 3.4 imply that the expected return of the basic CCT increases, but so does the expected return of the home BCT, which enters as a short position in the hybrid and the long-horizon CCT. Under segmented arbitrage, however, the effects of short-rate shocks on the CCT and BCT are disconnected because they driven by the risk aversion of different arbitrageurs. In particular, the expected return of the hybrid CCT can drop when the home short rate drops, while it always rises under global arbitrage.

4.3 Demand Shocks

Under global arbitrage, shocks to the demand for an asset class—foreign currency, home bonds, foreign bonds—affect all three asset classes. This is in contrast to segmented arbitrage, where only the asset class for which demand changes is affected (Proposition 3.5).

Proposition 4.4. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), and $\theta_j(\tau) > 0$ for all τ. A drop in investor demand for the bonds of country j (increase in $\beta_{jt}(\tau)$):*

- *Raises bond yields in country j.*
- *Raises bond yields in country $j' \neq j$ when the demand by currency traders is price-elastic ($\alpha_e > 0$).*
- *Causes the foreign currency to depreciate if $j = H$, and to appreciate if $j = F$.*

A drop in investor demand for home bonds depresses their prices, as in Proposition 3.5. Additionally, prices for foreign bonds drop and the foreign currency depreciates. The latter (cross) effects are driven by hedging of global arbitrageurs. Indeed, arbitrageurs accommodate the drop in demand for home bonds by holding more such bonds. Hence, they become more exposed to a rise in the home short rate r_{Ht} and less willing to hold assets that lose value when r_{Ht} rises. Foreign currency is such an asset, and hence it depreciates. Foreign bonds is another such asset (Proposition 4.2 shows that a rise in r_{Ht} drives foreign bond prices down when the demand by currency traders is price-elastic), and hence their prices drop. A drop in demand for foreign bonds has symmetric effects.

Proposition 4.5. *Suppose that arbitrage is global, arbitrageurs are risk-averse ($a > 0$), and $\theta_e > 0$ for all τ. A drop in currency traders' demand for foreign currency (increase in γ_t):*

- *Causes the foreign currency to depreciate.*
- *Raises bond yields in the home country.*
- *Lowers bond yields in the foreign country.*

A drop in currency traders' demand for foreign currency causes it depreciate, as in Proposition 3.5. Additionally, hedging by global arbitrageurs causes home bond prices to drop and foreign bond prices to rise. Indeed, arbitrageurs accommodate the drop in demand for foreign currency by holding more of it. Hence, they become more exposed to a rise in the home short rate r_{Ht} and to a decline in the foreign short rate r_{Ft}. This makes them less willing to hold home bonds, which lose value when r_{Ht} rises, and more willing to hold foreign bonds, which gain value when r_{Ft} drops.

5 Global Arbitrage and Demand Risk

In this section we allow the demand by preferred-habitat investors and currency traders to be stochastic. There are five risk factors: the two short rates $\{r_{jt}\}_{j=H,F}$ and the three demand risk factors $(\{\beta_{jt}\}_{j=H,F}, \beta_{et})$. In order to simplify notation, we collect the risk factors into a vector $\mathbf{y}_t = \begin{bmatrix} r_{Ht} & r_{Ft} & \beta_{Ht} & \beta_{Ft} & \beta_{et} \end{bmatrix}^\top$, and hence the dynamics of the model are given by the vector Ornstein-Uhlenbeck process

$$\mathrm{d}\mathbf{y}_t = -\mathbf{\Gamma}\left(\mathbf{y}_t - \overline{\mathbf{y}}\right)\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t\,. \tag{5.1}$$

5.1 Equilibrium

We conjecture that the equilibrium exchange rate and bond prices take the form

$$-\log P_{jt}^{(\tau)} = \mathbf{y}_t^T \mathbf{A}_j(\tau) + C_j(\tau), \tag{5.2}$$

$$-\log e_t = \mathbf{y}_t^T \mathbf{A}_e + C_e. \tag{5.3}$$

In order to solve with multiple stochastic demand factors, we turn to numerical solution methods. Our solution algorithm is described in Appendix Section B.

5.2 Calibration

In order to solve the model numerically, we need to calibrate the model parameters as well as take a stand on the functional form of the elasticity and demand functions $\alpha_j(\tau)$, $\theta_j(\tau)$.

Our calibration assumes the two countries are symmetric, except that demand shocks are larger in the Home country. This is captured by the demand functions $\theta_H(\tau)$ and $\theta_F(\tau)$, shown in the bottom two panels of Figure 1. Additionally, we set $\kappa_{rj} = 0.2$ and $\kappa_{\beta,j} = 0.35$, so that the shocks to the Home and Foreign short rate mean-revert more slowly than the Home bond, Foreign bond, and currency risk factors. The factors are all independent with standard deviation $\sigma = 0.02$.

Finally, we solve the model for different levels of risk aversion parameter, in order to explore how the model behaves as arbitrageur risk aversion increases. We set $a = \{0, 1.0, 8.0, 24.0\}$, which we call the "zero", "low," "medium," and "high" equilibria.

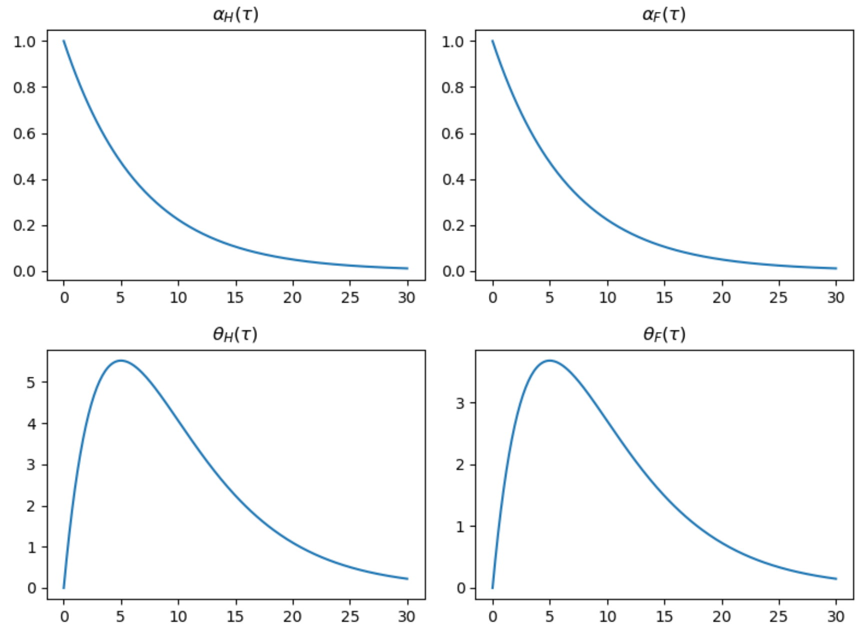

Figure 1: Habitat Elasticity and Demand Functions

Notes: Plots of the habitat elasticity and demand functions $\alpha_j(\tau)$, $\theta_j(\tau)$ across maturities τ.

We first explore how the model performs with respect to two common bond risk premia regressions: the Fama-Bliss (FB) regressions (measuring the relationship between the slope of the term structure and bond risk premia across maturities); and Campbell-Shiller (CS) regressions (measuring the relationship between the slope of the term structure and changes in bond yields across maturities). Figure 2 shows the model-implied regression coefficients of the two regressions across maturities, for the Home and Foreign countries.

When arbitrageurs are close to risk-neutral, the slope coefficients are constant across maturities and equal to 0 in the FB regression and 1 in the CS regression. As risk aversion increases, these coefficients deviate from the risk-neutral baseline. The FB coefficients in both countries becomes larger, and for high levels of risk aversion is increasing in maturity τ. For the CS regression coefficient, higher risk aversion pushes the coefficients below 1; when risk aversion is very high, the Home country CS coefficients drop below 0. Because the demand factor in the Foreign country is smaller than the Home country, the Foreign CS regression coefficient is slightly larger than the Home country.

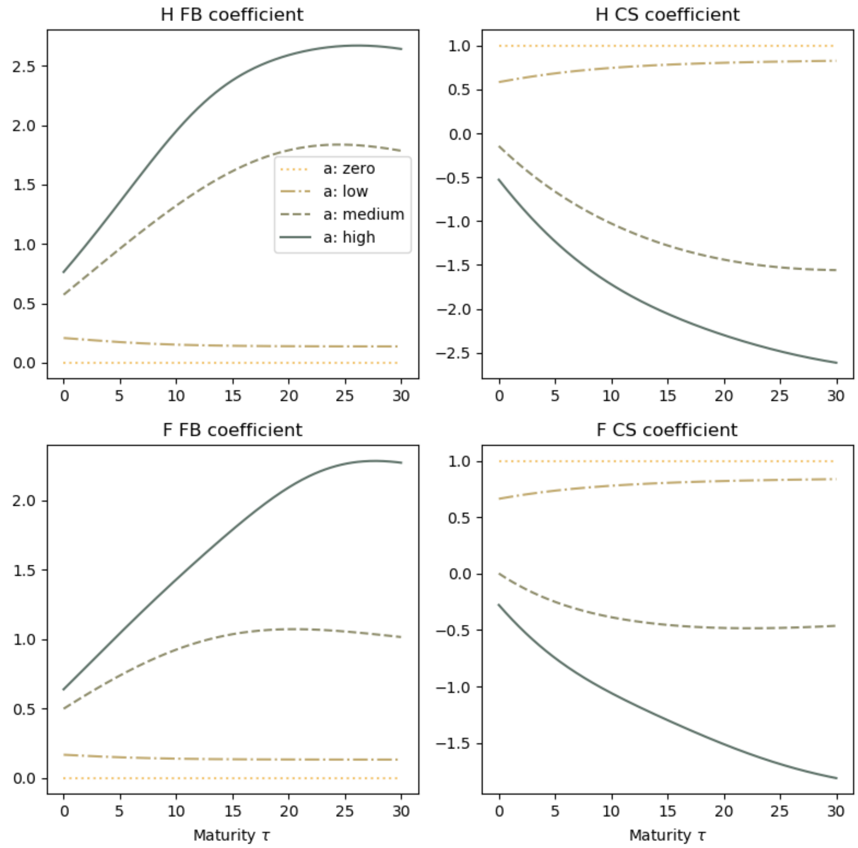

Figure 2: Model-Implied Term Structure Regression Coefficients

Notes: Plots of the model-implied regression coefficients across maturities τ. The top-left and bottom-left panels plot the slope coefficients of the Fama-Bliss regressions across maturities τ in the Home and Foreign country. The top-right and bottom-right panels plot the slope coefficients of the Campbell-Shiller regressions across maturities τ in the Home and Foreign country. The dashed, dotted, and solid lines correspond to low, medium, and high levels of risk aversion, respectively.

5.3 Shocks to Risk Factors

We now explore how the model reacts to shocks to the five different risk factors.

Figure 3 plots the change in the yield curve in the Home and Foreign countries, in response to changes in the risk factors. The dashed, dotted, and solid lines correspond to low, medium, and high levels of risk aversion, respectively. It is clear that both the quantitative and qualitative predictions of the model depend on the level of risk aversion. Figure 4 plots the corresponding change in arbitrageur portfolio allocations.

In order to understand why this is the case, we first start with the simplest case when arbitrageurs are nearly risk-neutral. Suppose the Home short rate r_{Ht} increases (the first row of panels in Figures 3 and 4). All else equal, expected returns on long-term bonds must also increase, and so this downward pressure on bond prices induces the Home (price-elastic) habitat investors to buy more long-term bonds. Arbitrageurs facilitate this by reducing their holdings of Home bonds; because they are nearly risk-neutral, they are happy to fully offset the shift in demand from habitat investors with little change in expected returns. Similarly, because the Foreign short rate r_{Ft} has not changed, all else equal the expected return on Foreign currency must increase. This leads to a depreciation of the Foreign currency. This induces (price-elastic) currency traders to hold more Foreign currency, and once again arbitrageurs are happy to accommodate this shift in demand. Hence, the change in the Home short rate leads to large changes in Home bond yields and the exchange rate, but there is little to no spillover to the Foreign bond market.

When instead arbitrageurs are risk-averse, they are less inclined to fully accomodate the shifts in demand from the price-elastic habitat investors. First, the fall in their holdings of Home bonds decreases their exposure to Home short rate risk. Hence arbitrageurs wish to increase their holdings of Home bonds relative to the risk-neutral baseline. This pushes the yield curve down relative to the risk-neutral baseline. For the same reason, decreased exposure to Home short rate risk also implies that arbitrageurs are less inclined to hold Foreign bonds and Foreign currency. This implies that the exchange rate falls by less than the risk-neutral baseline. It also implies that changes in the Home short rate spill into the Foreign yield curve. However, the top-right panel of Figure 3 shows that the shift in the Foreign yield curve changes shape when arbitrageurs move from moderately to highly risk-averse. To better understand this, we first turn to a discussion of how bond yields react to shifts in the demand factors.

The panels in the third rows of Figures 3 and 4 show the response to an decrease in the Home demand factor β_{Ht} (equivalently, an increase in supply). Again, it is useful to start with a baseline where arbitrageurs are close to risk-neutral. The immediate effect in an increase in the arbitrageurs' holdings of long-term bonds. Of course, when they are nearly risk neutral, they do not require changes in expected returns to accomodate this change in allocations, and hence there is no reaction in bond yields (in either country) or in the exchange rate. However, this shift in allocation towards Home long-term bonds implies that they are more exposed to Home short-rate risk. Hence, when they are risk averse they wish to hedge their exposure to this source of risk. They accomplish this by reducing their holdings of Foreign currency as well as Foreign bonds.

Now we return to the reaction of the Foreign yield curve to the increase in Home short rates when risk aversion is very high (the top-right panel of Figure 3). All else equal, arbitrageurs would like to decrease their holdings of Foreign bonds, due to their increased exposure to Home short rate risk. But from the above discussion of the demand factors, this decline in their allocation of Foreign bonds also reduces their exposure to the demand factors, in particular the Foreign demand factor. Hence, they are more willing to hold assets which are exposed to the demand factors, which in turn puts downward pressure on expected Foreign bond returns and yields. On net, this more complicated hedging behavior leads to a flattening of the Foreign yield curve in response to an increase in the Home short rate.

Intuitively, as arbitrageurs become more and more risk averse, they seek to limit their exposure to all sources of risk regardless of expected returns. The flattening of the Foreign yield curve arises because of the interaction of price-elastic bond traders (so that shifts in bond yields induce shifts in equilibrium portfolio allocations); price-elastic currency traders (so that shifts in the short rates spill across countries); and stochastic demand factors (so that arbitrageurs must hedge against shifts in their allocation of long-term bonds that arise for reasons unrelated to the short rate).

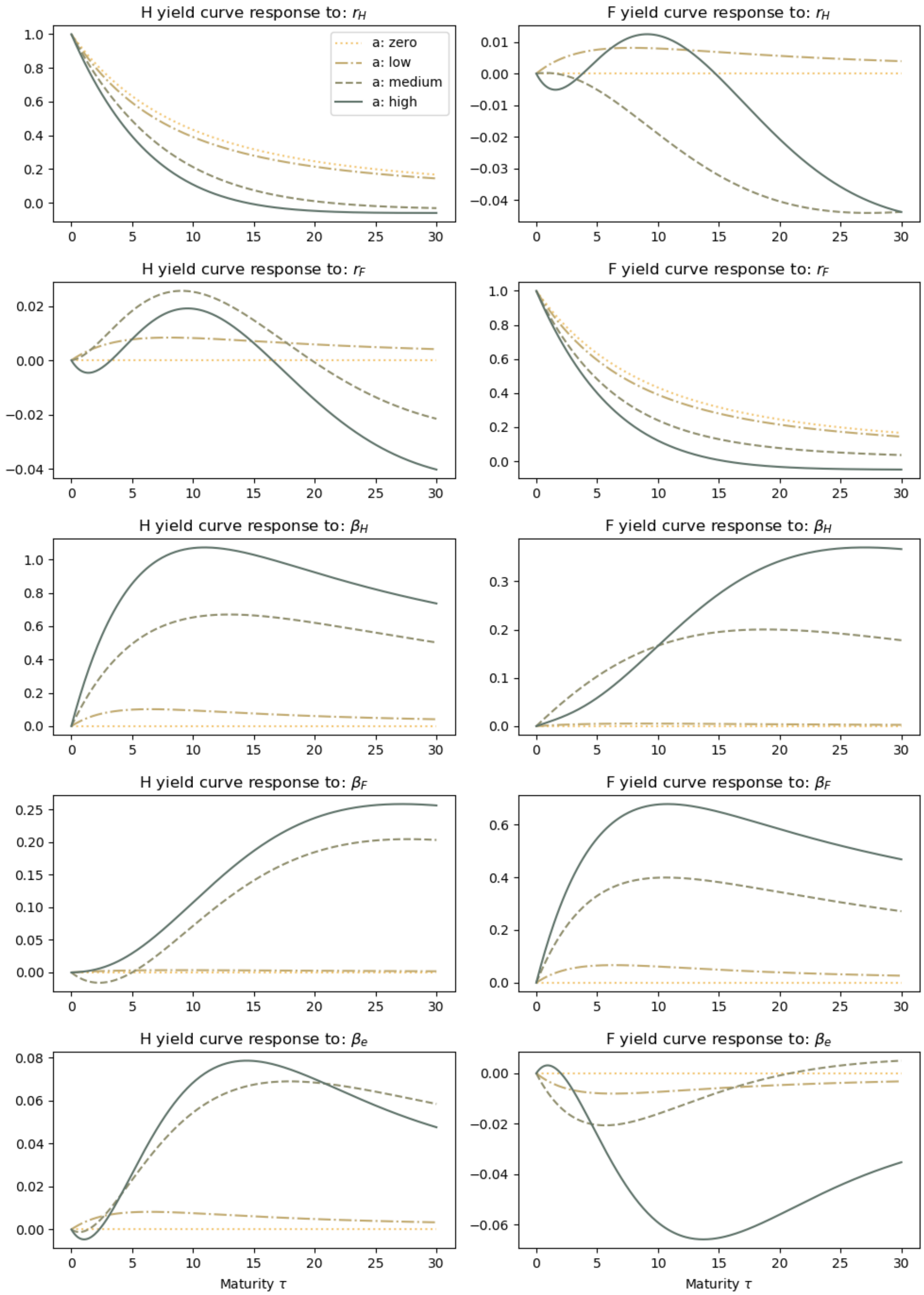

Figure 3: Response of Yield Curves to Risk Factors

Notes: Plots of the yield curve response in the Home and Foreign countries (left and right columns, respectively). The first row is in response to an increase in the Home short rate r_{Ht}, while the second row plots the responses to an increase in the Foreign short rate r_{Ft}. The third and fourth rows plot the response to an increase the Home and Foreign demand factors β_{Ht}, β_{Ft}. Finally, the bottom row shows the response to the currency risk factor β_e. The dashed, dotted, and solid lines correspond to low, medium, and high levels of risk aversion, respectively.

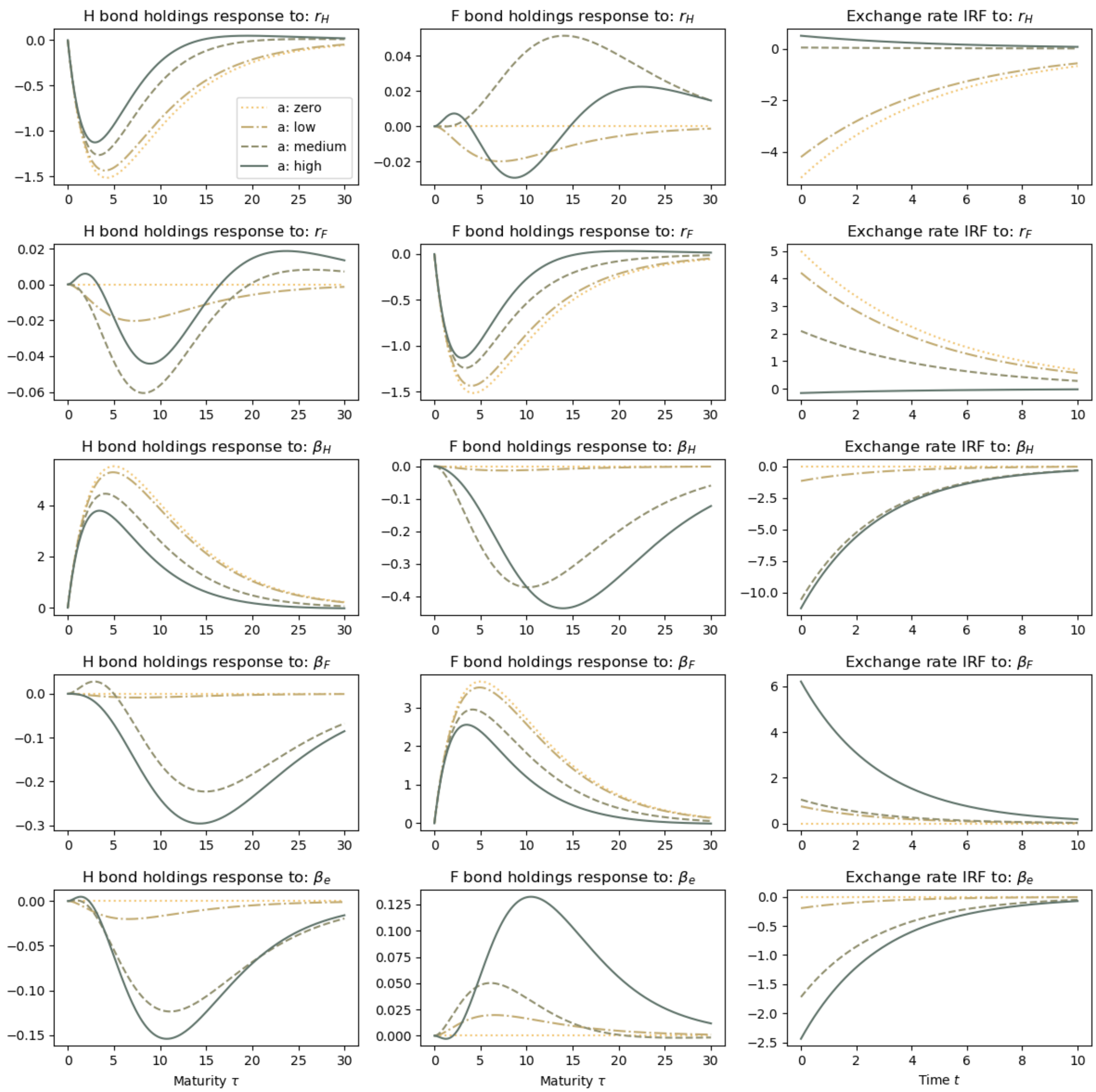

Figure 4: Response of Arbitrageur Allocations to Risk Factors

Notes: Plots of the change in arbitrageur holdings of Home bonds (left column) and Foreign bonds (middle column) across maturities τ. The final column plots the time series behavior of the exchange rate across time t. The first row is in response to an increase in the Home short rate r_{Ht}, while the second row plots the responses to an increase in the Foreign short rate r_{Ft}. The third and fourth rows plot the response to an increase the Home and Foreign demand factors β_{Ht}, β_{Ft}. Finally, the bottom row shows the response to the currency risk factor β_e. The dashed, dotted, and solid lines correspond to low, medium, and high levels of risk aversion, respectively.

Appendix

A Proofs

Proof of Proposition ??: We can rewrite (**??**) and (**??**) as

$$\left[\kappa_{rH} + a\alpha_e \left(\sigma_{rH}^2 A_{rHe}^2 + \sigma_{rF}^2 A_{rFe}^2\right)\right] A_{rHe} = 1, \tag{A.1}$$

$$\left[\kappa_{rF} + a\alpha_e \left(\sigma_{rH}^2 A_{rHe}^2 + \sigma_{rF}^2 A_{rFe}^2\right)\right] A_{rFe} = 1, \tag{A.2}$$

respectively. Since the terms in square brackets in (A.1) and (A.2) are positive, (A_{rHe}, A_{rFe}) are also positive. ∎

Proof of Corollary ??: When $a > 0$ and $\alpha_e > 0$, (**??**) implies that $\mu_{et} + r_{Ft} - r_{Ht}$ decreases in r_{Ht} and increases in r_{Ft} because $A_{rHe} = A_{rFe} > 0$. When $a = 0$ or $\alpha_e = 0$, (**??**) implies that $\mu_{et} + r_{Ft} - r_{Ht}$ is independent of (r_{Ht}, r_{Ft}). ∎

Proof of Proposition ??: We first determine the sign of $(\lambda_{rH}, \lambda_{rF})$. Equation (**??**) implies $\lambda_{rH} \leq 0$. Suppose, proceeding by contradiction, $\lambda_{rF} < 0$. Equations (**??**), (**??**) and the initial conditions $A_{rH}(0) = A_{rF}(0)$ imply $A'_{rH}(0) = 1 > 0$ and $A'_{rF}(0) = 0$. Moreover, differentiating (**??**), we find $A''_{rF}(0) = A'_{rH}(0)\lambda_{rF} < 0$. Hence, $A_{rH}(\tau) > 0$ and $A_{rF}(\tau) < 0$ for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{rH}(\tau') > 0 \text{ and } A_{rF}(\tau') < 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is finite, then (i) $A_{rH}(\tau_0) = 0$, $A'_{rH}(\tau_0) \leq 0$ and $A_{rF}(\tau_0) < 0$, or (ii) $A_{rH}(\tau_0) > 0$, $A_{rF}(\tau_0) = 0$ and $A'_{rF}(\tau_0) \geq 0$, or (iii) $A_{rH}(\tau_0) = A_{rF}(\tau_0) = 0$, $A'_{rH}(\tau_0) \leq 0$ and $A'_{rF}(\tau_0) \geq 0$. Cases (i) and (iii) yield a contradiction since (**??**), $A_{rH}(\tau_0) = 0$, $A_{rF}(\tau_0) \leq 0$ and $\lambda_{rF} < 0$ imply $A'_{rH}(\tau_0) \geq 1$. Case (ii) yields a contradiction since (**??**), $A_{rH}(\tau_0) > 0$, $A_{rF}(\tau_0) = 0$ and $\lambda_{rF} < 0$ imply $A'_{rF}(\tau_0) < 0$. Therefore, τ_0 is infinite, which means $A_{rH}(\tau) > 0$ and $A_{rF}(\tau) < 0$ for all $\tau > 0$. Equation (**??**) then implies $\lambda_{rF} \geq 0$, a contradiction. Hence, $\lambda_{rF} \geq 0$. Equations (**??**), $\lambda_{rH} \leq 0$ and $\lambda_{rF} \geq 0$ imply $A_{re} > 0$. To complete the proof, we distinguish the case $a > 0$ and $\alpha_e > 0$, and the case $a = 0$ or $\alpha_e = 0$.

Case $a > 0$ and $\alpha_e > 0$: If $\lambda_{rF} = 0$, then (**??**) and the initial condition $A_{rF}(0) = 0$ imply $A_{rF}(\tau) = 0$ for all τ. Since $A_{re} > 0$, (**??**) implies $\lambda_{rF} > 0$, a contradiction. Hence, $\lambda_{rF} > 0$.

We next show that $(A_{rH}(\tau), A_{rF}(\tau), A_{rH}(\tau) - A_{rF}(\tau))$ are positive. Since $A_{rH}(0) = A_{rF}(0) = A'_{rF}(0) = 0$, $A'_{rH}(0) = 1$ and $A''_{rF}(0) = A'_{rH}(0)\lambda_{rF} > 0$, $(A_{rH}(\tau), A_{rF}(\tau))$ are positive for τ close to zero. We define τ_0 by

$$\tau_0 \equiv \sup_{\tau}\{A_{rH}(\tau') > 0 \text{ and } A_{rF}(\tau') > 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ_0 is finite, then (i) $A_{rH}(\tau_0) = 0$, $A'_{rH}(\tau_0) \leq 0$ and $A_{rF}(\tau_0) > 0$, or (ii) $A_{rH}(\tau_0) > 0$, $A_{rF}(\tau_0) = 0$ and $A'_{rF}(\tau_0) \leq 0$, or (iii) $A_{rH}(\tau_0) = A_{rF}(\tau_0) = 0$, $A'_{rH}(\tau_0) \leq 0$ and $A'_{rF}(\tau_0) \leq 0$. Cases (i) and (iii) yield a contradiction since (**??**), $A_{rH}(\tau_0) = 0$, $A_{rF}(\tau_0) \geq 0$ and $\lambda_{rF} > 0$ imply $A'_{rH}(\tau_0) \geq 1$. Case (ii) yields a contradiction since (**??**), $A_{rH}(\tau_0) > 0$, $A_{rF}(\tau_0) = 0$ and $\lambda_{rF} > 0$ imply $A'_{rF}(\tau_0) > 0$. Therefore, τ_0 is infinite, which means $A_{rH}(\tau) > 0$ and $A_{rF}(\tau) > 0$ for all $\tau > 0$. Subtracting (**??**) from (**??**), and setting $\Delta A_r(\tau) \equiv A_{rH}(\tau) - A_{rF}(\tau)$, we find

$$\Delta A'_r(\tau) + \kappa_r \Delta A_r(\tau) - 1 = \Delta A_r(\tau)(\lambda_{rH} - \lambda_{rF}). \tag{A.3}$$

Equation (A.3) and the initial condition $\Delta A_r(0) = A_{rH}(0) - A_{rF}(0) = 0$ imply

$$\Delta A_r(\tau) = A_{rH}(\tau) - A_{rF}(\tau) = \frac{1 - e^{-[\kappa_r - (\lambda_{rH} - \lambda_{rF})]\tau}}{\kappa_r - (\lambda_{rH} - \lambda_{rF})}. \tag{A.4}$$

Hence, $A_{rH}(\tau) - A_{rF}(\tau) > 0$ for all $\tau > 0$.

We next show that $(A_{rH}(\tau), A_{rF}(\tau), A_{rH}(\tau) - A_{rF}(\tau))$ are increasing. Since $A'_{rH}(0) = 1$, $A'_{rF}(0) = 0$ and $A''_{rF}(0) > 0$, $(A'_{rH}(\tau), A'_{rF}(\tau))$ are positive for τ close to zero. We define τ'_0 by

$$\tau'_0 \equiv \sup_{\tau}\{A'_{rH}(\tau') > 0 \text{ and } A'_{rF}(\tau') > 0 \text{ for all } \tau' \in (0, \tau)\}.$$

If τ'_0 is finite, then (i) $A'_{rH}(\tau'_0) = 0$, $A''_{rH}(\tau'_0) \leq 0$ and $A'_{rF}(\tau'_0) > 0$, or (ii) $A'_{rH}(\tau'_0) > 0$, $A'_{rF}(\tau'_0) = 0$ and $A''_{rF}(\tau'_0) \leq 0$, or (iii) $A'_{rH}(\tau'_0) = A'_{rF}(\tau'_0) = 0$, $A''_{rH}(\tau'_0) \leq 0$ and $A''_{rF}(\tau'_0) \leq 0$. To show that Case (i) yields a contradiction, we use

$$A''_{rH}(\tau) + \kappa_r A'_{rH}(\tau) = A'_{rH}(\tau)\lambda_{rH} + A'_{rF}(\tau)\lambda_{rF}, \tag{A.5}$$

which follows by differentiating (**??**). Since $A'_{rH}(\tau'_0) = 0$, $A'_{rF}(\tau'_0) > 0$ and $\lambda_{rF} > 0$, (A.5) implies $A''_{rH}(\tau_0) > 0$. To show that Case (ii) yields a contradiction, we use

$$A''_{rF}(\tau) + \kappa_r A'_{rF}(\tau) = A'_{rH}(\tau)\lambda_{rF} + A'_{rF}(\tau)\lambda_{rH}, \tag{A.6}$$

which follows by differentiating (??). Since $A'_{rH}(\tau'_0) > 0$, $A'_{rF}(\tau'_0) = 0$ and $\lambda_{rF} > 0$, (A.6) implies $A''_{rF}(\tau_0) > 0$. Case (iii) yields a contradiction because the system of ODEs (A.5) and (A.6) in the functions $(A'_{rH}(\tau), A'_{rF}(\tau))$ with the initial condition $A'_{rH}(\tau'_0) = A'_{rF}(\tau'_0) = 0$ has a unique solution which must coincide with the zero solution. Hence, $(A_{rH}(\tau), A_{rF}(\tau))$ must be constants, and equal to (0,0) because $A_{rH}(0) = A_{rF}(0) = 0$. This is ruled out, however, from (??) and (??). Therefore, τ'_0 is infinite, which means $A'_{rH}(\tau) > 0$ and $A'_{rF}(\tau) > 0$ for all $\tau > 0$. Equation (A.4) implies $\Delta A'_r(\tau) = A'_{rH}(\tau) - A'_{rF}(\tau) > 0$ for all $\tau > 0$.

We next show that $\frac{A_{rF}(\tau)}{A_{rH}(\tau)}$ is increasing. The argument in the proof of Lemma 3 in Vayanos and Vila (2019) implies that the solution to the system of the two linear ODEs (??) and (??) with the initial conditions $A_{rH}(0) = A_{rF}(0) = 0$ is

$$A_{rH}(\tau) = \frac{1 - e^{-\nu_1 \tau}}{\nu_1} + \phi_{rH} \left(\frac{1 - e^{-\nu_2 \tau}}{\nu_2} - \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \right), \tag{A.7}$$

$$A_{rF}(\tau) = \phi_{rF} \left(\frac{1 - e^{-\nu_2 \tau}}{\nu_2} - \frac{1 - e^{-\nu_1 \tau}}{\nu_1} \right), \tag{A.8}$$

where (ν_1, ν_2) are the eigenvalues of the matrix

$$M = \begin{pmatrix} \kappa_r - \lambda_H & -\lambda_F \\ -\lambda_F & \kappa_r - \lambda_H \end{pmatrix},$$

and (ϕ_{rH}, ϕ_{rF}) are constant scalars. Since the matrix M is symmetric, the eigenvalues (ν_1, ν_2) are real. Without loss of generality, we assume $\nu_2 < \nu_1$. Since the function $(\nu, \tau) \longrightarrow \frac{1-e^{-\nu\tau}}{\nu}$ decreases in ν, the term in parenthesis in (A.8) is positive. The scalar ϕ_{rF} is also positive since $A_{rF}(\tau) > 0$. Since

$$\frac{A_{rH}(\tau)}{A_{rF}(\tau)} = \frac{\frac{1-e^{-\nu_1\tau}}{\nu_1}}{\phi_{rF}\left(\frac{1-e^{-\nu_2\tau}}{\nu_2} - \frac{1-e^{-\nu_1\tau}}{\nu_1}\right)} + \frac{\phi_{rH}}{\phi_{rF}} = \frac{1}{\phi_{rF}\left(\frac{\nu_1}{\nu_2}\frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}} - 1\right)} + \frac{\phi_{rH}}{\phi_{rF}},$$

and the function $(\nu_1, \nu_2, \tau) \longrightarrow \frac{1-e^{-\nu_2\tau}}{1-e^{-\nu_1\tau}}$ increases in τ because its derivative has the same sign as $\frac{e^{\nu_1\tau}-1}{\nu_1} - \frac{e^{\nu_2\tau}-1}{\nu_2}$, $\frac{A_{rH}(\tau)}{A_{rF}(\tau)}$ is decreasing, and hence $\frac{A_{rF}(\tau)}{A_{rH}(\tau)}$ is increasing.

Case $a = 0$ or $\alpha_e = 0$: If $\lambda_{rF} > 0$, then the argument in the previous case implies $A_{rH}(\tau) > 0$ and $A_{rF}(\tau) > 0$ for all $\tau > 0$. Since $a = 0$ or $\alpha_e = 0$, (??) implies $\lambda_{rF} \leq 0$, a contradiction. Hence, $\lambda_{rF} = 0$, which implies $A_{rF}(\tau) = 0$ for all τ. Equation (??) simplifies to

$$A'_{rH}(\tau) + \kappa_r A_{rH}(\tau) - 1 = A_{rH}(\tau)\lambda_{rH},$$

whose solution with the initial condition $A_{rH}(0) = 0$ is positive and increasing. Since $A_{rF}(\tau) = 0$, $A_{rH}(\tau) - A_{rF}(\tau)$ is also positive and increasing. ∎

Proof of Corollary ??: The derivative of $\mu_{jt}^{(\tau)} - r_{jt}$ with respect to r_{jt} is the left-hand (or right-hand) side of (**??**). When $a = 0$ or $\alpha(\tau) = \alpha_e = 0$ for all τ, (**??**) and (**??**) imply $\lambda_{rH} = \lambda_{rF} = 0$. Hence, the right-hand side of (**??**) is zero, and so is the left-hand side. To complete the proof of the corollary, we need to show that when $a > 0$ and either $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$ or $\alpha_e > 0$, the right-hand side of (**??**) is negative. We can write the right-hand side of (**??**) as

$$\begin{aligned} &A_{rH}(\tau)\lambda_{rH} + A_{rF}(\tau)\lambda_{rF} \\ &= A_{rH}(\tau)(\lambda_{rH} + \lambda_{rF}) - (A_{rH}(\tau) - A_{rF}(\tau))\lambda_{rF} \\ &= -a\sigma_r^2 A_{rH}(\tau) \int_0^T \alpha(\tau)\left(A_{rH}(\tau) + A_{rF}(\tau)\right)^2 d\tau - (A_{rH}(\tau) - A_{rF}(\tau))\lambda_{rF}, \end{aligned} \tag{A.9}$$

where the second step follows from (**??**) and (**??**). Since the function $A_{rH}(\tau)$ is positive and the function $A_{rF}(\tau)$ is non-negative, the first term in (A.9) is negative when $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$. Since, in addition, the function $A_{rH}(\tau) - A_{rF}(\tau)$ is positive, and $\lambda_{rF} > 0$ when $a > 0$ and $\alpha_e > 0$ (proof of Proposition **??**), the second term in (A.9) is negative under that condition. Therefore, when $a > 0$ and either $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$ or $\alpha_e > 0$, (A.9) is negative, and so is the right-hand side of (**??**). ∎

Proof of Corollary ??: The derivative of $\mu_{jt}^{(\tau)} - r_{jt}$ with respect to $r_{j't}$ is the left-hand (or right-hand) side of (**??**). When $a > 0$ and $\alpha_e > 0$, the left-hand side of (**??**) is positive because $A_{rF}(\tau)$ is positive and increasing in τ. When $a = 0$ or $\alpha_e = 0$, the left-hand side of (**??**) is zero because $A_{rF}(\tau) = 0$ for all τ. ∎

Proof of Corollary ??: Using (**??**)-(**??**) and (4.15), we can write $\mu_{Ft}^{(\tau)} - \mu_{Ht}^{(\tau)}$ as

$$[A_{rHe} + A_{rFH}(\tau) - A_{rHH}(\tau)]\lambda_{rHt} - [A_{rHe} + A_{rHF}(\tau) - A_{rFF}(\tau)]\lambda_{rFt}. \tag{A.10}$$

With symmetric countries, (A.10) becomes

$$[A_{re} + A_{rF}(\tau) - A_{rH}(\tau)]\left[(\lambda_{rH} - \lambda_{rF})(r_{Ht} - r_{Ft}) + \lambda_{rHC} - \lambda_{rFC}\right]. \tag{A.11}$$

Equations (**??**) and (**??**) imply

$$\lambda_{rH} - \lambda_{rF} = -a\sigma_r^2 \left[\int_0^T \alpha(\tau) \left(A_{rH}(\tau)^2 - A_{rF}(\tau)^2 \right) d\tau + 2\alpha_e A_{re}^2 \right]. \tag{A.12}$$

When $a = 0$ or $\alpha(\tau) = \alpha_e = 0$ for all τ, (A.12) implies $\lambda_{rH} - \lambda_{rF} = 0$, and hence (A.11) implies that $\mu_{Ft}^{(\tau)} - \mu_{Ht}^{(\tau)}$ is independent of $\{r_{jt}\}_{j=H,F}$. When $a > 0$ and either (i) $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$ or (ii) $\alpha_e > 0$, (A.12) implies $\lambda_{rH} - \lambda_{rF} < 0$. Equations (**??**) and (A.3) imply

$$A_{re} + A_{rF}(\tau) - A_{rH}(\tau) = \frac{e^{-[\kappa_r - (\lambda_{rH} - \lambda_{rF})]\tau}}{\kappa_r - (\lambda_{rH} - \lambda_{rF})}. \tag{A.13}$$

Since $\lambda_{rH} - \lambda_{rF} < 0$, $A_{re} + A_{rH}(\tau) - A_{rF}(\tau)$ is positive and decreases to zero when τ goes to infinity. Hence, (A.11) implies that $\mu_{Ft}^{(\tau)} - \mu_{Ht}^{(\tau)} + \mu_{et}$ decreases in r_{Ht} and increases in r_{Ft}, and that these effects decline with maturity and converge to zero when τ goes to infinity. ∎

Proof of Proposition 4.4: We start by computing the functions $(A_{\beta H}(\tau), A_{\beta F}(\tau))$ and the scalar $A_{\beta e}$ defined in Section **??**. For an unanticipated and one-off demand change, we set $\sigma_\beta = \sigma_{\beta e} = 0$ in [[**[INSERT REFERENCE]**]], and find

$$A'_{\beta H}(\tau) + \kappa_\beta A_{\beta H}(\tau) = A_{rH}(\tau)\lambda_{\beta H} + A_{rF}(\tau)\lambda_{\beta F}, \tag{A.14}$$

$$A'_{\beta F}(\tau) + \kappa_\beta A_{\beta F}(\tau) = A_{rH}(\tau)\lambda_{\beta F} + A_{rF}(\tau)\lambda_{\beta H}, \tag{A.15}$$

$$\kappa_\beta A_{\beta e} = A_{re}(\lambda_{\beta H} - \lambda_{\beta F}), \tag{A.16}$$

where

$$\lambda_{\beta H} = a\sigma_r^2 \left[\int_0^T [\theta(\tau) - \alpha(\tau) A_{\beta H}(\tau)] A_{rH}(\tau) d\tau - \int_0^T A_{\beta F}(\tau) A_{rF}(\tau) d\tau - \alpha_e A_{\beta e} A_{re} \right], \tag{A.17}$$

$$\lambda_{\beta F} = a\sigma_r^2 \left[\int_0^T [\theta(\tau) - \alpha(\tau) A_{\beta H}(\tau)] A_{rF}(\tau) d\tau - \int_0^T A_{\beta F}(\tau) A_{rH}(\tau) d\tau + \alpha_e A_{\beta e} A_{re} \right]. \tag{A.18}$$

Integrating (A.14) and (A.15) with the initial conditions $A_{\beta H}(\tau) = A_{\beta F}(\tau) = 0$, we find

$$A_{\beta H}(\tau) = \lambda_{\beta H} \int_0^\tau A_{rH}(\tau') e^{-\kappa_\beta(\tau - \tau')} d\tau' + \lambda_{\beta F} \int_0^\tau A_{rF}(\tau') e^{-\kappa_\beta(\tau - \tau')} d\tau', \tag{A.19}$$

$$A_{\beta F}(\tau) = \lambda_{\beta F} \int_0^\tau A_{rH}(\tau') e^{-\kappa_\beta(\tau - \tau')} d\tau' + \lambda_{\beta H} \int_0^\tau A_{rF}(\tau') e^{-\kappa_\beta(\tau - \tau')} d\tau'. \tag{A.20}$$

Substituting $\{A_{\beta j}(\tau)\}_{j=H,F}$ from (A.19) and (A.20) into (A.17) and (A.18), we find

$$\lambda_{\beta H} = a\sigma_r^2 \left[\int_0^T \theta(\tau) A_{rH}(\tau) d\tau - \lambda_{\beta H} \sum_{j=H,F} \int_0^T \alpha(\tau) \left(\int_0^\tau A_{rj}(\tau') e^{-\kappa_\beta(\tau-\tau')} d\tau' \right) A_{rj}(\tau) d\tau \right.$$
$$\left. -\lambda_{\beta F} \sum_{\substack{j,j'=H,F \\ j' \neq j}} \int_0^T \alpha(\tau) \left(\int_0^\tau A_{rj'}(\tau') e^{-\kappa_\beta(\tau-\tau')} d\tau' \right) A_{rj}(\tau) d\tau - \alpha_e A_{\beta e} A_{re} \right], \tag{A.21}$$

and

$$\lambda_{\beta F} = a\sigma_r^2 \left[\int_0^T \theta(\tau) A_{rH}(\tau) d\tau - \lambda_{\beta H} \sum_{\substack{j,j'=H,F \\ j' \neq j}} \int_0^T \alpha(\tau) \left(\int_0^\tau A_{rj'}(\tau') e^{-\kappa_\beta(\tau-\tau')} d\tau' \right) A_{rj}(\tau) d\tau \right.$$
$$\left. -\lambda_{\beta F} \sum_{j=H,F} \int_0^T \alpha(\tau) \left(\int_0^\tau A_{rj}(\tau') e^{-\kappa_\beta(\tau-\tau')} d\tau' \right) A_{rj}(\tau) d\tau + \alpha_e A_{\beta e} A_{re} \right], \tag{A.22}$$

respectively. Equations (A.16), (A.21) and (A.22) form a system of three linear equations in the three unknown scalars $(\lambda_{\beta H}, \lambda_{\beta F}, A_{\beta e})$.

Adding (A.21) and (A.22) yields an equation that involves $\lambda_{\beta H} + \lambda_{\beta F}$ as the only unknown. Solving that equation yields

$$\lambda_{\beta H}+\lambda_{\beta F} = \frac{a\sigma_r^2 \int_0^T \theta(\tau)(A_{rH}(\tau) + A_{rF}(\tau)) d\tau}{1 + a\sigma_r^2 \int_0^T \alpha(\tau) \left(\int_0^\tau (A_{rH}(\tau') + A_{rF}(\tau')) e^{-\kappa_\beta(\tau-\tau')} d\tau' \right) (A_{rH}(\tau) + A_{rF}(\tau)) d\tau}. \tag{A.23}$$

Subtracting (A.22) from (A.21), and using (A.16) to eliminate $A_{\beta e}$, yields an equation that involves $\lambda_{\beta H} - \lambda_{\beta F}$ as the only unknown. Solving that equation yields

$$\lambda_{\beta H}-\lambda_{\beta F} = \frac{a\sigma_r^2 \int_0^T \theta(\tau)(A_{rH}(\tau) - A_{rF}(\tau)) d\tau}{1 + a\sigma_r^2 \left[\int_0^T \alpha(\tau) \left(\int_0^\tau (A_{rH}(\tau') - A_{rF}(\tau')) e^{-\kappa_\beta(\tau-\tau')} d\tau' \right) (A_{rH}(\tau) - A_{rF}(\tau)) d\tau + \frac{2\alpha_e A_{re}^2}{\kappa_\beta} \right]}. \tag{A.24}$$

Since $(\theta(\tau), A_{rH}(\tau), A_{rH}(\tau) - A_{rF}(\tau))$ are positive and $A_{rF}(\tau)$ is non-negative, (A.23) and (A.24) imply that $(\lambda_{\beta H}+\lambda_{\beta F}, \lambda_{\beta H}-\lambda_{\beta F})$ are positive. Equation (A.16) implies $A_{\beta e} > 0$. Equation (A.19), written as

$$A_{\beta H}(\tau) = \lambda_{\beta H} \int_0^\tau (A_{rH}(\tau') - A_{rF}(\tau')) e^{-\kappa_\beta(\tau-\tau')} d\tau' + (\lambda_{\beta H} + \lambda_{\beta F}) \int_0^\tau A_{rF}(\tau') e^{-\kappa_\beta(\tau-\tau')} d\tau',$$

implies $A_{\beta H}(\tau) > 0$ for all τ provided that $\lambda_{\beta H} > 0$. The latter inequality holds since $(\lambda_{\beta H} + \lambda_{\beta F}, \lambda_{\beta H} - \lambda_{\beta F})$ are positive. Equation (A.20), written as

$$A_{\beta F}(\tau) = \lambda_{\beta F} \int_0^\tau (A_{rH}(\tau') - A_{rF}(\tau'))e^{-\kappa_\beta(\tau-\tau')}d\tau' + (\lambda_{\beta H} + \lambda_{\beta F}) \int_0^\tau A_{rF}(\tau')e^{-\kappa_\beta(\tau-\tau')}d\tau',$$

implies $A_{\beta F}(\tau) > 0$ for all τ provided that $\lambda_{\beta F} > 0$. The latter inequality holds when $\alpha(\tau)$ for all τ since (A.23) and (A.23) then imply $\lambda_{\beta H} + \lambda_{\beta F} > \lambda_{\beta H} - \lambda_{\beta F}$.

Since $A_{\beta H}(\tau) > 0$ for all τ, an increase in β_{jt} lowers $P_{jt}^{(\tau)}$. Since the derivative of $\mu_{jt}^{(\tau)} - r_{jt}$ with respect to β_{jt} is the right-hand (or left-hand) side of (A.14), which we can write as

$$A_{rH}(\tau)\lambda_{\beta H} + A_{rF}(\tau)\lambda_{\beta F} = [A_{rH}(\tau) - A_{rF}(\tau)]\lambda_{\beta H} + A_{rF}(\tau)(\lambda_{\beta H} + \lambda_{\beta F}) > 0,$$

an increase in β_{jt} raises $\mu_{jt}^{(\tau)} - r_{jt}$.

Since $A_{\beta e} > 0$, an increase in β_{jt} lowers e_t. Since the derivative of $\mu_{et} + \mu_{Ft}^{(\tau)} - \mu_{Ht}^{(\tau)}$ with respect to β_{jt} is

$$[A_{re} + A_{rF}(\tau) - A_{rH}(\tau)](\lambda_{\beta H} - \lambda_{\beta F})I(j),$$

where $I(j) = 1$ if $j = H$ and $I(j) = -1$ if $j = F$, an increase in β_{jt} raises $\mu_{et} + \mu_{Ft}^{(\tau)} - \mu_{Ht}^{(\tau)}$ when $j = H$ and lowers it when $j = F$. Since, in addition, $A_{re} + A_{rH}(\tau) - A_{rF}(\tau)$ is positive and decreases to zero when τ goes to infinity, the effects of β_{jt} on $\mu_{et} + \mu_{Ft}^{(\tau)} - \mu_{Ht}^{(\tau)}$ decline with maturity and converge to zero when τ goes to infinity.

Suppose finally that $\alpha(\tau) = 0$ for all τ. Since $A_{\beta F}(\tau) > 0$ for all τ, an increase in β_{jt} lowers $P_{j't}^{(\tau)}$. Since the derivative of $\mu_{j't}^{(\tau)} - r_{j't}$ with respect to β_{jt} is the right-hand (or left-hand) side of (A.15), which we can write as

$$A_{rH}(\tau)\lambda_{\beta F} + A_{rF}(\tau)\lambda_{\beta H} = [A_{rH}(\tau) - A_{rF}(\tau)]\lambda_{\beta F} + A_{rF}(\tau)(\lambda_{\beta H} + \lambda_{\beta F}),$$

an increase in β_{jt} raises $\mu_{j't}^{(\tau)} - r_{j't}$. ∎

B Numerical Solution Method

Define the following matrix

$$\begin{aligned}\mathbf{M} = \mathbf{\Gamma}^T - a\Bigg\{ &\int_0^T \left[-\alpha_H(\tau)\mathbf{A}_H(\tau) + \mathbf{\Theta}_H(\tau)\right] \mathbf{A}_H(\tau)^T \, \mathrm{d}\tau \\ &+ \int_0^T \left[-\alpha_F(\tau)\mathbf{A}_F(\tau) + \mathbf{\Theta}_F(\tau)\right] \mathbf{A}_F(\tau)^T \, \mathrm{d}\tau \\ &+ \left[-\alpha_e \mathbf{A}_e + \mathbf{\Theta}_e\right] \mathbf{A}_e^T \Bigg\} \mathbf{\Sigma}\end{aligned} \tag{B.1}$$

the following set of equations characterizing the solution to the affine functions $\mathbf{A}_j(\tau), \mathbf{A}_e$:

$$\mathbf{A}_j'(\tau) + \mathbf{M}\mathbf{A}_j(\tau) - \mathbf{e}_j = \mathbf{0} \tag{B.2}$$

$$\mathbf{M}\mathbf{A}_e - (\mathbf{e}_H - \mathbf{e}_F) = \mathbf{0} \tag{B.3}$$

with initial conditions $\mathbf{A}_j(0) = \mathbf{0}$.

Note that in general $\mathbf{M}$ depends on the solution to the affine functions. But treating $\mathbf{M}$ as fixed: if $\mathbf{M}$ is invertible and diagonalizable, with $\mathbf{M} = \mathbf{G}\mathbf{D}\mathbf{G}^{-1}$, then

$$\mathbf{A}_e = \mathbf{G}\mathbf{D}^{-1}\mathbf{G}^{-1}(\mathbf{e}_H - \mathbf{e}_F) \tag{B.4}$$

$$\begin{aligned}\mathbf{A}_j(\tau) &= \int_0^\tau \exp(-\mathbf{M}s) \, \mathrm{d}s \, \mathbf{e}_j \\ &= \mathbf{G} \int_0^\tau \exp(-\mathbf{D}s) \, \mathrm{d}s \, \mathbf{G}^{-1}\mathbf{e}_j \\ &= \mathbf{G}\mathbf{D}^{-1} \left[\mathbf{I} - \exp(-\mathbf{D}\tau)\right] \mathbf{G}^{-1}\mathbf{e}_j\end{aligned} \tag{B.5}$$

Hence we have

$$
\begin{aligned}
\mathbf{M} = \mathbf{\Gamma}^T - a\Bigg\{ \\
&\int_0^T \left[-\alpha_H(\tau)\left[\mathbf{G}\mathbf{D}^{-1}\left[\mathbf{I}-\exp(-\mathbf{D}\tau)\right]\mathbf{G}^{-1}\mathbf{e}_H\right] + \mathbf{\Theta}_H(\tau)\right] \\
&\qquad \left[\mathbf{G}\mathbf{D}^{-1}\left[\mathbf{I}-\exp(-\mathbf{D}\tau)\right]\mathbf{G}^{-1}\mathbf{e}_H\right]^T \mathrm{d}\tau \\
&+ \int_0^T \left[-\alpha_F(\tau)\left[\mathbf{G}\mathbf{D}^{-1}\left[\mathbf{I}-\exp(-\mathbf{D}\tau)\right]\mathbf{G}^{-1}\mathbf{e}_F\right] + \mathbf{\Theta}_F(\tau)\right] \\
&\qquad \left[\mathbf{G}\mathbf{D}^{-1}\left[\mathbf{I}-\exp(-\mathbf{D}\tau)\right]\mathbf{G}^{-1}\mathbf{e}_F\right]^T \mathrm{d}\tau \\
&+ \left[-\alpha_e\left[\mathbf{G}\mathbf{D}^{-1}\mathbf{G}^{-1}(\mathbf{e}_F-\mathbf{e}_H)\right] + \mathbf{\Theta}_e\right]\left[\mathbf{G}\mathbf{D}^{-1}\mathbf{G}^{-1}(\mathbf{e}_F-\mathbf{e}_H)\right]^T \\
\Bigg\}\mathbf{\Sigma}
\end{aligned}
\tag{B.6}
$$

Hence, if $\mathbf{y}_t \in \mathbb{R}^k$, then this is a fixed point problem in the $k \times k$ matrix $\mathbf{M}$, with k^2 equations and k^2 unknowns.

The solution depends on integrals of the functions $\alpha_j(\tau)$ and $\theta_j(\tau)$. In the numerical sections, we use the following parameterizations:

$$
\begin{aligned}
\alpha(\tau;\alpha_0,\alpha_1) &\equiv \alpha_0 \exp(-\alpha_1\tau) \\
\theta(\tau;\theta_0,\theta_1) &\equiv \theta_0\theta_1\tau \exp(-\theta_1\tau)
\end{aligned}
$$

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