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Variance (Non-)Causality:
A Bivariate GARCH-type model

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Abstract: In this paper I introduce a bivariate GARCH model with a structure that allows the estimation of a causality relation among variances and that solves some drawback of the existing methods. In particular the causality coefficients of the suggested formulation can be negative leading to detection of both the causality existence and direction. The stationarity restrictions are also provided. The causality is implemented using a multiplicative time dependent factor, while the conditional correlations are assumed to be constant. An application is also considered: it analyses the causality between the Nikkey225 and the SP100 stock market indexes.

Keywords: volatility, causality, multivariate GARCH

1 Introduction

In the last years there has been a growing interest in the study of the relation between variances and among prices and volumes, both from a theoretical point of view (as an example Blume, Easley and O'Hara, 1994) and from the empirical approach (see among others Karpoff, 1987). Most of the current empirical analysis considers different linear and non-linear specifications in order to verify and test the causal relations between the mentioned variables. However, most of them focus only the mean, restricting their attention on Granger's causality definition or to the study of a simultaneous relation.

In the last decades, with the emerging ARCH literature, different specifications of conditional heteroskedasticity have been taken into considerations and all of them allow for a deeper analysis on the causality topic. These efforts allow an adequate modelization of the relation between assets and market indices as well as among returns and volumes. These extensions can be thought both on one single asset case that in a much more general multivariate framework. The interest on multivariate heteroskedastic models maybe coupled with the necessity of an extension of the causality concept, which must considers the spillover effect among variances, and the in-mean GARCH effects. This will be the object of the present work.

After a brief review of the definitions of causality in mean and in variance (Engle and Granger (1986), then reviewed by Comte and Lieberman (2000)), this paper analyses in detail the different approaches that have been used up to this moment to identify the presence of causality among variances. There exist two different families of models: the first tries to explain the causality among variances through the correction of univariate models, it is worth mentioning the approach of Cheung and Ng (1996) who modify the univariate models observing the cross-correlations among the residuals and squared residuals; the second group of models is represented by the various multivariate GARCH formulations. All the up-to-date works in this field share a common problem: they can infer about the presence of causality but not on its direction, that is, given two assets A and B, assuming that there exist a causal relation among their variances, current model detect this relations but cannot tell us if an increase in the variance of A will imply an increase or a decrease in the variance of B. An interesting approach in this area is given by Hafner (2001), who provides a measure for causality in a multivariate GARCH framework. However, its study does not directly include the causal relation into a model. In this paper we will try to solve this problem in a multivariate framework, considering an extension of multivariate GARCH models that could be used to test both the existence of causality among variances and its direction. The suggested formulation will be then tested on an empirical basis, studying the relation between the Standard & Poor's 100 and the Nikkei 225 stock market indices.

The plan of the paper is as follow: in section 1 we review the current theoretical framework on causality both for the mean and the variance while section 2 focuses on different alternative models to verify second order causality. Section 3 is devoted to the case study, section 4 will concludes.

2 Causality in mean and in variance

We start introducing some notation and recalling well known concepts.

Define X_t as the n-dimensional set of variables of interest at time t, this set can be partitioned into $X_t = \{X_{1,t}\ ,\ X_{2,t}\}$, that have dimension n_1 and n_2, respectively. Moreover, denote by $I(X_t) = I_t(X) = I(X)$ the information set (a sigma algebra generated by the variable of interest or in general a Hilbert space) for the whole variables and with $I(X_{1,t}) = I_t(X_1) = I(X_1)$ the information set given by the first partition (similarly for X_2).

Proposition 1 *Granger (1980): X_2 does not cause X_1 in Granger sense, if and only if $E_t[X_{1,t}|I_{t-1}(X)] = E_t[X_{1,t}|I_{t-1}(X_1)]$.*

This is denoted by $X_2 \overset{G}{\nrightarrow} X_1$

The violation of the previous condition is normally referred as causality in the mean (usually defined also as Granger causality). However, a contemporaneous bi-directional relation is not included in the Granger definition, in this case Sims (1972) stated:

Proposition 2 *Sims (1972): there is no bidirectional causality between X_1 and X_2 if and only if $Cov[X_{1,t} - E_t[X_{1,t}|I_{t-1}(X)]\ ,\ X_{2,t} - E_t[X_{2,t}|I_{t-1}(X)]] = 0$.*

This is denoted by $X_2 \nleftrightarrow X_1$

However, an extension to these concepts is needed in dealing with time varying conditional variances and causal relation among these quantities. This topic has been analysed by Engle, Granger and Robins (1986) who provided the following proposition:

Proposition 3 *Engle, Granger and Robins (1986): X_2 does not second order cause X_1 in Granger sense, if and only if $E_t\left[(X_{1,t} - E_t[X_{1,t}|I_{t-1}(X)])^2 |I_{t-1}(X)\right] = E_t\left[(X_{1,t} - E_t[X_{1,t}|I_{t-1}(X)])^2 |I_{t-1}(X_1)\right]$.*

This is denoted by $X_2 \overset{G^2}{\nrightarrow} X_1$

The definition of second order non-causality does not presume any causal relation in the mean, however, this is not precisely a non-causality relation among variances. Note that the rigth hand expected value is not a conditional variance, the conditioning being different between the first and second order moments. Starting from this observation Comte and Lieberman (2000) gave a different definition:

Proposition 4 *Comte and Lieberman (2000): X_2 does not cause X_1 in variance, if and only if*

$V_t\left[X_{1,t}|I_{t-1}(X)\right] = V_t\left[X_{1,t}|I_{t-1}(X_1)\right]$.

This is denoted by $X_2 \overset{G_V}{\nrightarrow} X_1$

Note that the difference between second order non-causality and variance non-causality is only in the conditioning information sets. The two authors gave also the following relation:

Remark 5 : $X_2 \overset{G}{\nrightarrow} X_1 + X_2 \overset{G^2}{\nrightarrow} X_1 \Longleftrightarrow X_2 \overset{G_V}{\nrightarrow} X_1$

Proof. By substitution and with a direct application of the law of iterated expectation. ■

This last remark allows to note that non-causality in the variance exist if and only if there exist non-causality in the mean, moreover first and second order non-causality may combine in all possible pairs. A sequential testing scheme is therefore possible, check at first causality in the mean, then if there is no relation, we can test for second order non causality. If and only if both tests lead to a no-relation result we can conclude that there is non-causality among variances.

Researchers are also interested in verifying the previous relations from an empirical point of view. This is possible analysing the restrictions implied by first and second order non-causality in a very general framework, using as a reference model the VARMA-GARCH.

The benchmark model can be represented as follow: given the variables of interest X_t consider the following VARMA(p,q)-GARCH($\overline{p}$,$\overline{q}$) model

$$\begin{aligned} X_t &= A(L) X_t + B(L) E_t \qquad E_t \tilde{} iid(\mathbf{0}, H_t) \\ H_t &= \boldsymbol{\omega} + C(L) H_t + D(L)[E_t E_t'] \end{aligned} \tag{1}$$

where $A(L) = \sum_{i=1}^{p} A_i L^i$, $B(L) = \sum_{i=1}^{q} B_i L^i$, $C(L) = \sum_{i=1}^{\overline{p}} C_i L^i$, $D(L) = \sum_{i=1}^{\overline{q}} D_i L^i$, and $\boldsymbol{\omega}$, A_i, B_i, C_i, D_i, are all square matrices of dimension n, while p, q, $\overline{p}$ and $\overline{q}$ are interager numbers. Assume also that the model is stationary and invertible. A similar approach was also used by Comte and Lieberman (2000) and Boudjellaba, Dufour and Roy (1992 and 1994) in giving a set of parametric restrictions and tests for causality, we recall in the following their results. For the purpose of testing first order noncausality it is convenient to transform the VARMA(p,q) into its VAR(∞) representation (given the invertibility assumption)

$$\begin{aligned} [B(L)]^{-1}[1 - A(L)] X_t &= W(L) X_t = E_t \\ where\ W(L) &= \sum_{i=0}^{\infty} W_i L^i \end{aligned} \tag{2}$$

then $X_2 \overset{G}{\nrightarrow} X_1$ if and only if $[W_i]_{12} = 0$ for all lags i, that is the coefficients that link the variables included in the two partitions on X, are identically equal to zero (Boudjellaba et al. 1992). As noted by Comte and Lieberman (2000) there will always be second order noncausality dropping the GARCH part of the model and considering a simple VARMA process with constant variance-covariance matrix. For the GARCH part, similarly to the VARMA case, in a first step the model is converted into its ARCH(∞) representation:

$$H_t = [1 - C(L)]^{-1} \boldsymbol{\omega} - [1 - C(L)]^{-1} D(L)[E_t E_t'] = \overline{\boldsymbol{\omega}} + U(L)[E_t E_t'] \tag{3}$$
$$where\ U(L) = \sum_{i=0}^{\infty} U_i L^i$$

A causality restriction similar to the one of VARMA models holds here: $X_2 \overset{G^2}{\nrightarrow} X_1$if and only if $[U_i]_{12} = 0$ for all lags i. In this framework noncausality of the first and of the second order can independently exist, however, variance noncausality is associated with their contemporaneous existence. An example of a VARMA(1,1)-GARCH(1,1) case is included in the paper of Comte and Lieberman (2000). None of the previous cited papers deal with the case of a VARMA-GARCH-M model, where an additional source of causality is added: the one of conditional variances on the mean of the process. In this case how are modified the conditions for first and second order noncausality? or more precisely, are the implications of remark (5) still valid or do they need an update? Consider the following extension of equation (1):

$$\begin{aligned} X_t &= A(L)X_t + B(L)E_t + GVech(H_t) \qquad E_t\tilde{\ }iid(\mathbf{0}, H_t) \\ H_t &= \boldsymbol{\omega} + C(L)H_t + D(L)[E_tE_t'] \end{aligned} \tag{4}$$

where the operator $Vech$ stacks the lower triangular element of H_t, therefore the vector $Vech(H_t)$ is of dimension $r = n(n+1)/2$ and G of dimension $n \times r$. Consider, at first, second order noncausality: given the previous definition, that does not presume any causal or noncausality relation on the mean, the restrictions are the same as in the previous case, that is, rewriting the model in its ARCH(∞) representation, again $X_2 \overset{G^2}{\nrightarrow} X_1$ if and only if $[U_i]_{12} = 0$. The difference is in the first order noncausality: there is now dependence of returns from variance-covariance matrix, in principle two cases are identified, depending on the existence of second order noncausality.

Remark 6 *In a stationary and invertible VARMA-GARCH-M model, with* $\left|[G]_{i,j}\right| \geq 0$ $i = 1, ...n_1$ $j = 1, ...r$, *and at least one coefficient for which strict inequality hold, if there is second order causality there is also first order causality.*

Proof. Consider the condition for noncausality in the mean: $E_t[X_{1,t}|I_{t-1}(X)] = E_t[X_{1,t}|I_{t-1}(X_1)]$, substituting $X_{1,t}$ with its expression from (4)

$$E_t\left[[A(L)X_t + B(L)E_t + GVech(H_t)]_{1,1}|I_{t-1}(X)\right]$$

the first two components maybe measurable with respect to the information set restricted to the past of X_1 but this is not true for H_1 where, given the presence of second order noncausality, H_1 is measurable only on the whole information set. This is true if at least one of the coefficients linking the variables in X_1 with the variance-covariance matrix is different from zero, in the opposite case we could write a restricted VARMA model for X_1 without the in-mean component, returning to a situation similar to the VARMA-GARCH approach. ■

Assume now that there is second order noncausality, in this case:

Remark 7 *Consider a stationary and invertible VARMA-GARCH-M model, where, for the sake of exposition, the in-mean component is reparameterized as follows*

$$Vech\left(H_t\right) \rightarrow Vech\left([H_t]^{T2}\right)$$

and $[\cdot]^{T2}$ *represent a transpose with respect to the secondary diagonal of a matrix. In this model,* $X_2 \overset{G}{\nrightarrow} X_1$ *if the following conditions are satisfied: i)* $[W_i]_{12} = 0$ *for all lags i; ii)* $[Z_l]_{i,j} = 0$, $i = 1, ...n_1$, $j = 1, ...(r - n_1(n_1 + 1)/2)$, *for all lags l. Where* $W(L)$ *is defined as in (2) and* $Z(L)$ *is defined as* $[B(L)]^{-1} G = \sum_{i=0}^{\infty} Z_i L^i = Z(L)$, *is a sequence of matrices of dimension* $n \times r$

Proof. Again referring to the measurability with respect to the information sets, violating one of the previous condition will imply non-measurability with respect to the restricted information set of X_1, i) concern with dependance from the variables included in X_2 while ii) is devoted to the dependance of X_1 only from its own variance covariance matrix. The reordering allow us to concentrate the element of the variance covariance matrix of X_1 at the end of the vector of in-mean effects. ■

We can now summarize our findings extending remark (5) in the following way:

Remark 8 *In a stationary and invertible VARMA-GARCH-M the following relations hold* *i)* $X_2 \overset{G}{\nrightarrow} X_1 + X_2 \overset{G^2}{\nrightarrow} X_1 \Longleftrightarrow X_2 \overset{G_V}{\nrightarrow} X_1$

ii) $X_2 \overset{G^2}{\rightarrow} X_1 \Longrightarrow X_2 \overset{G}{\rightarrow} X_1$ *if* $\left|[G]_{i,j}\right| \geq 0 \;\; i = 1, ...n_1 \;\; j = 1, ...r$ *with* $\left|[G]_{i,j}\right| > 0$ *for at least one* (i, j)

iii) $X_2 \overset{G}{\rightarrow} X_1 \nLeftrightarrow X_2 \overset{G^2}{\rightarrow} X_1$

iv) $X_2 \overset{G^2}{\nrightarrow} X_1 \nRightarrow X_2 \overset{G}{\nrightarrow} X_1$

This remark provides a general set of resctictions and relations for the existence of mean and variance causality in a general VARMA-GARCH-M model.

3 Multivariate analysis and causality

Different works introduced methods for the detection of second order causality without including it directly into a multivariate model; among the others we mention the approach of Cheung and Ng (1996), which considers the cross-correlation among univariate model residuals and the measures of variance causality of Hafner (2001). A second part of the literature focused on the GARCH-type models, a direct and intuitive structure within testing variance causality restrictions. In the previous section no assumption was posed on the GARCH structure, even if in the literature there are different parameterisations, which includes the BEKK and Vech of Engle and Kroner (1995), and the Conditional Correlation of Bollerslev (1990) two of the most used and known.

3.1 Traditional models: BEKK, Vech and their drawbacks

Most empirical works dealing with second order causality considered multivariate GARCH in the BEKK and Vech representations. Engle and Kroner (1995) showed that the two formulations can be derived one from the other with an adequate reparameterization. For the moment we assume that the specification chosen is the BEKK, represented as:

$$H_t = \boldsymbol{\omega} + \sum_{i=1}^{p} C_i H_{t-i} C_i' + \sum_{j=1}^{q} D_j E_{t-j} E_{t-j}' D_j'$$

where C_i and D_j are $n \times n$ matrices and $\boldsymbol{\omega}$ is a symmetric positive definite $n \times x$ matrix. The existence of any causal relation among the variances and covariances included in H_t imply that (at least some of) the off-diagonal coefficients of C_i and D_j are different from zero. If all the parameter matrices are diagonal the model collapses into a particular case in which all conditional variances and covariances follow a GARCH(p,q) process. The most important feature of the BEKK model is that it can explain causality relation among both variances and covariances. Moreover, various nonlinear relations can be imposed with a limited number of parameters which are also free of any constraints since are implemented in a quadratic form. However, the number of parameters greatly increases with the number of variables, creating a series of problems on convergence of the estimation algorithm, reliability of the estimates and last but not least CPU time.

Within this framework the causality relation among variances can be tested with a set of zero restrictions on parameters. In this case, an additional remark is also needed: the model postulate that causality among variances (excluding covariances for the moment) act only in one direction, that is the positive one. Consider as an example a shock to a bivariate system (x_t,y_t), that will affect only the second variable y_t, causing an increase in its variance $\sigma^2_{y,t}$. As a consequence, it will necessarily cause an increase in the variance of the first variable $\sigma^2_{x,t}$. The possibility that the first variance $\sigma^2_{x,t}$ decreases is in principle not contemplated. This particular situation might be very difficult to realize in financial markets, but we cannot a priori exclude it.

The quadratic parameter structure implies also another problem: only combinations of parameters are responsible for the non-linear relations between variables, we cannot therefore directly interpret the estimates of a BEKK formulation. In addition, significativity tests run on the BEKK parameter are no more valid in testing the significance in the single equation GARCH.

The Vech formulation can be represented as

$$Vech\left(H_t\right) = \omega + \sum_{i=1}^{p} C_i Vech\left(H_{t-1}\right) + \sum_{j=1}^{q} D_j Vech\left(E_{t-1}E'_{t-1}\right) \tag{5}$$

where the $Vech\left(M\right)$ operators stacks the columns of the lower triangular matrix M and all parameter matrices are of dimension $n(n+1)/2$. This representation allows, as the BEKK, a shock transmission among variances and covariances, but the parameters increase with respect to the BEKK formulation. Moreover, we face an additional problem, parameters must be bounded in such a way that they are all positive and guarantee the positive definiteness of the variance covariance matrix.

3.2 The CCC and DCC GARCH

Another commonly used multivariate GARCH model is the Constant Conditional Correlation (CCC). This parameterization can be represented as:

$$H_t = Q_t R Q'_t$$

$$R = \begin{bmatrix} 1 & \cdots & \rho_{i1} & \cdots & \rho_{n1} \\ \vdots & \ddots & \vdots & & \vdots \\ \rho_{i1} & \cdots & 1 & \cdots & \rho_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ \rho_{n1} & \cdots & \rho_{in} & \cdots & 1 \end{bmatrix}$$

$$Q = [\sigma_{1,t}\ \sigma_{2,t}...\sigma_{n,t}]'$$

$$\sigma^2_{j,t}\tilde{}GARCH(p,q)$$

The CCC-GARCH can be easily generalised to allow for variance causality. We can therefore model the pure variance process in the following way

$$\begin{bmatrix} \sigma^2_{1,t} \\ \vdots \\ \sigma^2_{n,t} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} + \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{bmatrix} \begin{bmatrix} \varepsilon^2_{1,t-1} \\ \vdots \\ \varepsilon^2_{n,t-1} \end{bmatrix} \tag{6}$$

$$+ \begin{bmatrix} \beta_{1,1} & \cdots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,n} \end{bmatrix} \begin{bmatrix} \sigma^2_{1,t-1} \\ \vdots \\ \sigma^2_{n,t-1} \end{bmatrix} \tag{7}$$

where parameters will have to be bounded above zero to ensure positivity of variances. Stationarity of the process is then affected by this structure, and we need to impose additional restrictions. The model con be reformulated in a companion VARMA representation: define $\nu_{i,t} = \varepsilon^2_{i,t-1} - \sigma^2_{i,t}$ for $i = 1, 2...n$ we can write

$$\begin{bmatrix} \varepsilon^2_{1,t} \\ \vdots \\ \varepsilon^2_{n,t} \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \vdots \\ \omega_n \end{bmatrix} + \left(\begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{bmatrix} \right. \tag{8}$$

$$\left. + \begin{bmatrix} \beta_{1,1} & \cdots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,n} \end{bmatrix} \right) \begin{bmatrix} \varepsilon^2_{1,t-1} \\ \vdots \\ \varepsilon^2_{n,t-1} \end{bmatrix} +$$

$$- \begin{bmatrix} \beta_{1,1} & \cdots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,n} \end{bmatrix} \begin{bmatrix} \nu_{1,t-1} \\ \vdots \\ \nu_{n,t-1} \end{bmatrix} + \begin{bmatrix} \nu_{1,t} \\ \vdots \\ \nu_{n,t} \end{bmatrix}$$

and recalling that $\nu_{i,t}$ is a martingale difference sequence we can state that the stationarity conditions for (6) are exactly the same of a VARMA(1,1) model, we need that all the eigenvalues of the matrices be outside the unit circle.

The CCC model, as the BEKK or the Vech, detects only positive causality and implies an elevate number of parameters.

A recent extension of the CCC model is due to Engle and Sheppard (2001): they suggest to introduce a limited dynamic structure in the correlations (then the name Dynamic Conditional Correlations or DCC). This approach faces the same problems as the previous one and, in addition, the assumption of a common dynamic on all correlations seems questionable. It clearly provides an interesting framework adding a limited number of parameters but, if we model, for example, the variances of a group of stocks and a group of exchange rate products, we can presume a different dynamics among the correlations of these two groups of intruments.

3.3 A new approach: the EC-GARCH

A common drawback of traditional GARCH models applied to second order causality testing is the elevate number of parameters, which is also normally coupled with the necessity of constraints ensuring positivity of conditional variances. These points lead to complex numerical evaluations in the estimation of parameters that transfer in increasing and often unrealistic CPU time. This influence the researchers in the choice of simple models that do not consider causality among variances, think of the traditional CCC-GARCH models applied in finance, an area where timeliness is fundamental. In this paper we suggest an alternative methodology that can be used in testing for the presence of second order causality. We try to solve the problem imposed by the constraint on parameters via a multiplicative effect between variances. This extension has been suggested by a group of papers dealing with switching GARCH and threshold models; among these it is worth mentioning Hamilton (1994) and McAleer (2001). The model is mainly derived from the ideas of the first author that proposed a switching structure for ARCH models in a simple way, pre-multiplying the ARCH equation by a state dependant factor. In its framework the state variable was unobserved and driven by a Markov chain.

Differently, our approach focuses on an observable factor influencing the causality relation: assume that there exist a causal relation among variances and that the correlations are modelled with a CCC structure. The suggested bivariate GARCH can be represented as:

$$\begin{aligned} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} &= \begin{bmatrix} \mu_{1,t}\left(I^{t-1}\right) \\ \mu_{2,t}\left(I^{t-1}\right) \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} \\ & \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix} \tilde{}\, iid\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_{1,t}^2 & \rho\sigma_{1,t}\sigma_{2,t} \\ \rho\sigma_{1,t}\sigma_{2,t} & \sigma_{2,t}^2 \end{bmatrix} \right) \end{aligned} \tag{9}$$

where the mean dynamic is not specified and time dependence is based on the information set up to time $t-1$, $\left(I^{t-1}\right)$, therefore we do not rule out GARCH-in-mean effects. The variances are represented as

$$\sigma_{1,t}^2 = \exp\left[f_1\left(I^{t-1}\right)\right]\left[\omega_1 + \sum_{j=1}^{p}\beta_{1,j}\sigma_{1,t-j}^2 + \sum_{j=1}^{q}\alpha_{1,j}\varepsilon_{1,t-j}^2\right] \tag{10}$$

$$\sigma_{2,t}^2 = \exp\left[f_2\left(I^{t-1}\right)\right]\left[\omega_2 + \sum_{j=1}^{p}\beta_{2,j}\sigma_{2,t-j}^2 + \sum_{j=1}^{q}\alpha_{2,j}\varepsilon_{2,t-j}^2\right] \tag{11}$$

where a modified standard GARCH structure is included (in brackets) but there are no constraints to consider any possible GARCH formulation such as FI-GARCH, the leverage GARCH of Glosten, Jagannathan and Runkle (1993) or the asymmetric power ARCH of Ding, Granger and Engle (1993) as well as the structure of the two variances is not constrained to be identical. The causal relation is modeled by the functions $f_1\left(I^{t-1}\right)$ and $f_2\left(I^{t-1}\right)$ which depend on the information sets up to time $t-1$. We suggest the following specification

$$f_i\left(I^{t-1}\right) = \gamma_i z_{j,t-1}^2 \qquad i,j = 1,2 \quad i \neq j \tag{12}$$

Where the squared standardised residuals are used as an indicator of causality between the two variables. In fact, the squared residuals can be thought as the "true" variance shocks or innovations. In this setup the multiplicative effect, driven by the parameters γ_i, allow for positive and negative causality, in the sense that an increase in the variance of the second series imply an increase in the variance of the first series only if the function $f_i(\cdot)$ is greater than 1 (the parameter greater than zero), otherwise the variance decreases. Non causality is then associated with a zero parameter. Moreover, parameters need not to be constrained given the exponential formulation. Therefore, a significativity test on the parameters γ_i will indicate the existence or not of a causal relation between the variances of the two series, while its sign can be interpreted as the causality direction. The model is labelled Exponential Causality GARCH (EC-GARCH) by the structure of the causal relation among variances. Moreover, the stationarity of the model can also be verified. The following theorem holds:

Theorem 9 *The model represented by equations (9)-(12) is stationary if* $\omega > 0$, $\alpha_{i,j}, \beta_{i,l} > 0$ $j = 1...p$, $l = 1...q$

$$\gamma_i < 1/2 \quad and \quad (1-2\gamma_i)^{1/2} - \sum_{j=1}^{p} \alpha_{i,j} - \sum_{l=1}^{q} \beta_{i,l} > 0 \qquad i = 1,2 \tag{13}$$

under the assumption of conditional normality of the standardised residuals $z_{i,t-1}$, $i = 1,2$

Proof. The stationarity of this model will be proofed serching the constraints which bound the unconditional variance. The fundamental point is the computation of the unconditional variance itself

$$E\left(\sigma_{1,t}^2\right) = E\left[\exp\left(\gamma_1 z_{2,t-1}^2\right)\left(\omega_1 + \beta_1 \sigma_{1,t-1}^2 + \alpha_1 \varepsilon_{1,t-1}^2\right)\right] \tag{14a}$$

where a simple GARCH(1,1) has been considered but in general any GARCH structure can be, in principle, employed. A similar equation, not necessarily with the same GARCH structure, can be written for the second variable of interest. The analysis is now concentrated only on one variable. First of all, it is worth recalling that $z_{2,t-1}^2$ is independent both from $\sigma_{1,t-1}^2$ and $\varepsilon_{1,t-1}^2$ (and any other past values of these two quantities) since $\sigma_{1,t-1}^2$ depends on the information set up to time $t-2$ while $\varepsilon_{1,t-1}^2$ depends on $z_{1,t-1}^2$ and on the past values of z_2^2 up to time $t-2$. Given these relations the following equality can be obtained

$$E\left(\sigma_{1,t}^2\right) = E\left[\exp\left(\gamma_1 z_{2,t-1}^2\right)\right] E\left[\omega_1 + \beta_1 \sigma_{1,t-1}^2 + \alpha_1 \varepsilon_{1,t-1}^2\right] \tag{15}$$

and recognizing in the first expected value the moment generating function of the variable $z_{2,t-1}^2$ we are almost done. One additional assumption is only needed, it concerns the distribution of $z_{2,t-1}^2$, for the simplicity of the exposition a normal standardized residual distribution is considered. Therefore, recalling that the squared standardized residuals are distributed as a χ^2 with one degree of freedom, and that the moment generating function of the $\chi^2(k)$ distribution is

$$mgf(t) = \left[\frac{1}{1-2t}\right]^{k/2} \qquad t < 1/2$$

the equation (15) can be rewritten as

$$E\left(\sigma_{1,t}^{2}\right)=\left[\frac{1}{\left(1-2\gamma_{1}\right)^{1/2}}\right]\left(\omega_{1}+\beta_{1}E\left[\sigma_{1,t-1}^{2}\right]+\alpha_{1}E\left[\sigma_{1,t-1}^{2}\right]\right)\qquad\gamma_{1}<1/2 \tag{16}$$

Where the independence between $\sigma_{1,t-1}^{2}$ and $z_{1,t-1}^{2}$, was also considered. Therefore

$$\begin{aligned}E\left(\sigma_{1,t}^{2}\right) &= \frac{\omega_{1}}{\left(1-2\gamma_{1}\right)^{1/2}}+\frac{\alpha_{1}+\beta_{1}}{\left(1-2\gamma_{1}\right)^{1/2}}E\left[\sigma_{1,t-1}^{2}\right] \qquad (17)\\ E\left(\sigma_{1,t}^{2}\right) &= \frac{\omega_{1}}{\left(1-2\gamma_{1}\right)^{1/2}-\alpha_{1}-\beta_{1}}\end{aligned}$$

and the stationarity conditions are These are similar to the one of the GARCH(1,1), however, much more narrower, at least when γ_1 is positive and close to its limit. The derivation of the stationarity restrictions for a GARCH(p,q) is straightforward, leading to the following inequality

$$\gamma_{1}<1/2\quad\text{and}\quad\left(1-2\gamma_{1}\right)^{1/2}-\sum_{j=1}^{p}\alpha_{j}-\sum_{i=1}^{q}\beta_{i}>0$$

■

Remark 10 *When $\gamma_1 = 0$ the model collapse on a GARCH(1,1) with its well know stationarity restriction.*

Remark 11 *The stationarity can be easily verified also for the IGARCH(1,1) case, and results to be $\gamma_1 < 0$, therefore the interpretation of the sign of the coefficient is no more possible. Therefore, the IGARCH(1,1) parameterization should be avoided in this framework since it allows only a detection of the causality existence.*

One may object that the squared residuals $\varepsilon_{i,t-1}^{2}$ or the conditional variances $\sigma_{i,t}^{2}$ should be used instead of the squared standardised residuals. We discarded the first solution because a problem of stationarity arises: while the conditional variances are bounded the same in not true for unconditional variances who can easily diverge to infinity. We consider then the contemporaneous conditional variance since it is measurable with respect to the information set up to time $t-1$, while the lagged conditional variances are measurable up to time $t-2$. In addition, the contemporaneous and direct causality among conditional variances create a different problem: the system cannot be transformed in a simple reduced form causing a drawback in the implementation of the model: a numerical evaluation algorithm is needed to calculate each of the two conditional variances for any time value t. A simple solution could have been the use of lagged conditional variances, however, we stress that, in this last hypothesis, the conditioning will not be on the information set up to $t-1$ while up to $t-2$, and therefore it has not been pursued.

A final note: while the mean residuals of equation (9) have the conditional variance-covariance matrix represented in brackets, the equations (10) and (11) do not represent conditional variances of the mean residuals, given the dependance of each equation from an additional source of noise.

The model can be simply extended to a dimension higher than 2, allowing a CCC structure across all covariances and extending (12) to

$$f_i\left(I^{t-1}\right)=\sum_{\substack{j=1\\ j\neq i}}^{n}\gamma_{i,j}z_{j,t-1}^2 \tag{18}$$

We must note that the causality test is now on an elevate number of parameters. Therefore, this structure should be implemented on small systems, such as the volume-volatility study considered in the following application, or to the causal relation existent across financial markets such as Europe, New York and Tokyo, in the setup suggested by Pojarlev and Polasek (2000). Additional possible bivariate (and then multivariate) extensions can be obtaining modifying the causality function as follows

$$\begin{aligned}
f_1\left(I^{t-1}\right) &= \gamma_1 z_{2,t-1} \\
f_1\left(I^{t-1}\right) &= \gamma_{1,1} z_{2,t-1} + \gamma_{1,2}\left(\left|z_{2,t-1}\right| - E\left|z_{2,t-1}\right|\right) \\
f_1\left(I^{t-1}\right) &= \gamma_{1,1} z_{2,t-1} + \gamma_{1,2} I_{z_{2,t-1}<0} z_{2,t-1} \\
&\vdots
\end{aligned}$$

where only some of the possible formulae are reported, mirroring the EGARCH and the GJR-GARCH. However, the stationarity conditions should be recalculated for all these specifications, and it is not sure that a close formula exist. A deeper analysis of stationarity of the model under different specifications of the causality function and with different assumptions on the distribution of standardized residuals will be the object of future researches.

Finally, another extension regards possible alternative functions $f_i(\cdot)$, we restricted the attention to the exponential specification, however, a logistic function can be easily introduced. In that case, the stationarity conditions should be evaluated numerically.

4 Variance causality between SP100 and Nikkey225

This section focus on an application of the EC-GARCH model to the Standard & Poor's and the Nikkey indices. These two variables have been chosen because they do not averlap and we can in principle expect a causality relation from Standard & Poor's to the Nikkey, at least in the mean. Our purpose is to verify and measure the causality relation among the conditional variances of these two stock market indices.

The period covered starts in 1988 up to mid-february of 2003, data have been obtained from Datastream and are collected at a daily frequency. The original data are at first filtered from holydays and closed market days. Since these events are not common in the two markets we simply deleted the corresponding days in both series as normal practice in the contagion analysis. The obtained series are graphed in the following figures, both for their level and logarithmic differences.

Figure 1: Nikkey225

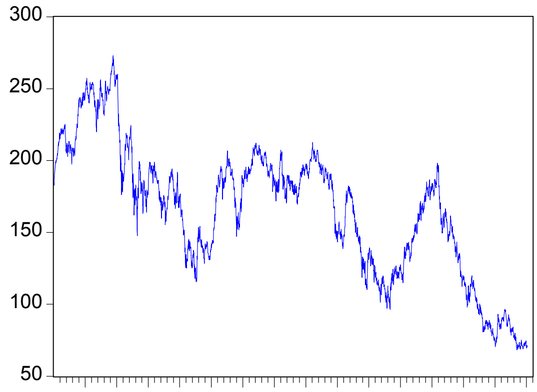


Figure 2: SP100

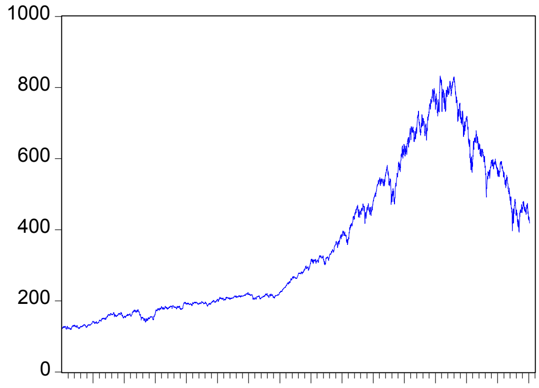


Figure 3: Nikkey225 log-returns

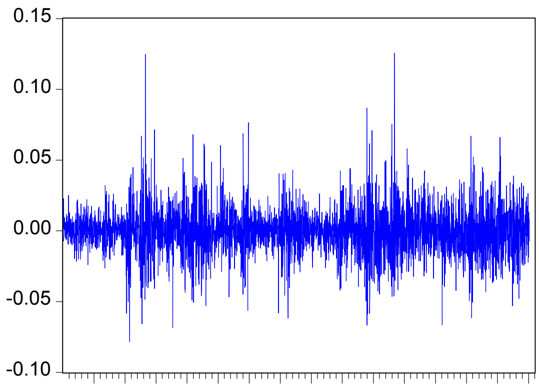
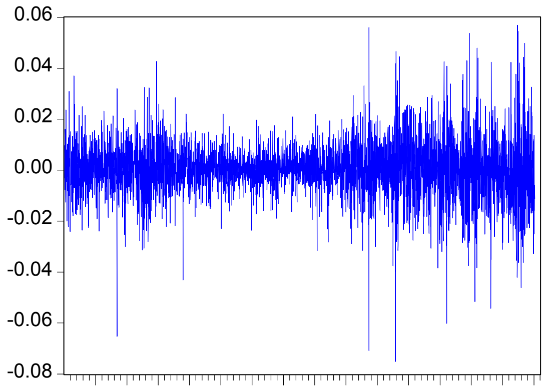


Figure 4: SP100 log-returns



The log-returns of both inidices evidence the well known volatility clustering effect. Beside the purpose of testing the causality relation among variances we are also interested in comparing our model with an alternative specification, the Constant Conditional Correlation GARCH, where the univariate models include both long and short memory parameterizations. In order to cover this additional point the analysis has been structured as follow: at first we filter out the mean effect with a VAR(1) model; then we turn to the estimation of the volatility structure. We chose to work on a two stage model in order to reduce the parameter vector dimension and the time requested by the estimation of the various models. The estimation result of the VAR(1) model is reported in the following table:

Table 1: VAR(1) estimation output

	NIK_DL	SP_DL
NIK_DL(-1)	-0.018499	-0.018187
	(0.01599)	(0.01105)
	(-1.15726)	(-1.64601)
SP_DL(-1)	0.328790	-0.011267
	(0.02370)	(0.01638)
	(13.8704)	(-0.68763)
C	-0.000370	0.000328
	(0.00026)	(0.00018)
	(-1.42517)	(1.82729)
R-squared	0.048776	0.000908
Adj. R-squared	0.048269	0.000375

The table reports the estimation of a VAR(1) model with the log-returns of the Nikkey225 and the SP100 indices, together with a constant. Parameter estiamtion, standard errors and t-statistics are reported. Moreover, the R^2 and adjusted R^2 are included.

As it could be expected, the lagged Standard & Poor's index has a relevant impact on the Nikkey, while the other coefficients result to be non-significant. Moreover, the following graphs report the autocorrelation funtions of both VAR(1) residuals series (for log-returns, lower line, and absolute log-returns, upper line) and evidence that a long term correlation exist in both series. This behavior is particularly evident in the Standard & Poor index where the autocorrelation oscillate around a value of 0.1, far above the significativity confidence bounds. One may object that such a behavior may be caused by the deletion of holidays and closed market day, however we replicate that the correlation in the original series is closer to the one here reported. This lon-memory behavior will be modelled, where possible, by Fractionally Integrated GARCH models.

Figure 5: ACF Nikkey225 residuals

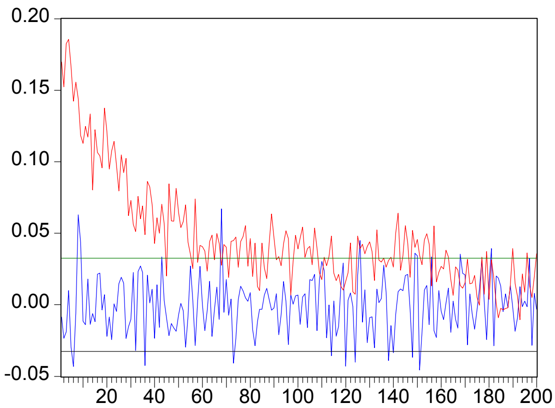
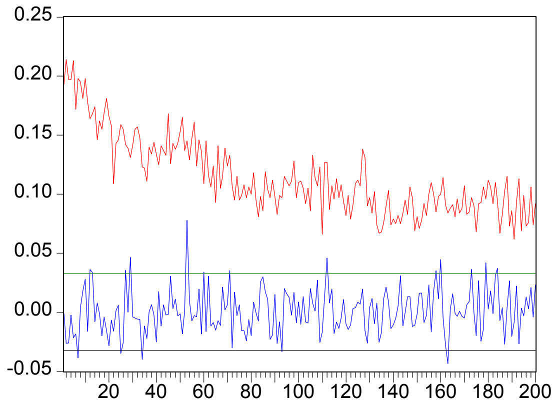


Figure 6: ACF SP100 residuals



For the variance part of our model we considered the following alternative specifications: a conditional constant correlation model where the univariate GARCH can be specified as GARCH(1,1), IGARCH(1,1), FIGARCH(1,d,0) and FIGARCH(1,d,1);the EC-GARCH(1,1). Other alternative specifications have been in principle considered, the BEKK-GARCH and the Vech-GARCH. However, the software packages we used for the estimation provided unsatisfactory results due to problems of convergence, constraint definitions and unexpected errors. We report in the following the variance-covariance structure of the two fitted models:

Constant Conditional Correlation - CCC-GARCH

$$\begin{aligned}
H_t &= \begin{bmatrix} \sigma^2_{NK,t} & \sigma_{NK-SP,t} \\ \sigma_{NK-SP,t} & \sigma^2_{SP,t} \end{bmatrix} = \begin{bmatrix} \sigma^2_{NK,t} & \rho\sigma_{NK,t}\sigma_{SP,t} \\ \rho\sigma_{NK,t}\sigma_{SP,t} & \sigma^2_{SP,t} \end{bmatrix} \\
\sigma^2_{j,t} &= \begin{cases} \omega_j + \alpha_j \varepsilon^2_{j,t-1} + \beta_j \sigma^2_{j,t-1} & GARCH(1,1) \\ \omega_j + \left(1-\beta_j\right)\varepsilon^2_{j,t-1} + \beta_j \sigma^2_{j,t-1} & IGARCH(1,1) \\ \omega_j + \beta_j \sigma^2_{j,t-1} + \left[1 - \beta_j L - (1-L)^{d_j}\right]\varepsilon^2_{j,t} & FIGARCH(1,d,0) \\ \omega_j + \beta_j \sigma^2_{j,t-1} + \left[1 - \beta_j L - \left(1-\psi_j L\right)(1-L)^{d_j}\right]\varepsilon^2_{j,t} & FIGARCH(1,d,1) \end{cases} \\
j &= SP, NK
\end{aligned}$$

Exponential Causality - EC-GARCH

$$\begin{aligned}
H_t &= \begin{bmatrix} \sigma^2_{NK,t} & \rho\sigma_{NK,t}\sigma_{SP,t} \\ \rho\sigma_{NK,t}\sigma_{SP,t} & \sigma^2_{SP,t} \end{bmatrix} \\
\sigma^2_{j,t} &= \exp\left(\gamma_j z^2_{i,t-1}\right)\left(\omega_j + \beta_j \sigma^2_{j,t-1} + \alpha_j \varepsilon^2_{j,t-1}\right) \\
j &= SP, NK \qquad i = SP, NK \qquad i \neq j
\end{aligned}$$

The results of model estimation are reported in table 2, where we included also the value of the log-likelihood, a set of information criteria and the Box-Pierce statistics for residual correlation. We compare the EC-GARCH with constant correlation models. As we previously specified, we consider a set of possible alternative specifications for the univariate volatilitied in the CCC models. All possible combinations have been estimated and compared on the basis of information criteria. The preferred CCC model turned out to be the CCC-FIGARCH with a long-memory model on both series. The CCC-GARCH is included since it represents the EC-GARCH with a zero restriction on causality parameters.

Comparing the information criteria of all the included models we note that all similarly explain the long term correlation, the worst result is provided by the CCC-GARCH, an expected results. What is interesting is the estimation of EC-GARCH where the causality parameter measuring the Standard & Poor's innovation effect on Nikkey index volatility is significant. Moreover, there is not evidence of the reverse cusal relation, from Nikkey to the Standard & Poor's. The EC-GARCH model provide in any case an improvement on the CCC-GARCH without any specific long-memory modelisation. Moreover, the conditional variances produced by both models (EC-GARCH and CCC-FIGARCH) are really close. The final conditional variances are reported in Figures 7 and 8.

Table 2: multivariate GARCH estimation

	EC-GARCH		CCC-FIGARCH		CCC-GARCH	
	NK-log-res	SP-log-res	NK-log-res	SP-log-res	NK-log-res	SP-log-res
μ	0.000318	0.000205	0.000350	0.000219	0.000356	0.000218
	0.000086	0.000085	0.000227	0.000099	0.000301	0.000119
ω	0.000005	0.000001	0.000006	0.000003	0.000005	0.000001
	0.000001	0.000000	0.000003	0.000003	0.000002	0.000000
d			0.662207	0.478713		
			0.129988	0.176724		
α	0.107002	0.056281			0.110707	0.056967
	0.006633	0.004829			0.015104	0.013410
β	0.869680	0.933569	0.687489	0.586735	0.873308	0.936911
	0.006048	0.004836	0.126131	0.236068	0.016974	0.014650
ϕ			0.130172	0.176683		
			0.056146	0.097912		
γ	0.007365	0.003408				
	0.003089	0.002864				
ρ	0.079757		0.080937		0.080412	
	0.009044		0.019995		0.019190	
LogL	-11963.924		-11955.731		-11965.359	
Akaike	6.378		6.374		6.378	
Hannan-Quinn	6.385		6.380		6.383	
Schwarz	6.374		6.369		6.374	
Shibata	6.378		6.374		6.378	
Q(5)	2.817	7.061	3.093	8.555	2.802	7.201
Q(10)	17.161	11.939	17.435	13.991	17.580	12.101
Q(20)	23.396	26.674	23.571	29.780	23.755	26.954
Q(50)	58.678	50.651	59.822	56.057	58.727	51.103
Q(100)	109.432	104.583	109.669	107.341	109.051	104.651
$Q^2(5)$	3.462	2.225	2.551	4.698	3.642	2.215
$Q^2(10)$	7.250	2.438	7.615	5.369	7.268	2.443
$Q^2(20)$	13.527	8.090	13.787	11.499	13.860	8.163
$Q^2(50)$	46.417	37.566	46.930	43.702	47.809	37.294
$Q^2(100)$	100.984	75.124	99.247	82.816	102.979	74.907

Quasi Maximum Likelihood estimation of the EC-GARCH and of the benchmark models CCC-FIGARCH and CCC-GARCH. The Table reports the estimated parameters with the corresponding standard errors. The last lines contain in the order: the value of the Log-Likelihood, the information criteria of Akaike, Hannan-Quinn, Schwarz and Shibata, the Lijung-Box Q test performed on standardised and squared standardised residuals. The degrees of freedom of this last test are reported in parenthesis and are equal to the maximum correlation used in the test. It is well known that the test asymptotic distribution is χ^2 a whose critical 5% values are: $\chi^2(5) = 11.1$, $\chi^2(10) = 18.3$, $\chi^2(20) = 31.4$, $\chi^2(50) = 67.5$, $\chi^2(100) = 124$, while at 1%: $\chi^2(5) = 15.1$, $\chi^2(10) = 23.2$, $\chi^2(20) = 37.6$, $\chi^2(50) = 76.2$, $\chi^2(100) = 136$.

Figure 7: Nikkey225 estimated variances variances

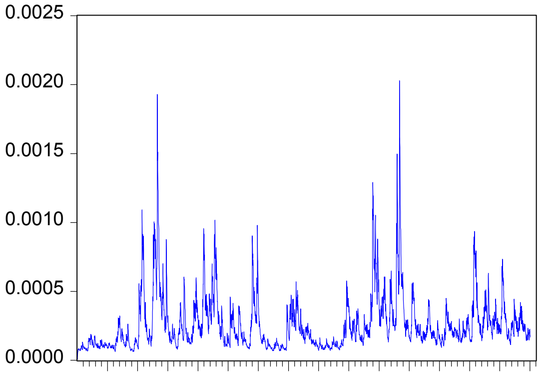
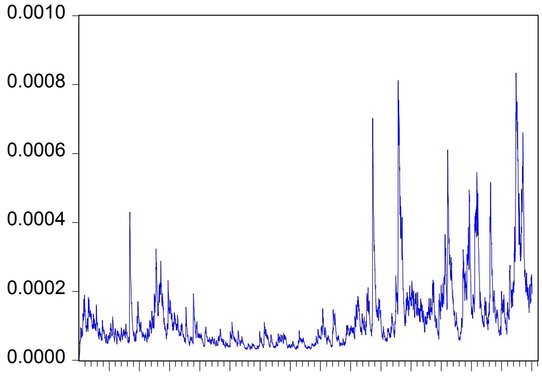


Figure 8: SP100 estimated variances



5 Conclusions

In this paper we introduce a new kind of bivariate GARCH model which can be used for the detection of a causality relation among conditional variances. The causality is added to the model via an exponential factor which multiplies the traditional GARCH equation. By this approach both causality existence and direction can be detected. This models solves some cdrawback of currently used specifications including the non-interpretability of parameters or the necessity of imposing positivity restrictions on the causality relation. The specification we propose allow for extensions both on the structure of the causal relation and in the system dimension. The model is also applied on an empirical excercise analysing the relation between the Standard & Poor 100 and the Nikkey225. The estimations performed evidence the existence of a causal relation among the variances and that this relation explain very well a clear long memory pattern evidenced by the series. The possible extension of the model and its relation with the long memory behavior represent an interesting area for future researches.

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