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Keywords: Long-Run Income Risk, Cointegration, Investment, Retirement

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1 Introduction

We note two empirical facts from the Health and Retirement Study (HRS) dataset (Figure 1):[1]: (i) stock investment increases with retirement age, i.e., individuals who retire early invest less in the stock market, and (ii) the individual portfolio share (defined as proportion of wealth invested in risky assets) before retirement is an increasing and concave function of wealth.[2]

In contrast to these observations, standard retirement literature predicts that portfolio share is a decreasing function of wealth and thus, individuals who retire early with less wealth have to invest more in the stock market (Farhi and Panageas, 2007, hereafter FP; DL; Jang *et al.*, 2013; Bensoussan *et al.*, 2016; Jang *et al.*, 2020).[3] The source of this prediction is that the existing literature specifies labor income dynamics in such a way that they act more like a riskless bond than a risky stock (Jagannathan and Kocherlakota, 1996; Heaton and Lucas, 1997). As such, because individuals choose to retire when they have far enough financial wealth compared to their human capital, if this human capital is bond-like, their implicit position in bonds through their human capital is high. As a result, they will compensate by holding a relatively larger fraction of financial wealth in stocks, thereby offsetting the larger implicit bond position they already have through their human capital. They should, thus, take a far more aggressive position in stocks with little wealth and shift their portfolio position away from stocks as they accumulate wealth and approach retirement.

[Insert Figure 1 here.]

We now have one research question to address: if labor income was specified so that it had more stock-like features than bond-like features, could theory and observation be reconciled? We investigate this question by proposing a model of optimal portfolio choice over the life cycle in an economy in which an individual has to decide on her allocation between a stock and a risk-free asset, as well as decide on the time when she goes into retirement, and where labor income

[1]Refer to the Appendix for the HRS data description.

[2]This empirical result is consistent with Surveys of household finances through which Wachter and Yogo (2010) observe an empirical result that household portfolio shares rise in wealth.

[3]For instance, FP has shown that the increased stock market exposure with a booming stock market between 1995 and 2000 in the U.S. is followed by early retirement of individuals.

dynamics are specified so that they are cointegrated with the stock market. The pivotal role of the long-run income risk in individuals' financial decisions has been has been documented by a great deal of empirical and anecdotal evidence (Baxter and Jermann, 1997; Menzly *et al.*, 2004; Santos and Veronesi, 2006; Davis and Willen, 2013).[4] As Dybvig and Liu (2010, hereafter DL) claim

> It would be nice to add more state variables to the model. For example, it has long been known that wages are sticky and it is reasonable that they respond to shocks in the stock market, but with a delay.

We find that the long-run income risk modeled as a cointegration between log labor income and log stock price changes the retirement strategy and the optimal risky asset allocation mix.

We first shed new light on the retirement decision especially in today's rapidly changing labor market situation. This paper endogenizes retirement decision as a function of wealth and income and long-run income shock. We demonstrate that retirement decision is likely to be determined by the levels of wealth. Having identified and numerically derived the wealth threshold for retirement, we find that it is lower when income will fall than when it will rise in the long run (Figure 2 in Section 4). Given that labor share has been declining on the aggregate level due to various reasons (e.g., technological change, increased globalization, changing composition of the workforce etc.),[5] if future income is expected to decline further, there will be then more incentive to retire earlier.

We also derive the target wealth (before approaching the retirement threshold) under which the individual's optimal decision is to not invest in the stock market, whereas above which she finds it optimal to increase her portfolio share as wealth increases (Figure 10 in Section 4). These model results help explain the empirical evidence that stock investment increases with

[4]We acknowledge that there are many other important determinants for an individual's life-cycle portfolio choices and retirement decision. The main reason of our focus on the long-run income risk is that it is suggestive of being significantly disruptive to a large cross-section of our society as is evidenced from both the COVID-19 pandemic and technological innovation. In particular, half of jobs in the world today are susceptible to becoming automated in the future (Frey and Osborne, 2017), so that insecurity and volatility levels around earnings will be greater in the long run.

[5]Even though certain skilled workers may earn higher wages and salaries, labor share has shown its apparent decline partly due to the composition of demand for skilled and unskilled labors change. For more details, refer to Grossman and Oberfield (2022).

retirement age by demonstrating that individuals who retire early with less wealth due to the long-run income risk invest less in the stock market. They also help explain the lack of stock-market participation by young individuals and a risky asset-allocation share that is increasing and concave in wealth.

The key mechanism that derives the portfolio choices stated above is that long-run labor income is cointegrated with stock price. Here, the stock price is modeled as a random walk with a positive drift. This implies that while the overall stock price will go up over long-run, for any specific time window corresponding to the individual's life, stock price may still go down even though her labor earnings profile remains unaltered. Further, stock price process is infinitely lived while the individual's labor income time span is finite. As a result of the cointegration channel, young individuals would be then reluctant to hold stocks because they already have high exposure to long-run labor income risk which also co-varies with stock price. This, thus, leads to the predicted portfolio choices with respect to the individual's wealth levels.

Interestingly, we find that the individual's retirement decision and life-cycle portfolio choices are interdependent in a nontrivial way. Contrary to the literature, retirement flexibility rather makes the optimal portfolio invest less in the stock market with the long-run income risk (Figure 11 in Section 4). The flexibility of supplying labor for a longer time rather exposes individuals to the greater income risk in the long run, thus reducing the stock investment. To manage risk exposure to the long-run income risk, the optimal portfolio should be more tilted towards riskless bonds compared to the case in which retirement flexibility is not allowed.

Our work further draws on the baseline optimal consumption/savings and investment models especially with endogenously determined (optimal) retirement decision. Since the seminal work of Merton (1969, 1971), Bodie *et al.* (1992), FP, DL, Chai *et al.* (2011), Jang *et al.* (2013), Bensoussan *et al.* (2016), Jang *et al.* (2020) have addressed the interactions between savings, portfolio choice, and retirement over the life cycle. As such, these studies have relied on either the absence of income risk or the assumption of a geometric Brownian motion when modeling and interpreting short-run income risk (Heaton and Lucas, 1997; Viceira, 2001; Cocco *et al.*, 2005), overlooking the effects of long-run income risk. However, the large body of empirical literature has documented the economic significance of the long-run income risk, thereby causing substantial deviations from the geometric Brownian motion assumption of labor income modeling. To our best knowledge, this paper is a first attempt to consider both Brownian-type short-run income risk and income-stock cointegration channel (long-run income risk) in the optimal retirement problem.

The solution to a life-cycle model has been, in general, attained by using either martingale pricing approach (MPA) or dynamic programming approach (DPA). The classical MPA of Cox and Huang (1989) has been applied to various life-cycle models in complete markets where all the risk can be diversifiable. When markets are complete, the uniqueness of state price density (or pricing kernel; stochastic discount factor) is guaranteed and thus, the unique risk-neutral measure can be successfully established for pricing purposes (Ross, 1978). However, when markets are incomplete, i.e., when some risk cannot be diversifiable, there are infinitely many state price densities, so the set of equivalent martingale measures is also infinite. Thus, the MPA may not be applicable to incomplete markets unless one can find a way to determine the unique state price density.[6]

Instead of the MPA, the DPA of Merton (1969, 1971) can be used to solve a life-cycle model in complete or incomplete markets. When markets are incomplete, however, one should confront highly non-linear Hamilton-Jacobi-Bellman (HJB) equations, which are typically almost impossible to be solved analytically. Basically, the value function we aim to solve in incomplete markets is highly likely to be degenerate and hence, it does not need to be smooth. In this case, the viscosity solutions techniques of Duffie *et al.* (1997) are available.[7] However, those still require to develop a method for computing the optimal strategies. In the absence of labor income and its risk, Garlappi and Skoulakis (2010), Jin and Zhang (2012), and Jin *et al.* (2017) have suggested numerical methods to solving the consumption/savings and investment models in incomplete markets. In the presence of labor income and its undiversifiable long-run risk, ours is a

[6]The incomplete-market MPA of Karatzas *et al.* (1991) is tricky to be used as it requires to add fictitious assets for market completion, which may not be easy to be justified in the reality. Liu *et al.* (2003) and Branger *et al.* (2017) have established a dynamically completed market by adding derivatives.

[7]When using the viscosity solutions techniques to deal with incomplete markets models, the first step is to approximate the value function by a sequence of smooth functions that are the value function of non-degenerate problems, where solutions are regular. One then needs to verify whether the limit of this sequence is indeed the value function, i.e., that the viscosity solution of the HJB equation is unique and the unique solution is the value function to be derived.

first attempt to propose a convenient and efficient numerical scheme for computing the optimal strategies in incomplete markets.

One of the main difficulties of our analysis lies in that allowing for an extra dimension of income risk causes considerable challenges in solving the optimal retirement model: “Unfortunately, models with additional state variables seem almost impossible to be solved analytically given current tools and numerical solution is also very difficult (DL)." Existing convex-duality approaches of Cox and Huang (1989), Karatzas *et al.* (1991), FP, DL, Jang *et al.* (2013), Bensoussan *et al.* (2016), and Jang *et al.* (2020) cannot be applicable to our retirement problem with the long-run income risk. This is because those approaches are capable of managing one-dimensional state variable (wealth) only, but our problem is a problem with two state variables (wealth and long-run income risk). We numerically solve the retirement model with the long-run income risk where a human-capital-to-total wealth ratio (human capital divided by total wealth) plays a crucial role in determining optimal retirement. Indeed, the work region and retirement region are characterized by a joint consideration of wealth and long-run income risk variables.

Limited Stock Market Participation Puzzle and Risky Asset-Allocation Share. Contrary to the theoretical prediction by Merton (1969, 1971), the rates of stock market participation have been low even though the stock has a positive risk premium. Merely 52% U.S. households invest in the stock market directly or indirectly through other substitutes (Gomes and Michaelides, 2005). Such a gap between theory and reality gives rise to a so-called non-participation puzzle. There is an extensive literature in an attempt to resolve the non-participation puzzle as follows: ongoing participation costs (Vissing-Jorgensen, 2002), fixed entry costs (Hong *et al.*, 2004; Guiso and Jappelli, 2005; Gomes and Michaelides, 2005), cointegrated labor income with the stock market (Benzoni *et al.*, 2007), existence of a large, negative wealth shock and insufficient insurance against the shock (Gormley *et al.*, 2010), and countercyclical volatility and procyclical mean of U.S. labor income (Lynch and Tan, 2011). On the other hand, Polkovnichenko (2007), Wachter and Yogo (2010), and Calvet and Sodini (2014) support that household portfolio shares defined as proportion of wealth invested in risky assets rise in wealth. This paper finds that adding the long-run income risk helps explain both the lack of stock-market participation by young agents and the risky asset-allocation share that is increasing and concave in wealth.

Related Literature. There are four papers closely related to the consumption/savings and investment models especially with retirement and income risk. Benzoni *et al.* (2007) make the point that with cointegrated (long-run) labor income risk, young investors are already exposed to stock market risk through their labor income and thus, might choose not invest in the stock market. Benzoni *et al.* (2007)'s model, however, faces a major limitation by neglecting one of the most important life-cycle dimensions: the retirement decision.[8] Our goal is distinct from Benzoni *et al.* (2007) in that we isolate and very closely investigate the long-run income risk issues not only on the consumption/savings and investment decisions, but uniquely, on the retirement decision as well.

DL studies a retirement model with income risk modeled by a geometric Brownian motion. Instead of diffusive and continuous income shocks of DL, Jang *et al.* (2013), and Bensoussan *et al.* (2016) incorporate a discrete and jump income shock in a retirement model, which has been supported by a great number of empirical studies such as Low *et al.* (2010) and Guvenen *et al.* (2015). The major difference of this paper from these three retirement papers is that we consider in the retirement model both diffusive and continuous income shocks and jump income shocks, and more importantly, incorporate the long-run income risk in the model via a cointegration between log labor income and log stock price. Specifically, we add one more state variable for the long-run income risk which captures the concept of long-run dependence between the stock and labor markets: If the log difference between labor income and stock price is below (above) the long-run mean, the labor income will then increase (decrease) in the long run.

Outline. The paper is organized as follows. In Section 2, we propose our baseline model. In Section 3, we provide analytic results for an approximate case. In Section 4, we carry out in-depth quantitative analysis with reasonable parameter values. In Section 5, we conclude the paper. For the numerical solution and the verification argument for the derived optimal strategies, refer to the Appendix.

[8] FP, DL, Chai *et al.* (2011), Jang *et al.* (2013), Bensoussan *et al.* (2016), Bensoussan *et al.* (2000) arguably state that retirement is one of the most important life-cycle decisions. Indeed, early (optimal) retirement was quite popularly opted for by individuals especially in the stock market booms like those observed in the late 1990's (Gustman and Steinmeier, 2002; Gustman *et al.*, 2010).

2 The Model

Short-Run Income Risk. For the short-run risk associated with labor income, we consider a geometric Brownian motion process with exogenously driven Poisson jumps (Wang *et al.*, 2016):

$$dI_t^S = \mu_I I_{t-}^S dt + \sigma_I I_{t-}^S d\tilde{\mathcal{B}}_t - (1-\kappa) I_{t-}^S dN_t, \;\; I_0^S = I^S > 0, \tag{1}$$

where $\mu_I > 0$ and $\sigma_I > 0$ are the expected rate and volatility of income growth, respectively, $\tilde{\mathcal{B}}_t$ is a standard one-dimensional Brownian motion, κ follows a power distribution over $[0,1]$ with parameter $\nu > 0$,[9] and N_t is a pure Poisson jump process with intensity δ_D. We assume that diffusive and continuous shocks (represented by a Brownian motion) and discrete and jump shocks (represented by a Poisson jump process) are independent for technical convenience.[10] The Brownian income risk and Poisson-jump income risk cannot be fully diversified and thus, the income risks are all idiosyncratic and uninsurable (Cocco *et al.*, 2005).[11]

The possibility of large, negative income shocks driven by the Poisson jump N_t is the following: for time $t \geq 0$,

$$\mathbb{P}\{\tau_D \leq t\} = 1 - e^{-\delta_D t},$$

where τ_D is the time when the large, negative income shock occurs and $\delta_D > 0$ is the income shock (Poisson) intensity. Whenever an individual experiences an unexpected, exogenous, and permanent income shock, her income plummets to $\kappa I_{\tau_D -}$ $(\kappa \in [0,1))$ from $I_{\tau_D -}$. As long as κ is positive, the individual obtains some positive income in the aftermath of the disastrous income shock. This positive income can be funded by social security program or subsistence (such as public welfare or unemployment allowances).[12]

[9] The probability density function for κ with parameter ν is given by $P_\kappa(z) = \nu z^{\nu-1}$, where $0 \leq z \leq 1$. This specification is appropriate when adopting a well-behaved distribution for κ. An expected income loss decreases with respect to an increase of ν due to the relationship of $E[1-\kappa] = 1/(\nu+1)$.

[10] Some correlations between those can be possibly considered by an additional stochastic process for the probability distribution of jump size κ.

[11] The currently available explicit insurance markets are not sufficient to perfectly hedge against the risks associated with labor income.

[12] In case of unemployment, Carroll *et al.* (2003) assume that post-unemployment income is about 20% of permanent labor income, which can be financed by a safety net such as formal or informal insurance markets.

Financial Market. There are two tradable assets in the financial market: a riskless bond and a risky stock. The bond price grows at a constant rate $r > 0$. The stock price, S_t, follows a geometric Brownian motion process:

$$dS_t = \mu S_t dt + \sigma S_t d\mathcal{B}_t^1, \tag{2}$$

where $\mu > r$ and $\sigma > 0$ are the expected rate and volatility of stock returns, and $\mathcal{B}_t^1$ is a standard one-dimensional Brownian motion with an instantaneous correlation $\rho \in [-1, 1]$ with the labor income process given in (1), i.e., $d\mathcal{B}_t^1 d\tilde{\mathcal{B}}_t = \rho dt$. We assume that r, μ, σ are constant, i.e., the investment opportunity is constant.

Long-Run Income Risk. Following Section 6 of DL, we add one more state variable to consider the long-run income risk. We assume that labor income process I_t follows

$$I_t = S_t e^{Z_t},$$

subject to the large, negative income shocks driven by the Poisson jump N_t, where Z_t is the difference between the logs of labor income and stock price. It is assumed to follow a mean reverting process

$$dZ_t = -\alpha(Z_t - \overline{z})dt - \sigma_z d\mathcal{B}_t^1 + \sigma_I d\mathcal{B}_t^2,$$

where $\alpha > 0$ measures the degree of mean reversion, $\overline{z}$ denotes the long-term mean, σ_z and σ_I measure the conditional volatilities of the difference change dZ_t, and $\mathcal{B}_t^2$ is a standard one-dimensional Brownian motion independent of $\mathcal{B}_t^1$. Thus, returns to labor income follow

$$dI_t/I_{t-} = \{\mu_I - \alpha(Z_t - \overline{z})\}dt + (\sigma - \sigma_z)d\mathcal{B}_t^1 + \sigma_I d\mathcal{B}_t^2 - (1-\kappa)dN_t, \;\; I_0 = I > 0, \tag{3}$$

where $\mu_I = \mu + \frac{1}{2}\sigma_z^2 + \frac{1}{2}\sigma_I^2 - \sigma\sigma_z$. The drift in returns to labor income pushes them up or down depending upon market/economic conditions, i.e., relying on the expected stock return μ, the stock volatility σ, the income volatility σ_I, and the volatility σ_z of the difference change dZ_t.

The dynamics of returns to labor income given in (3) can capture two empirical facts well. Firstly, by setting $\sigma = \sigma_z$, the risk exposure of labor income to stock market becomes zero, thereby reflecting that the contemporaneous correlation between returns to labor income and market returns should be nearly zero (Cocco *et al.*, 2005; Davis and Willen, 2013). Secondly, labor income responds with a delay to shocks in the stock market in the long run.[13] When $Z_t - \overline{z} < 0$, the labor income is expected to increase in the long term, whereas when $Z_t - \overline{z} > 0$, the labor income decrease.

Credit Market. Individuals are not allowed to borrow by capitalizing their human capital, i.e., they are exposed to a borrowing constraint (Cocco *et al.*, 2005; DL; Jang *et al.*, 2013, 2019, 2020). The borrowing constraint is typically imposed due to market frictions such as informational asymmetry, agency conflicts, limited enforcement etc. Also, they are exposed to a short sale constraint because of legal and institutional restrictions in the US equity markets (Bai *et al.*, 2006).

In the presence of the short sale and borrowing constraint, both bond investment x_t and stock investment y_t are all nonnegative. Hence, financial wealth W_t that is the sum of x_t and y_t is also nonnegative:

$$x_t \geq 0, \;\; y_t \geq 0, \;\; W_t \equiv x_t + y_t \geq 0, \tag{4}$$

which evolves by the following dynamics:

$$dW_t = (rW_t - c_t + I_t)dt + y_t\sigma(d\mathcal{B}_t^1 + \theta dt), \;\; W_0 = w \geq 0,$$

where c_t is the per-period consumption and $\theta = (\mu - r)/\sigma$ is the Sharpe ratio. Let $\mathcal{A}(w, I, z)$ denote the set of admissible policies such that short sale and borrowing constraints (4) are satis-

[13] This specification for labor income that its growth relies on the expected stock return in the long run is particularly relevant to the literature on stock return being forward-looking because it incorporates information on future economic activities. Similar to CAY, a variable constructed from consumption and wealth relation, which is empirically plausible our wealth variable may be able to play a role in predicting stock returns as an imputed measure of income.

fied.

A Retirement Problem. Following FP and DL, the individual's optimal retirement problem is to maximize her constant relative risk aversion (CRRA) utility preference by optimally controlling per-period consumption c, risky investment y, and voluntary retirement time τ. That is, the value function is given by

$$V(w,I,z) \equiv \sup_{(c,y,\tau)\in\mathcal{A}(w,I,z)} \mathbb{E}\Big[\int_0^{\tau\wedge\tau_D} e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta(\tau\wedge\tau_D)}\int_{\tau\wedge\tau_D}^{\infty} e^{-\beta(t-\tau\wedge\tau_D)}\frac{(Bc_t)^{1-\gamma}}{1-\gamma}dt\Big], \tag{5}$$

where $\mathbb{E}$ is the expectation taken at time 0, $\beta > 0$ is the subjective discount rate, and $\gamma > 0$ ($\gamma \neq 1$) is the constant coefficient of relative risk aversion.[14] Here, the parameter $B > 1$ stands for the leisure preference after retirement. We assume that these is no income source after retirement, i.e., labor income I_t becomes zero for $t \geq \tau$. We also assume that there is no bequest motive for simplicity. The optimal retirement considered in this paper excludes forced or involuntary retirement due to professions or health shock.

The value function (5) properly incorporates a labor-leisure trade-off in a very reduced-form way that would push the individual to, at some point, stop working to enjoy her free time after retirement. The incentive to enter voluntary retirement results from more leisure preferences by not working anymore than in the workforce. The exogenous parameter B, which enters like the bequest function in the value function, suggests that retirees value consumption more than when they are employed, and thereby the marginal utility of consumption after retirement is larger than that before retirement. The decision to enter retirement simply entails a point where one stops receiving the labor income stream, at which point one continues on living forever and consuming and investing from the accumulated wealth.[15]

[14]Throughout the paper, we assume that $\gamma > 1$, which is consistent with the data.

[15]To enrich the retirement model, more plausible economic trade-offs that lead individuals to choose to go into retirement and give up their day-to-day job can be considered by some physical and legal constraints. The individual's productivity would decline with age or the effort-cost of working would increase. This could be a crucial factor in the decision to stop working and would affect the target retirement wealth level. There might be

We denote by $R(W_\tau)$ the maximal utility value after retirement that has the following form (Merton, 1969, 1971):

$$R(w) = \frac{B^{1-\gamma}\overline{K}^{-\gamma}}{1-\gamma} w^{1-\gamma}, \;\; \overline{K} = \frac{\beta}{\gamma} - \frac{1-\gamma}{\gamma}\Big(r + \frac{\theta^2}{2\gamma}\Big).$$

By the principle of dynamic programming, we can then restate the value function (5) after integrating out τ_D as

$$V(w, I, z) \equiv \sup_{(c,y,\tau)\in\mathcal{A}(w,I,z)} \mathbb{E}\Big[\int_0^\tau e^{-(\beta+\delta_D)t}\Big\{\frac{c_t^{1-\gamma}}{1-\gamma} + \delta_D V(W_t, \kappa I_t, Z_t)\Big\}dt + e^{-(\beta+\delta_D)\tau} R(W_\tau)\Big]. \tag{6}$$

The value function (6) has distinctive features compared to conventional life-cycle models without large, negative income shocks or endogenous retirement. In case of $\delta_D > 0$, i.e., when there is the possibility of large, negative income shocks, the value function V itself has a recursive structure driven by the term involving $\delta_D V$, in addition to the intermediate CRRA utility value of consumption. This recursive structure differs from traditional life-cycle framework without the income shocks in which the value function is the maximal utility of consumption only. Further, in case of $\tau < +\infty$, the maximal utility value R after retirement influences the optimal strategies before retirement due to the irreversible decision of retirement timing τ which is a non-linear option-type element and plays a pivotal role in the optimal strategies. This option structure differs from existing framework without endogenous retirement in which the post-retirement value function does not affect the pre-retirement value function.

3 Analytic Results: Approximate Case

The three-dimensional problem (6) coupled with consumption, investment, and retirement does not seem to admit a simple analytic solution. Even though we will provide the graphical illus-

some legal constraints such as mandatory retirement where time also plays an explicit role in determining retirement. In some cases, mandatory retirement may be optimal (Lazear, 1979).

trations for the optimal strategies later based on a numerical solution given in the Appendix, it is still worth deriving analytic results for an approximate case in order to obtain economic intuition and motivate the paper's focus on the income-stock cointegration channel. The income-stock cointegration channel can be approximately considered without the cointegration by substantially increasing the risk exposure of the individual in the stock market.[16] That is, we consider the approximate case where income varies significantly with the stock market.

One measure that is helpful for investigating the individual's risk exposure in the stock market is the sensitivity of income to the stock market conditions, σ_I/θ, and the sensitivity of stock investment to the stock market conditions, $1/\gamma$.[17] We can then obtain the following result showing when income varies significantly with the stock market: If $\sigma_I/\theta > 1/\gamma$, then income varies significantly with the stock market. Here, the condition $\sigma_I/\theta > 1/\gamma$ can represent the economic situation where income becomes more sensitive to the stock market conditions relative to the sensitivity of stock investment to the stock market conditions.[18] Throughout this section, we, thus, mainly focus on the following case:

Assumption 3.1. *The income-stock cointegration channel can be approximately considered by assuming that* $\sigma_I/\theta > 1/\gamma$.

For the purpose of illustrating the effects of cointegrated income and stock on the retirement strategy and the optimal risky asset allocation in an intuitive way, we proceed pedagogically with our analysis and develop insights by solving four models, which are sorted by retirement flexibility, borrowing constraints, and discrete and jump income shocks as follows:

Model 1. Consumption and risky asset allocation only (Bodie *et al.*, 1992).

Model 2. Consumption and risky asset allocation with retirement flexibility (FP).

[16] The approximate case may not be empirically plausible in most cases because it is well known that the instantaneous correlation between the labor and stock markets should be nearly zero (Cocco *et al.*, 2005; Davis and Willen, 2013), implying that the risk exposure of labor income to stock market becomes zero. However, this approximate case helps us with analytic solutions to intuitively understand the effects of income-stock cointegration channel on the retirement strategy and the optimal risky asset allocation.

[17] Basak *et al.* (2006) have made use of these quantities for examining the risk management with benchmarking.

[18] Although the reversed condition $\sigma_I/\theta < 1/\gamma$ generally holds given the relatively low income volatility σ_I, entrepreneurs are those who have significant exposure to the stock market risk (Heaton and Lucas, 2000), possibly satisfying $\sigma_I/\theta > 1/\gamma$.

Model 3. Consumption and risky asset allocation with both retirement flexibility and borrowing constraints (DL).

Model 4. Consumption and risky asset allocation with retirement flexibility, borrowing constraints, and discrete and jump income shocks.[19]

Below are the four model formulations stated above.

Model 1. The individual aims to maximize her expected discounted total CRRA utility from consumption by optimally controlling consumption c and investment y:

$$V(w, I) \equiv \sup_{(c,y)} \mathbb{E}\Big[\int_0^{\infty} e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \Big],$$

which is subject to

$$dI_t = \mu_I I_t dt + \sigma_I I_t d\tilde{\mathcal{B}}_t, \;\; I_0 = I > 0, \tag{7}$$

$$dW_t = (rW_t - c_t + I_t)dt + y_t \sigma (d\tilde{\mathcal{B}}_t + \theta dt), \;\; W_0 = w > -\frac{I}{\beta_1}, \tag{8}$$

$$W_t \geq -\frac{I_t}{\beta_1}, \tag{9}$$

where $\beta_1 \equiv r - \mu_I + \sigma_I \theta$.

Model 2. The individual aims to maximize her expected discounted total CRRA utility from consumption by optimally controlling consumption c, investment y, and retirement time τ:

$$V(w, I) \equiv \sup_{(c,y,\tau)} \mathbb{E}\Big[\int_0^{\tau} e^{-\beta t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + e^{-\beta \tau} \int_{\tau}^{\infty} e^{-\beta(t-\tau)} \frac{(Bc_t)^{1-\gamma}}{1-\gamma} dt \Big],$$

which is subject to (7), (8), and (9).

Model 3. The same as Problem 2, except that the wealth constraint (9) is replaced by borrowing constraints

$$W_t \geq 0, \;\; W_0 = w \geq 0. \tag{10}$$

Model 4. The same as Problem 3, except that income dynamics (7) is replaced with discrete and

[19]This model is a close relative of the Bensoussan *et al.* (2016) model with forced unemployment risk. The model considered in this paper (Model 4) is more challenging to be solved because it adds more realistic real-world complications represented by both borrowing constraints and discrete and jump income shocks rather than only one-time jump income shock in Bensoussan *et al.* (2016).

jump income shocks by (1).

Model 1 is the case of Bodie *et al.* (1992). Moving to Model 2 isolates the effects of retirement flexibility on risky asset allocation. Subsequently moving to Model 3 isolates the effects of borrowing constraints on risky asset allocation. Lastly, moving to Model 4 closely investigates the issues associated with discrete and jump income shocks on risky asset allocation.

Theorem 3.1. *(Model 1). The optimal risky asset allocation is*

$$y = \frac{\theta}{\gamma\sigma}w + \frac{\theta}{\sigma}\Big(\frac{1}{\gamma} - \frac{\sigma^I}{\theta}\Big)\frac{I}{\beta_1}.$$

With our approximately considered income-stock cointegration channel by 3.1 where $\sigma_I/\theta > 1/\gamma$, Theorem 3.1 demonstrates that the portfolio share (the proportion of financial wealth invested in the stock market), y/w, increases with wealth. The literature already offers explanations for such an increasing pattern of the portfolio share in wealth (Polkovnichenko, 2007; Wachter and Yogo, 2010; Calvet and Sodini, 2014). The income-stock cointegration channel can be regarded as complementary to these. In particular, the long-run income risk modeled by the cointegration has a first-order effect on the optimal risky asset allocation of individuals by exposing them to the stock market risk substantially.

Theorem 3.2. *(Model 2). The optimal risky asset allocation is*

$$y = \frac{\theta}{\gamma\sigma}w + \frac{\theta}{\sigma}\Big(\frac{1}{\gamma} - \frac{\sigma^I}{\theta}\Big)\Big[\frac{I}{\beta_1} + (\gamma\alpha - 1)IC\lambda^{-\alpha}\Big], \tag{11}$$

where $\alpha > 1$ is a constant satisfying the following characteristic equation:

$$CE(x) \equiv -\frac{1}{2}\beta_3^2 x(x-1) + (\beta_2 - \beta_1)\alpha + \beta_1 = 0, \tag{12}$$

$C > 0$ is a constant determined with wealth threshold $\overline{w}$ for retirement by

$$C = \Big[\Big(B^{1/\gamma-1}\overline{K}^{-1} - \frac{1}{\hat{A}}\Big)\underline{\lambda}^{-1/\gamma} + \frac{1}{\beta_1}\Big]\underline{\lambda}^{\alpha},$$

where

$$\overline{w} = IB^{1/\gamma-1}\overline{K}^{-1}\underline{\lambda}^{-1/\gamma},$$

$$\underline{\lambda} = \left(\left[B^{1/\gamma}\overline{K}^{-1}\left\{\frac{\beta_2}{1-\gamma} - \left(\beta_1 + \frac{\beta_3^2\alpha}{2}\right)\right\} - \frac{\gamma}{1-\gamma} + \frac{\beta_3^2}{2\hat{A}}\left(\alpha - \frac{1}{\gamma}\right)\right] \Big/ \left(1 + \frac{\beta_3^2\alpha}{2\beta_1}\right)\right)^{\gamma},$$

$$\hat{A} = \frac{\gamma-1}{\gamma}\left(\beta_1 + \frac{\beta_3^2}{2\gamma}\right) + \frac{\beta_2}{\gamma},$$

$$\beta_2 = \beta - \mu_I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)\sigma_I^2,$$

$$\beta_3 = \gamma\sigma_I - \theta,$$

and $\lambda > \underline{\lambda}$ *is a dual variable of wealth-to-income ratio* $z = w/I$ *defined as the first derivative of* $\phi(z)$ *which has a relation with value function* $V(w, I)$ *in Model 2 as*

$$\lambda(z) \equiv \phi'(z),$$

$$V(w, I) = I^{1-\gamma}\phi(z), \;\; z \equiv \frac{w}{I}.$$

As long as $\sigma_I/\theta > 1/\gamma$ as in Assumption 3.1, i.e., with the approximately considered income-stock cointegration channel, Theorem 3.2 arguably states that retirement flexibility (the additional term involving C on the right-hand side of (11)) rather decreases the risky investment in the stock market, not increases as standard retirement literature predicted. For those who have already significant exposure to the stock market risk due to the cointegration, the flexibility of supplying labor for a longer period rather exposes them to the greater income risk as well in the long run, diminishing the stock investment to reduce total risk exposure.

Theorem 3.3. *(Model 3). The optimal risky asset allocation is*

$$y = \frac{\theta}{\gamma\sigma}w + \frac{\theta}{\sigma}\left(\frac{1}{\gamma} - \frac{\sigma^I}{\theta}\right)\left[\frac{I}{\beta_1} + (\gamma\alpha - 1)IC\lambda^{-\alpha} + (\gamma\alpha^* - 1)IC^*\lambda^{-\alpha^*}\right], \tag{13}$$

where $\alpha > 1$ *and* $-1 < \alpha^* < 0$ *are two constants satisfying the characteristic equation (12) given in Theorem 3.2,* $C > 0$ *and* $C^* > 0$ *are two constants determined with* $\underline{\lambda}$ *and* $\overline{\lambda}$ *by the* *following system of four algebraic equations:*

$$\begin{cases} \Big\{\beta_2 \dfrac{1}{1-\gamma} B^{1/\gamma - 1}\overline{K}^{-1} - \Big(\beta_1 + \dfrac{\beta_3^2}{2\gamma}\Big)\dfrac{1}{\hat{A}} - \dfrac{\gamma}{1-\gamma}\Big\}\underline{\lambda}^{-1/\gamma} \\ = \Big(\beta_1 + \dfrac{1}{2}\beta_3^2\alpha\Big)C\underline{\lambda}^{-\alpha} + \Big(\beta_1 + \dfrac{1}{2}\beta_3^2\alpha^*\Big)C^*\underline{\lambda}^{-\alpha^*}, \\ \Big\{B^{1/\gamma-1}\overline{K}^{-1} - \dfrac{1}{\hat{A}}\Big\}\underline{\lambda}^{-1/\gamma} + \dfrac{1}{\beta_1} = C\underline{\lambda}^{-\alpha} + C^*\underline{\lambda}^{-\alpha^*}, \\ \dfrac{1}{\hat{A}}\overline{\lambda}^{-1/\gamma} + C\overline{\lambda}^{-\alpha} + C^*\overline{\lambda}^{-\alpha^*} = \dfrac{1}{\beta_1}, \\ -\dfrac{1}{\gamma\hat{A}}\overline{\lambda}^{-1/\gamma} - \alpha C\overline{\lambda}^{-\alpha} - \alpha^* C^*\overline{\lambda}^{-\alpha^*} = 0. \end{cases}$$

In the presence of the long-run income risk by assuming $\sigma_I/\theta > 1/\gamma$ as in Assumption 3.1, borrowing constraints (the extra term involving C^* on the right-hand side of (13)) have a positive impact on the amount of risky investment in the stock market. The individuals who have already significant exposure to the stock market risk because of the long-run income risk surprisingly choose a larger exposure to the stock market than without borrowing constraints. This is because they are constrained to maintain their wealth above zero in all states and hence, their optimal decision may be to take on large equity positions to finance a high wealth level as long as the positive risk premium is guaranteed, while allowing for large losses resulting from an adverse shock in the stock market.

Theorem 3.4. *(Model 4). The optimal risky asset allocation is*

$$\begin{aligned} y = \frac{\theta}{\gamma\sigma}w + \frac{\theta}{\sigma}\Big(\frac{1}{\gamma} - \frac{\sigma^I}{\theta}\Big)\Big[\frac{I}{\beta_1} &+ (\gamma\alpha_{\delta_D} - 1)I(C_{\delta_D} - \textit{PS1}(\lambda;\delta_D))\lambda^{-\alpha_{\delta_D}} \\ &+ (\gamma\alpha^*_{\alpha_{\delta_D}} - 1)I(C^*_{\delta_D} - \textit{PS2}(\lambda;\delta_D))\lambda^{-\alpha^*_{\delta_D}} \\ &+ \frac{2\gamma\delta_D}{\beta_3\lambda}E[\kappa^{1-\gamma}\phi(z/\kappa)]\Big], \end{aligned} \tag{14}$$

where $\alpha_{\delta_D} > 1$ *and* $-1 < \alpha^*_{\delta_D} < 0$ *are two constants satisfying the following characteristic equation:*

$$CE(x;\delta_D) \equiv -\frac{1}{2}\beta_3^2 x(x-1) + (\beta_2 + \delta_D - \beta_1)\alpha + \beta_1 = 0,$$

PS1$(\lambda;\delta_D)$ *and PS2*$(\lambda;\delta_D)$ *are given by*

$$PS1(\lambda;\delta_D) = \frac{2\delta_D(\alpha_{\delta_D}-1)}{\beta_3^2(\alpha_{\delta_D}-\alpha^*_{\delta_D})}\int_{\underline{\lambda}}^{\lambda}\mu^{\alpha_{\delta_D}-2}E\Big[\kappa^{1-\gamma}\phi\big((G(\mu)-1/\beta_1)/\kappa\big)\Big]d\mu < 0,$$

$$PS2(\lambda;\delta_D) = \frac{2\delta_D(\alpha^*_{\delta_D}-1)}{\beta_3^2(\alpha_{\delta_D}-\alpha^*_{\delta_D})}\int_{\lambda}^{\overline{\lambda}}\mu^{\alpha^*_{\delta_D}-2}E\Big[\kappa^{1-\gamma}\phi\big((G(\mu)-1/\beta_1)/\kappa\big)\Big]d\mu < 0,$$

$G(\lambda)$ *satisfies the following non-linear differential equation:*

$$-\frac{1}{2}\beta_3^2\lambda^2G''(\lambda)-\{\beta_3^2+\beta_2+\delta_D-\beta_1\}\lambda G'(\lambda)+\beta_1G(\lambda)-\lambda^{-1/\gamma} = -\delta_D E[\kappa^{-\gamma}\lambda(z/\kappa)]G'(\lambda),$$

and C_{δ_D}, $C^*_{\delta_D}$, $\underline{\lambda}$, $\overline{\lambda}$ *are four positive constants to be determined appropriately according to value matching and smooth pasting conditions specified in the Appendix.*

The discrete and jump income shocks turn out to reinforce the effects of retirement flexibility and borrowing constraints as we analyzed in Theorem 3.2 and 3.3 by having extra two terms involving $PS1(\lambda;\delta_D)$ and $PS2(\lambda;\delta_D)$ on the right-hand side of (14); (retirement flexibility effects) C_0 same as C in Theorem 3.2 increases to $C_{\delta_D}-PS1(\lambda;\delta_D)$ and (borrowing constraints effects) C_0^* same as C^* in Theorem 3.3 increases to $C^*_{\delta_D}-PS2(\lambda;\delta_D)$.

Summarizing, all four models solved above can explain with the approximately considered income-stock cointegration channel (Assumption 3.1) the empirically plausible result on the optimal risky asset allocation that the portfolio share increases in wealth. Further, Model 2 implies that retirement flexibility decreases the stock market exposure of individuals. Model 3 and Model 4 confirms this result even with borrowing constraints, and discrete and jump income shocks.

Interestingly, contrary to standard retirement literature, our model (particularly, Model 4) has a potential to explain the empirical evidence that stock investment increases with retirement age (i.e., those who retire early decrease their stock market exposure). This evidence would be explained if one can show that individuals optimally retire early with less wealth than without the income-stock cointegration channel. Put differently, those retiring early optimally have less wealth, thus decreasing their stock market exposure in the presence of the cointegration.

Having understood such a clear need for further quantitative exploration on the retirement strategy and the optimal risky asset allocation, we will solve in the next section the originally established long-run income risk model by (6) numerically, aiming to investigate whether individuals retire early with less wealth than without the long-run income risk.

4 Quantitative Analysis

We carry out in-depth quantitative analysis of the baseline model with reasonable parameter values. Technically, for graphical illustration we apply the penalty method with finite difference discretization (Dai and Zhong, 2010) to our retirement problem (5). Further details for the verification for the optimal strategies are provided in the Appendix.

We set the baseline parameter values as follows:

$$\beta = 0.04, r = 0.01, \mu = 0.05, \sigma = 0.18, \gamma = 3, B = 2, \mu_I = 0.005, \sigma_I = 0.10,$$
$$I = 1, \kappa = 0.8, \delta_D = 0.05, \sigma_z = \sigma, \alpha = 0.15, \overline{z} = 0.$$

Our parameter choice is supported by existing literature. The parameter values in asset returns (μ, r, σ) are taken from DL. The subjective discount rate β of 4% is consistent with existing life-cycle literature (Cocco *et al.*, 2005; Gomes and Michaelides, 2005; Wachter and Yogo, 2010; Wang *et al.*, 2016). The higher value of 4% than the risk-free rate of 1% can be thought of as the mortality-risk-perceived subjective discount rate, and it makes people relatively impatient in the bond market, so the wealthier people tend to save less in the bond market. The coefficient 3 of relative risk aversion γ is appropriate as it is absolutely lower than the upper bound 10 of risk aversion suggested by Mehra and Prescott (1985). The leisure preference after retirement B of 2 is exactly the same as DL.

We carefully choose the baseline parameter values in labor income dynamics as follows. Similar to Deaton (1991), Carroll (1992), DL, and Wang *et al.* (2016), we set the income growth's expected rate μ_I and volatility σ_I to 0.5% and 10%, respectively. In this case, the expected change $(\mu_I - \sigma_I^2)/2$ of logarithmic income level becomes zero. We normalize the labor income I as one. The recovery parameter κ after large, negative income shocks is assumed to be constant and set to 80%. The Poisson intensity δ of 5% for large, negative income shocks is exactly the same as Wang *et al.* (2016). By setting $\sigma = \sigma_z$, we reflect the fact that the contemporaneous correlation between returns to labor income and market return is zero (Campbell *et al.*, 2001; Cocco *et al.*, 2005; Gomes and Michaelides, 2005; Davis and Willen, 2013). Instead, labor income responds with a delay to shocks in the stock market in the long run: when $Z_t - \bar{z} < 0$ ($Z_t - \bar{z} > 0$), labor income will rise (fall) in the long run. The degree of mean reversion α and long-term mean $\bar{z}$ are set to 15% and zero, respectively. The initial value of Z_t is $Z_0 = 0$, which is the steady state value.

The benchmark model against which we compare our results is DL, where the contemporaneous correlation ρ between the stock and labor markets is zero and there is no long-run income risk.

Optimal Retirement Strategy. We endogenize retirement decision as a function of wealth and income and long-run income risk. We, thus, develop a two-dimensional retirement model with the long-run income risk in which the wealth threshold for retirement is a function of the extent of the long-run income risk. We introduce a human-capital-to-total-wealth ratio ξ, which is the present value of future labor income I/r divided by the total wealth $w + I/r$. In Figure 2, we characterize the work region and retirement region by a joint consideration of the threshold of ξ (or the threshold of w) and the state variable z (which represents the long-run income risk). For the fixed extent of the long-run income risk, i.e., for the fixed value of z, there exists the threshold of ξ (or the threshold of w) under (or over) which it is optimal to enter voluntary retirement (FP; DL; Chai *et al.*, 2011; Jang *et al.*, 2013; Bensoussan *et al.*, 2016; Jang *et al.*, 2020).

[Insert Figure 2 here.]

We demonstrate that retirement decision is likely to be determined by the levels of wealth. In particular, the income-to-wealth ratio is inversely related to individual wealth. The income is a major staple of the relatively low-wealth people and it could well be that the income-to-wealth ratio is high which the model predicts that the individual should optimally remain working. While the income is a smaller staple of the high-wealth people, so they could be in a situation that their income-to-wealth ratio is low, predicting that they should optimally retire soon. This does make sense in real life in that they may choose to retire because they have sufficient wealth to maintain their life style and allow to pursue other interests in life.

The distinct feature of our optimal strategy from the existing literature is that the wealth threshold itself varies significantly with the long-run income risk. In the presence of the long-run income risk, labor income responds with a delay to shocks in the stock market, depending upon the sign of $z - \overline{z}$: when $z - \overline{z} < 0$ ($z - \overline{z} > 0$), labor income will rise (fall) in the long run. In Figure 2, with the baseline parameter value of $\overline{z} = 0$, when $z < 0$ ($z > 0$) labor income will rise (fall) in the long run. We find that the wealth threshold for retirement is lower when income will fall ($z > 0$) than when it will rise ($z < 0$) in the long run.

[Insert Figure 3 here.]

Interestingly, for the positive values of z, the wealth threshold for retirement further decreases as z increases, i.e., when a larger negative income shock is expected in the long run (Figure 3). We arguably state that existing guidance on retirement without considering the long-run income risk represents an overly simplified situation. For example, if the individual over-targets the wealth threshold as suggested by the existing literature without the long-run income risk and thus, follows the corresponding suboptimal retirement strategy under the wrong target by working longer, when the larger negative income shock occurs she is in danger of being caught up in financial distress. This would leave individuals vulnerable to not having enough resources towards the end of their life cycle.

The intuition is as follows. The individual with the high target for retirement wealth has a relatively weaker retirement demand than with the low target, because she is more far from her goal of optimal retirement with the longer distance to the target retirement wealth. So, when receiving a unit of wealth windfall, the individual with the higher target wealth for retirement optimally consumes a larger fraction of the windfall. However, when the larger negative income shock occurs such a strong consumption demand is no longer covered by her current income. Therefore, the individual with the larger negative income shock in the long run would be better off lowering her target wealth than without the long-run income risk, thereby consuming fewer of the wealth windfall and brining her optimal retirement closer. This shows the economic importance of correctly taking the long-run income risk into account the retirement plan when considered today's increased concern about potentially catastrophic future income uncertainty driven by earnings insecurity and volatility in the long run.

[Insert Figure 4 here.]

While existing literature predicts that risk aversion tends to increase the wealth threshold for optimal retirement (FP; DL; Chai *et al.*, 2011; Jang *et al.*, 2013), we show that risk aversion rather decreases the wealth threshold, making individuals perceiving the long-run income risk intensely (Figure 4). That is, the more risk averse individuals feel their labor income stream riskier so that they choose to give it up to not face the labor income risk any longer.

[Insert Figure 5 here.]

If the investment opportunity improves, or equivalently, if the expected return on the stock is high or the volatility of the return is low, or both, the lower wealth threshold for retirement the individuals tends to target for, implying earlier retirement (Figure 5). This is consistent with the empirically observed retirement behaviors of individuals who have opted for early retirement, especially during the stock market boom between 1995 and 2000 in the U.S. (Gustman and Steinmeier, 2002; Gustman *et al.*, 2010).

[Insert Figure 6 here.]

The standard precautionary savings demand against income fluctuations is shown in Figure 6. Naturally, the savings demand becomes larger when income growth volatility is higher, resulting in the lower wealth threshold for retirement.

[Insert Figure 7 here.]

A higher degree of mean reversion, the more transitory income shocks the individual is exposed to, thereby delaying her participation in the stock market and hastening her timing for retirement (Figure 7). In general, when income growth is more transitory the precautionary savings demand is lower. In this case, chances are individuals are inclined to participate in the stock market earlier and delay their retirement than as originally planned when the savings demand is higher. The seemingly counter-intuitive result may be explained as follows. The income-stock cointegration channel changes particularly the optimal retirement strategy in that the wealth threshold for retirement is lower when the income will fall in the long run than when it will rise. The effects of the cointegration on retirement are strengthened when income growth is more transitory with the higher mean revision. Hence, the retirement wealth threshold becomes smaller than when income growth is less transitory, with less need to amass a large amount of wealth, thus suggesting no rush in the stock market participation.

Human Capital. We define the value of human capital as the marginal rate of substitution between labor income and financial wealth (Koo, 1998). In other words, it can be regarded as the individual's subjective marginal value of her labor income as follows:

$$\frac{\partial V(w,I,z)}{\partial I}\Big/\frac{\partial V(w,I,z)}{\partial w}.$$

The value of human capital can be a proxy for a demand for retirement. The retirement demand increases as the human capital value decreases. It becomes the strongest when the human capital value approaches zero, and thereby the individual finds it optimal to enter retirement.

[Insert Figure 8 here.]

Figure 8 generates the empirically plausible hump shape of the human capital value (Carroll and Samwick, 1997; Cocco *et al.*, 2005; Benzoni *et al.*, 2007), which is a salient wealth-related profile. In reality, labor earnings tend to reach the peak between age 50 and 60 and then decline thereafter. A large drop of labor earnings occurs at the individual's retirement. We also demonstrate this fact by the value of human capital. Financially, the importance of receiving labor income by working increases up to some early point in wealth because when wealth is small income is a major source for future consumption, which explains the upward-sloping part of the hump shape. As wealth increases, the total resources for future consumption increase as well, so the human capital value naturally decreases, which explains the downward-sloping part. The individual would then optimally enter retirement especially upon the human capital value becomes zero.

Since the individual targets her retirement wealth at the lower level with the long-run income risk, the peak of the labor income profile is observed at earlier point in wealth.

Marginal Propensity to Consume. Figure 9 shows that the marginal propensities to consume (MPC) out of financial wealth, $\partial c/\partial w$, are not constant, but rather decreasing as wealth increases, which confirms the concavity of consumption function (Carroll and Kimball, 1996).[20]

[Insert Figure 9 here.]

Compared to the existing studies such as DL, the distinct feature of the MPC with the long-run income risk is that we could see not only the overall decrease, but also the important discontinuity and dramatic decrease in the MPC especially when wealth is small. The best way to understand why the MPC is much lower than without the long-run income risk is to associate the effects of the short sale and borrowing constraint with Friedman's (1957) permanent income hypothesis (PIH). In the context of the PIH, when future unhedgeable income shocks are possible, people should save for precautionary reasons (Bewley, 1977; Campbell, 1987). Here, the precautionary savings motive against unhedgeable long-run income risk plays a central and unifying role in our current analysis of the MPC behaviors.

Suppose that an individual's wealth is likely to be expected to decrease in the long run by an amount d because of an unexpected income shock. In the PIH framework, the individual is able to absorb the income shock by using her precautionary savings, thereby recovering the loss of d units of wealth. How could she attain such enough savings? The ability to self-insure for precautionary purposes, of course, depends crucially on the available financial resources that can be used for financing people's current or future consumption needs. However, people considered

[20]Our MPCs are consistent with the standard buffer-stock savings literature (Deaton, 1991; Carroll, 1992) that range from 0.04 to 0.07.

in this paper are borrowing and short sale constrained, so they are limited to borrow or short sell to secure such enough financial resources in preparation for satisfying their consumption needs. The amount of present consumption they would be willing to give up now to receive one more unit of future consumption becomes larger than without the long-run income risk. Therefore, in a prompt response to the reduced total resources, they choose to cut down on consumption itself significantly for amassing enough wealth. This would make the consumption function much less concave and hence, result in the much lower MPC with the long-run income risk than without it.

Interestingly, contrary to Wang *et al.* (2016), the effects of downward jumps in labor income could influence a decrease in the MPC rather than its increase, regardless of the presence of the long-run income risk. In addition to the long-run income risk, the other major departure of our model from Wang *et al.* (2016) is that we consider one more dimension among important financial decisions: investment in the stock market. Not only idiosyncratic and unhedgeable jump risk in the labor market, but systematic risk in the stock market also are all uninsurable, so the inability to hedge increases the precautionary savings motive further. Due to the strong precautionary savings motive, the individual with these two risk sources relatively consumes less out of the additional value of wealth than with jump income risk only. The strong consumption demand with a high MPC as in Wang *et al.* (2016) seems too good to be true in the joint consideration of jump income risk and market risk.

Optimal Risky Investment. We investigate the effects of the long-run income risk on the optimal decision with retirement flexibility to buy more or fewer risky assets. We find that the risky investment decision crucially depends on both the extent of the long-run income risk and retirement decision itself.

Figure 10 represents the proportion of financial wealth invested in the stock market (or the portfolio share) as a function of financial wealth. Existing life-cycle models have demonstrated that the portfolio share is a decreasing function of wealth (Cocco *et al.*, 2005; FP; DL; Jang *et al.*, 2013; Bensoussan *et al.*, 2016; Jang *et al.*, 2019; Jang *et al.*, 2020). Labor income has been regarded as a substitute for (implicit) riskless asset holdings (Jagannathan and Kocherlakota, 1996; Heaton and Lucas, 1997), so an individual's resources available for risky investment are relatively larger than without labor income. Hence, the individual typically is able to take on risk in the stock market even wealth is small, thereby relying on her total resources (financial wealth + human capital). Indeed, in DL without the long-run income risk, the portfolio share decreases with wealth (Figure 10).

[Insert Figure 10 here.]

However, the conventional tendency of portfolio share is no longer applicable when we incorporate the long-run income risk in the investment decision. Consistent with the empirical observations that the portfolio share rather increases in wealth (Polkovnichenko, 2007; Wachter and Yogo, 2010; Calvet and Sodini, 2014; Figure 1), in our model with the long-run income risk, there exists a target wealth (before approaching the retirement threshold) under which individuals do not invest in the stock market, whereas above which they increase the portfolio share as wealth increases (Figure 10). The former investment behavior explains the non-participation puzzle and the latter one generates the empirically plausible portfolio share that is an increasing and concave function of wealth. A large body of literature provides various explanations for either the non-participation puzzle or the risky asset-allocation result that is increasing and concave in wealth.[21] To our best knowledge, our work is a first attempt to explain these two empirical observations jointly especially through the precautionary savings channel with the long-run income risk.

In our previous analysis of the MPC, we have obtained the much lower MPC than without the long-run income risk due to the strong demand for precautionary savings. Given income is a major staple of the relatively low-wealth people, they should concern themselves with diversifying the negative effects of the long-run income risk on their expected future income by saving more for precautionary purposes, thereby absorbing part of the income shock. Thus, precaution makes the wealth-poor much more conservative than the wealthy when taking on risk in the stock market. As long as substantial precautionary savings are required, the low-wealth people find it optimal to not invest in the stock market and to choose to save in the bond market up to

[21]As to the resolution of the non-participation puzzle, please refer to Vissing-Jorgensen (2002), Hong *et al.* (2004), Guiso and Jappelli (2005), Gomes and Michaelides (2005), Benzoni *et al.* (2007), Gormley *et al.* (2010), and Lynch and Tan (2011). Concerning the increasing and concave portfolio share, please see Polkovnichenko (2007), Wachter and Yogo (2010), and Calvet and Sodini (2014).

the target wealth. Relative to the wealth-poor, the income is a smaller staple of the high-wealth people, so they have greater tolerance for taking risk in the stock market than the low-wealth people. Further, such a risk taking is compensated to increase their expected returns on the total investment and thus, the negative effects of the long-run income risk can be partially absorbed by investing in the stock market. Hence, the wealthy above the target wealth would rather not be concerned with diversification anymore and find it optimal to participate in the stock market, increasing their risky portion as wealth increases.

We also obtain the reversed result about the effects of retirement flexibility on the risky investment (Figure 11). Taking on more risk in the stock market by adjusting retirement timing (FP; DL; Chai *et al.*, 2011; Jang *et al.*, 2013; Bensoussan *et al.*, 2016) is no longer applicable with the long-run income risk. Rather, retirement flexibility makes the optimal portfolio invest less in the stock market. The flexibility of supplying labor for a longer period of time to hedge against stock market risk rather exposes individuals to the greater income risk in the long run, reducing the stock investment. This is because income itself fluctuates substantially with the market in the long run. To manage risk exposure to the long-run income risk, the optimal portfolio should be more geared towards riskless bonds compared to the case in which retirement flexibility is not allowed.

[Insert Figure 11 here.]

The investment opportunity affects the individual's optimal portfolio share (Figure 12). As expected, the individual is willing to increase her portfolio share if the investment opportunity improves, i.e., when the expected return on the stock is high or the volatility of the return is low, or both.

[Insert Figure 12 here.]

Changes in risk aversion also affect the optimal risky asset allocation (Figure 13). Reduced risk aversion leads the individual to increase her portfolio share. For individuals with low risk aversion, the long-run income risk is relatively likely to be negligible, so that their optimal risky investment decision becomes increasingly aggressive as her wealth increases. In contrast, individuals with high risk aversion perceives the long-run income risk is perceived intensely, so their optimal decision becomes conservative for most levels of wealth, suggesting non-participation in the stock market.

[Insert Figure 13 here.]

The effects of income growth volatility on the optimal portfolio share are not always obvious, but rather are non-monotonic, depending on the presence of the long-run income risk (Figure 14). The portfolio share decreases as income growth volatility increases without the long-run income risk, whereas the portfoio share increases in income growth volatility with the long-run income risk. The increased income growth volatility leads to the increased background risk, giving rise to a demand for hedging against the unhedgeable income risk (Bodie *et al.*, 1992; Heaton and Lucas, 1997; Koo, 1998). Such an increased demand for hedging can be determined by two considerations: a precautionary savings motive that decreases risky investment and a risk diversification motive that increases risky investment. It turns out that the precautionary savings motive dominates the risk diversification motive in the absence of the long-run income risk, while the opposite is true in the presence of the long-run income risk.

[Insert Figure 14 here.]

The effects of mean reversion on the portfolio share are that a larger degree of mean reversion makes the income risk more transitory (not permanent), reducing the risk diversification motive and thus, decreasing risky investment (Figure 15).

[Insert Figure 15 here.]

Recursive Utility Preferences. Extending the expected utility with CRRA, we now separate risk aversion from the elasticity of intertemporal substitution (EIS) by considering non-expected recursive utility (Epstein and Zin, 1989; Weil, 1990). The continuous-time formulation of this non-expected utility is given by (Duffie and Epstein, 1992)

$$f(c, V) = \frac{\beta}{1 - \psi^{-1}} \Big(c^{1-\psi^{-1}} \{(1-\gamma)V\}^{\frac{\psi^{-1}-\gamma}{1-\gamma}} - (1-\gamma)V \Big),$$

where $\psi > 0$ is the coefficient of EIS. When $\psi = 1/\gamma$, the recursive utility reduces to the standard CRRA utility.

The effects of changes in EIS show that a higher EIS, the lower wealth thresholds for stock market participation and voluntary retirement (Figure 16) the individual optimally targets to set. Without the EIS consideration, the individual with a higher risk aversion typically delays her stock market participation due to the increased precautionary savings motive. The individual with a higher EIS is willing to have less consumption now reflecting the precautionary savings motive as in the case without the EIS consideration, but rather than riskless savings she tends to participate earlier and invest in the stock market to accumulate more wealth for earlier retirement in the future.

[Insert Figure 16 here.]

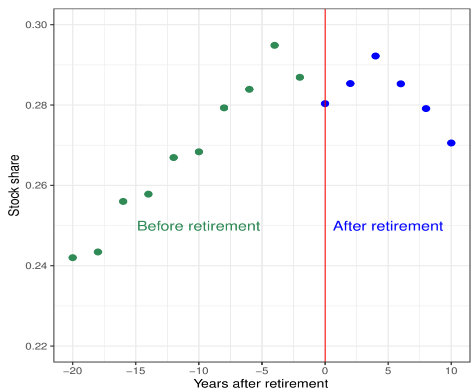
5 Conclusion

This paper examines an individual's life-cycle portfolio choices and retirement decision in the presence of long-run labor income risk. It models long-run labor income as being co-integrated with stock price. It claims that the risk associated with long-run income and stock price offers an explanation for stock investment to increase with retirement age and low stock market participation for young individuals and an increasing concave stock proportion in wealth.

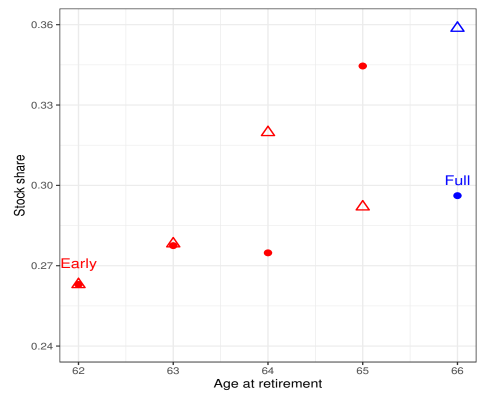
An important missing component in the model is housing asset. While it varies over time, in general, about two-thirds of U.S. households own their primary residence. For many of them, housing asset is the largest single asset in their household portfolio. Therefore, the observed pattern in life-cycle portfolio choices, stock market participation, and the retirement decision as well as their relation to household wealth could well be driven by their homeownership decision and status.

Secondly, the short-run labor income risk considered in the paper only allows a negative or downward large income drop having the jump component. However, in real life an individual may also receive a positive or upward large income jump when she changes jobs. In particular, it happens after she completes her education or professional training. It would be, thus, of utmost importance to further investigate the implications of the rigidity of wages for life-cycle portfolio choices and retirement decision.

Lastly, some institutional details on retirement benefits will have important implications on retirement decision. In practice, retirement decision is determined by replacement income after retirement in addition to the levels of individual wealth. While the age for receiving full social security benefits is 67 currently in the U.S., individuals may choose to receive their social security payment starting at age 59 1/2 at a partial benefit. On the other hand, an individual can delay and receive increasing retirement benefit up to age 70. It would be great if future research can show an age-profile of retirement for the U.S. population.



(a)



(b)

Figure 1: **Stock investment near retirement (using HRS dataset).** The left panel (Figure 1 (a)) plots the stock share at different year near retirement. The year after retirement is calculated as the difference between current age and the age at retirement. Positive value on x-axis shows the age after retirement while negative value shows the age before retirement. It is observed that the stock share before retirement is an increasing and concave function of wealth. The right panel (Figure 1 (b)) plots the stock share at different retirement age. In the U.S., the current full retirement (benefit) age is 66 years, and early retirement is 62 years. The dots show the stock investment at retirement with respect to different retirement ages. The upper triangles show the stock investment at age 62 with respect to different retirement ages. For example, the average stock share at retirement for those individuals who choose to retire early at age 62 is 0.263, while it increases to 0.345 for those individuals who choose to retire at age 65. It is observed that the stock share increases with retirement age, i.e., individuals who retire early invest less in the stock market.

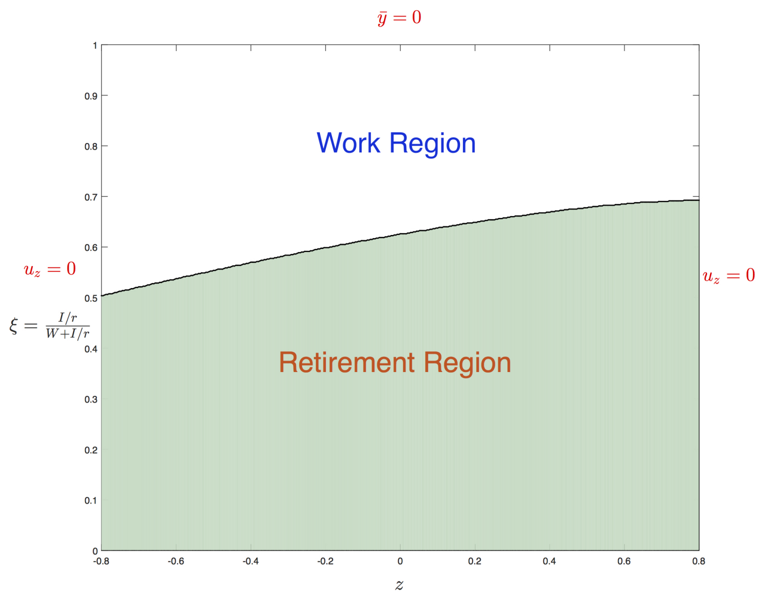
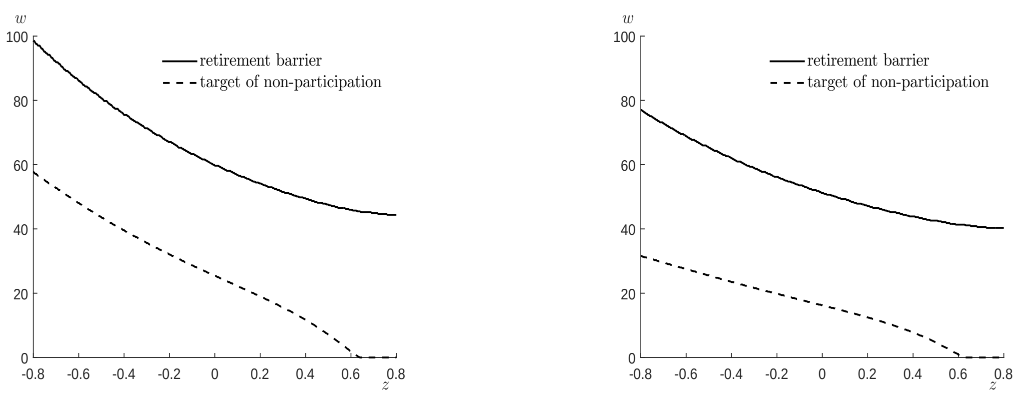


Figure 2: **Work and retirement regions.**

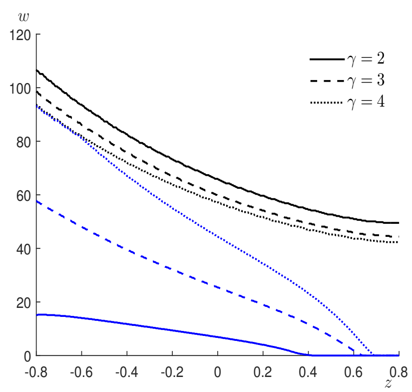


(a) without downward jumps in labor income

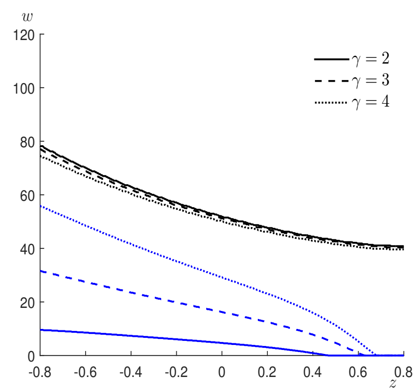
(b) with downward jumps in labor income

Figure 3: **Sensitivity analysis of threshold wealth for voluntary retirement (retirement barrier) and target wealth-to-income ratio for stock market participation (target of non-participation) with respect to changes in initial value z of additional state variable representing the long-run income risk.**

Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $B = 2$ (post-retirement leisure preference), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income)



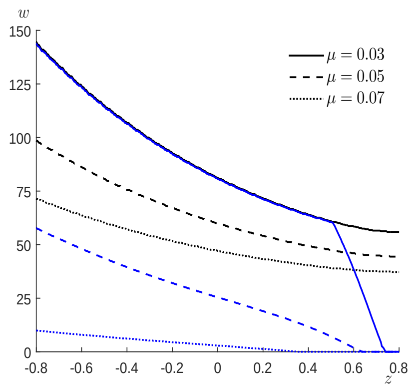
(a) without downward jumps in labor income



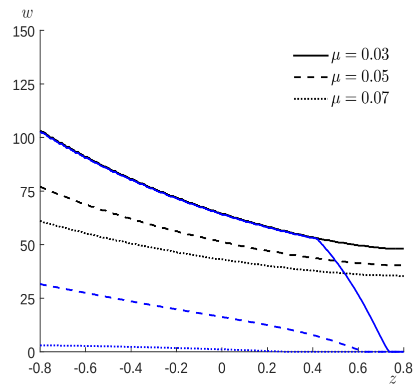
(b) with downward jumps in labor income

Figure 4: **Sensitivity analysis of threshold wealth for voluntary retirement (retirement barrier) and target wealth-to-income ratio for stock market participation (target of non-participation) with respect to changes in risk aversion.** The black lines and the blue lines represent, respectively, retirement barrier and target of non-participation.

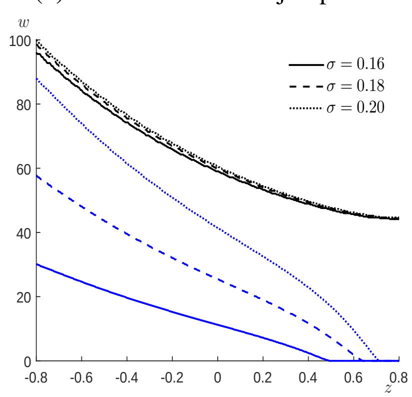
Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $B = 2$ (post-retirement leisure preference), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\bar{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).



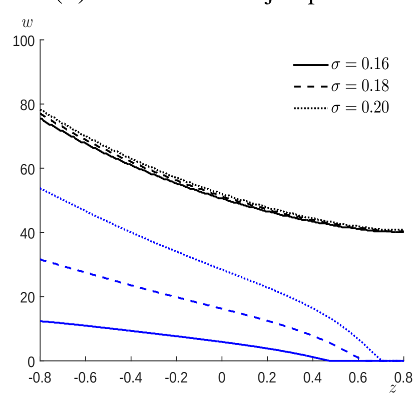
(a) without downward jumps in labor income



(b) with downward jumps in labor income



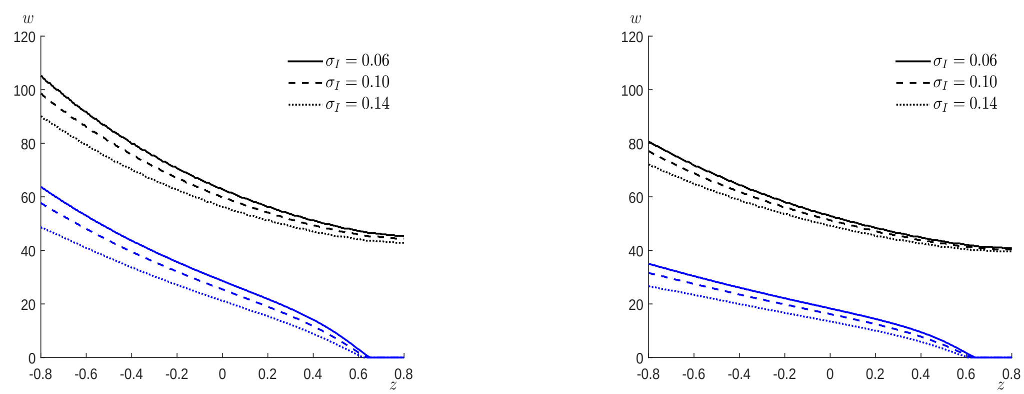
(c) without downward jumps in labor income



(d) with downward jumps in labor income

Figure 5: **Sensitivity analysis of threshold wealth for voluntary retirement (retirement barrier) and target wealth-to-income ratio for stock market participation (target of non-participation) with respect to changes in investment opportunity.** The black lines and the blue lines represent, respectively, retirement barrier and target of non-participation.

Basic parameters: $r = 1\%$ (risk-free interest rate), $\gamma = 3$ (relative risk aversion), $B = 2$ (post-retirement leisure preference), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\bar{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).

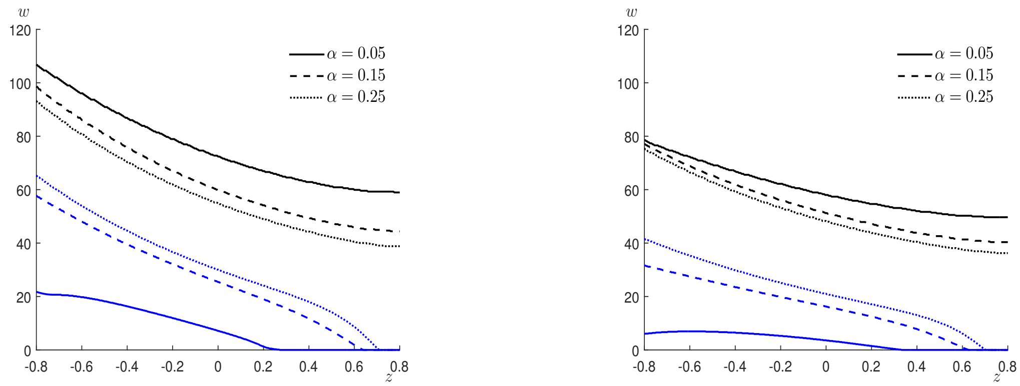


(a) without downward jumps in labor income

(b) with downward jumps in labor income

Figure 6: **Sensitivity analysis of threshold wealth for voluntary retirement (retirement barrier) and target wealth-to-income ratio for stock market participation (target of non-participation) with respect to changes in income growth volatility.** The black lines and the blue lines represent, respectively, retirement barrier and target of non-participation.

Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $B = 2$ (post-retirement leisure preference), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\bar{z} = 0$ (long-term mean).

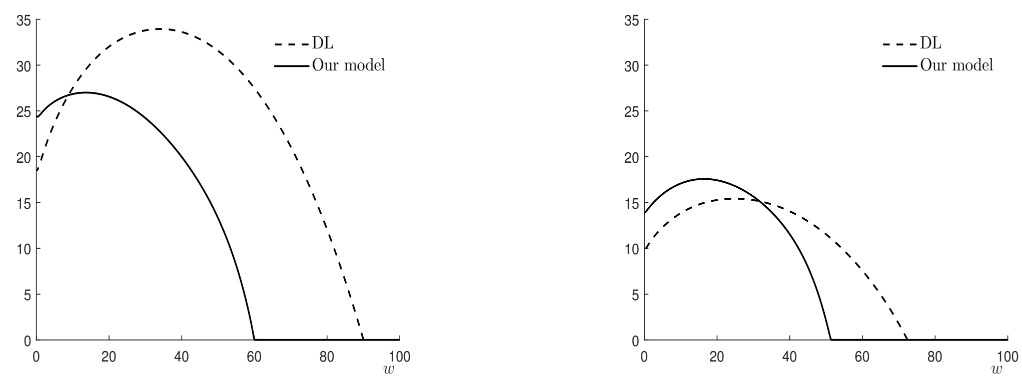


(a) without downward jumps in labor income

(b) with downward jumps in labor income

Figure 7: **Sensitivity analysis of threshold wealth for voluntary retirement (retirement barrier) and target wealth-to-income ratio for stock market participation (target of non-participation) with respect to changes in mean reversion.** The black lines and the blue lines represent, respectively, retirement barrier and target of non-participation.

Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $B = 2$ (post-retirement leisure preference), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\overline{z} = 0$ (long-term mean).

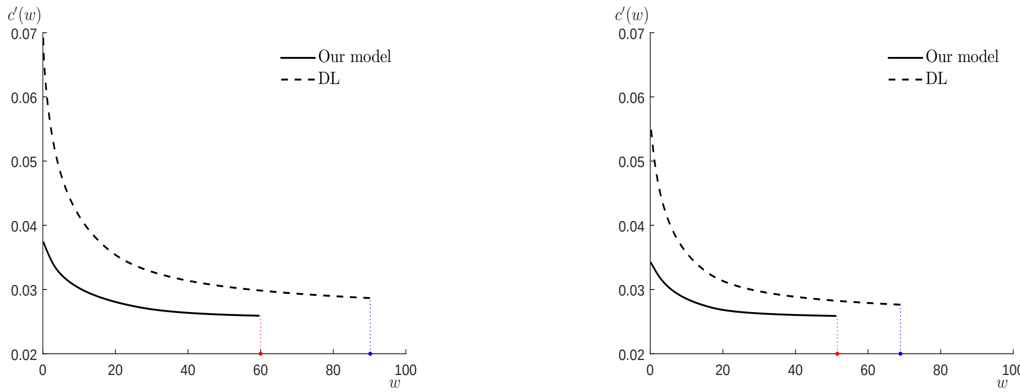


(a) without downward jumps in labor income

(b) with downward jumps in labor income

Figure 8: **Human capital value as a function of wealth-to-income ratio.**

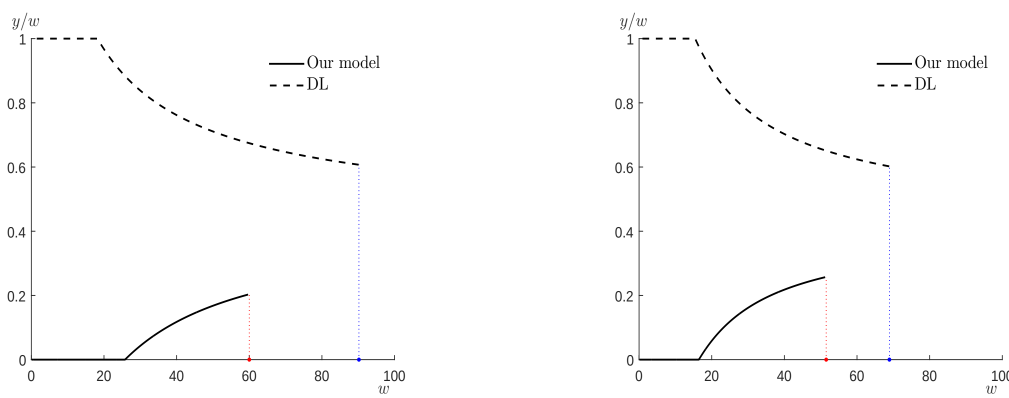
Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).



(a) without downward jumps in labor income (b) with downward jumps in labor income

Figure 9: **Marginal propensities to consume out of financial wealth.**

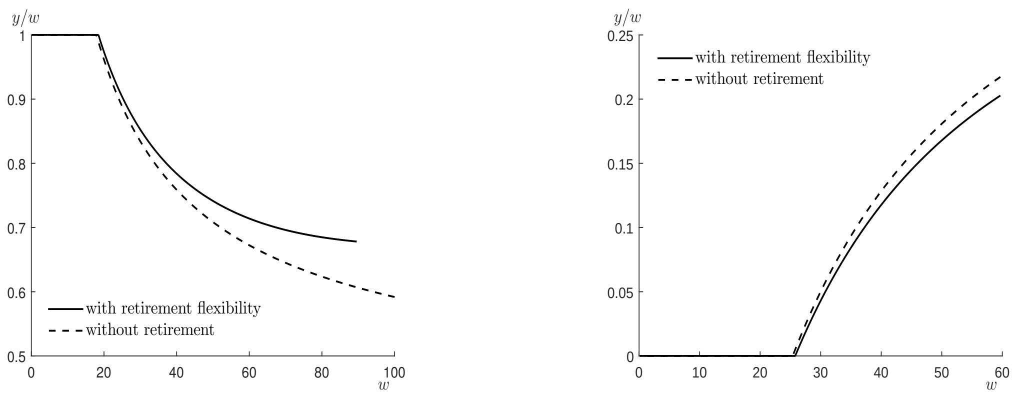
Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $B = 2$ (post-retirement leisure preference), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).



(a) without downward jumps in labor income (b) with downward jumps in labor income

Figure 10: **Proportion of financial wealth invested in the stock market (or portfolio share) as a function of wealth-to-income ratio.**

Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $B = 2$ (post-retirement leisure preference), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).

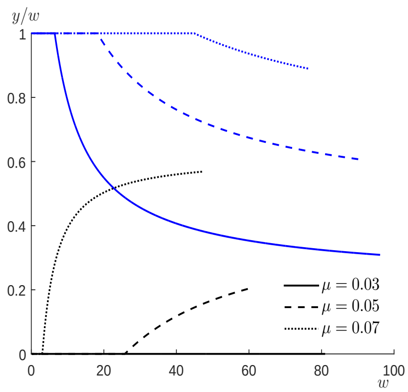


(a) without cointegration

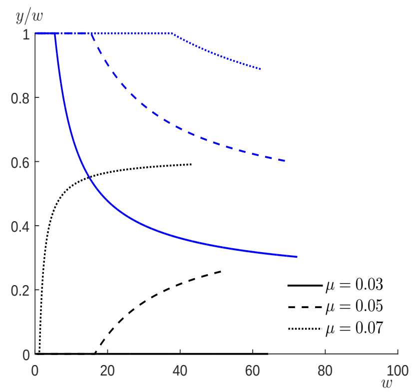
(b) with cointegration

Figure 11: **Effects of retirement flexibility on portfolio share.**

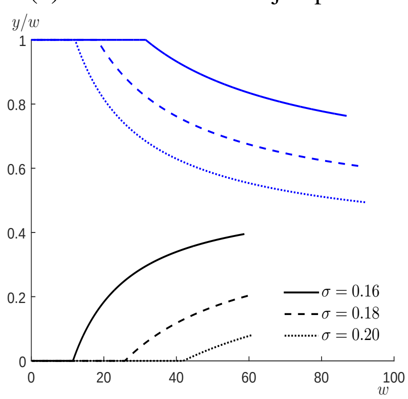
Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\delta_D = 0$ (disastrous labor income shock intensity, $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).



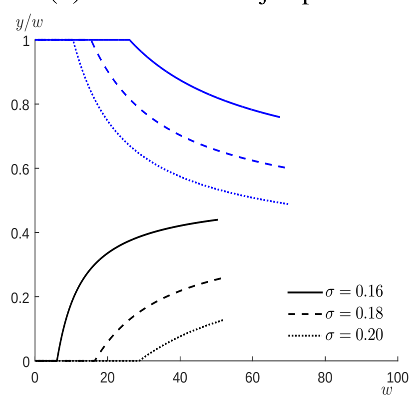
(a) without downward jumps in labor income



(b) with downward jumps in labor income



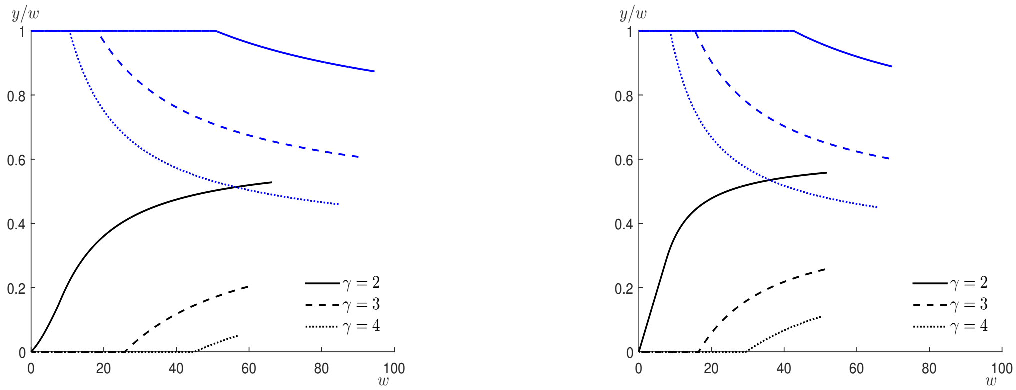
(c) without downward jumps in labor income



(d) with downward jumps in labor income

Figure 12: **Effects of investment opportunity on portfolio share.** The blue colored lines represent DL, while the black colored lines represent our model.

Basic parameters: $r = 1\%$ (risk-free interest rate), $\gamma = 3$ (relative risk aversion), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\delta_D = 0$ (disastrous labor income shock intensity, $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).

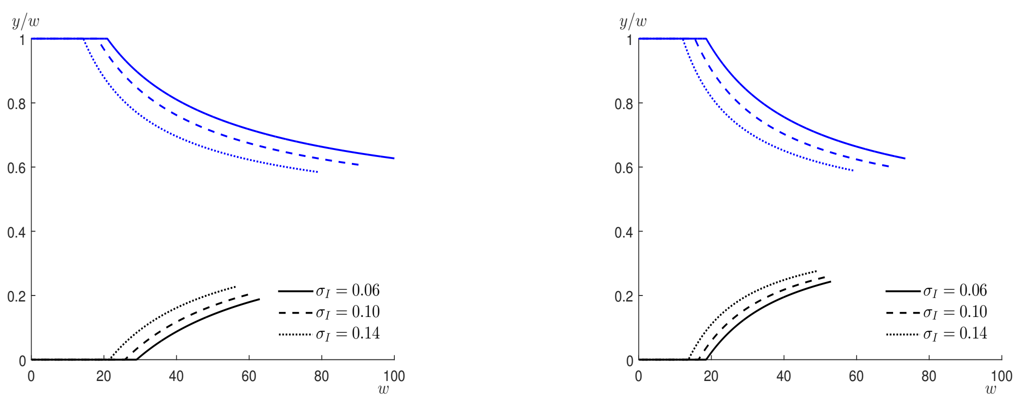


(a) without downward jumps in labor income

(b) with downward jumps in labor income

Figure 13: **Effects of risk aversion on portfolio share.** The blue colored lines represent DL, while the black colored lines represent our model.

Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\delta_D = 0$ (disastrous labor income shock intensity, $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).

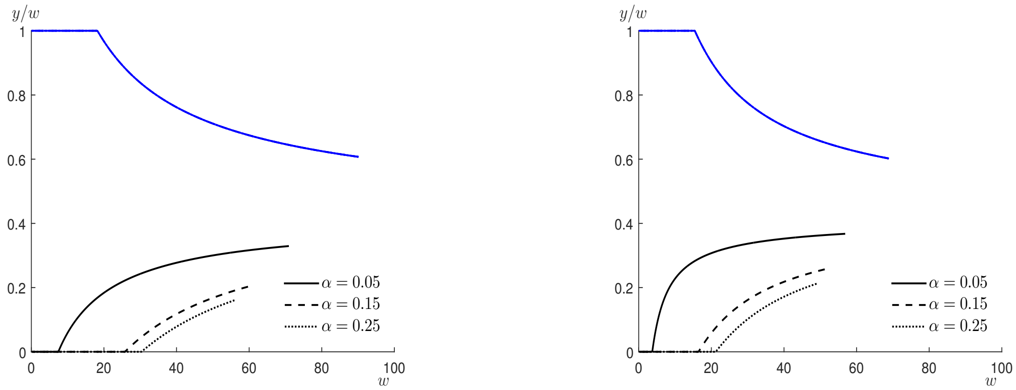


(a) without downward jumps in labor income

(b) with downward jumps in labor income

Figure 14: **Effects of income growth volatility on portfolio share.** The blue colored lines represent DL, while the black colored lines represent our model.

Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\delta_D = 0$ (disastrous labor income shock intensity, $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).

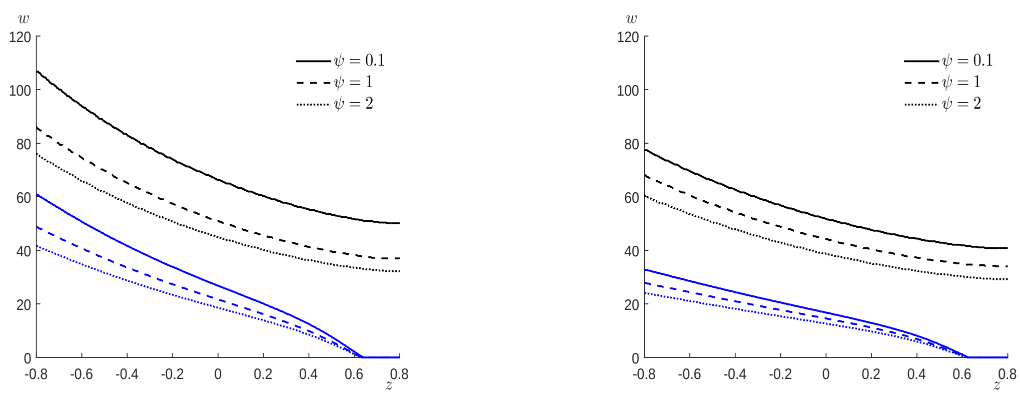


(a) without downward jumps in labor income

(b) with downward jumps in labor income

Figure 15: **Effects of mean reversion on portfolio share.** The blue colored lines represent DL, while the black colored lines represent our model.

Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\delta_D = 0$ (disastrous labor income shock intensity, $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income).



(a) without downward jumps in labor income

(b) with downward jumps in labor income

Figure 16: **Sensitivity analysis of threshold wealth for voluntary retirement (retirement barrier) and target wealth-to-income ratio for stock market participation (target of non-participation) with respect to changes in elasticity of intertemporal substitution.** The black lines and the blue lines represent, respectively, retirement barrier and target of non-participation.

Basic parameters: $r = 1\%$ (risk-free interest rate), $\mu = 5\%$ (expected rate of stock return), $\sigma = 18\%$ (stock volatility), $\gamma = 3$ (relative risk aversion), $B = 2$ (post-retirement leisure preference), $\beta = 4\%$ (subjective discount rate), $\mu_I = 0.5\%$ (expected rate of income growth), $\sigma_I = 10\%$ (volatility on income growth), $I = 1$ (annual rate of labor income), $\kappa = 80\%$ (recovery parameter), $\sigma_z = \sigma$ (volatility on difference between the logs of stock price and labor income), $\alpha = 15\%$ (degree of mean reversion), $\overline{z} = 0$ (long-term mean), $\delta_D = 5\%$ (intensity of downward jumps in labor income)

Appendix

The Health and Retirement Study (HRS) Dataset Description

To analyze individual investment behavior near retirement, we use the Health and Retirement Study (HRS) dataset. The HRS dataset contains large number of interviewers, the majorities of which are old people. Therefore it becomes appropriate when analyzed the investment behavior near retirement. The concrete data set we use is RAND HRS Longitudinal File 2016 (v.1), which incorporates HRS core interviews from 1992 to 2016. We use total non-housing wealth (HwATOTN) and net value of stocks, mutual funds, and investment trusts (HwASTCK) as the wealth and equity investment, respectively. The status of retirement is obtained by the self retirement report (RwSAYRET). Using the age at 1992 interview (R1AGEY_B), individual ID (HHIDPN), and the number of waves, we can retrieve the age of interviewers in each interview wave. Then we can obtain the age at retirement for each interviewer. It is also necessary to keep track of status of death. All statistic calculations are conditional on survival. There are 2,636 different individuals with valid self reported age of retirement, or 31,662 individual-year observations.

Proofs of Theorems

The proofs of Theorem 3.1, 3.2, 3.3 are straightforward if we prove Theorem 3.4, so we will focus on the proof of Theorem 3.4.

We apply the convex-duality approach developed by Bensoussan *et al.* (2016) to solve Model 4. For a fixed stopping time τ, we define

$$V_\tau(w,I) \equiv \sup_{(c,y,\tau)} \mathbb{E}\Big[\int_0^\tau e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt + e^{-\beta\tau}\int_\tau^\infty e^{-\beta(t-\tau)}\frac{(Bc_t)^{1-\gamma}}{1-\gamma}dt\Big],$$

which is subject to (1), (8), and (10). Then the value function $V(w,I)$ for Model 4 is

$$V(w,I) \equiv \sup_\tau V_\tau(w,I), \tag{15}$$

which is the so-called optimal stopping problem. The variational inequality approach (Bensoussan and Lions, 1982; Øksendal, 2007) results in the following inequality associated with the optimal stopping problem (15): for any $w \geq 0$, $I \geq 0$,

$$\max_{(c,y)}\Big\{\mathcal{L}V(w,I), \quad R(w) - V(w,I)\Big\} = 0, \tag{16}$$

where the differential operation $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}V =& -\beta V + \{rw - c + I\}V_w + y\sigma\theta V_w + \frac{1}{2}y^2\sigma^2 V_{ww}\\ &+ \mu_I I V_I + \frac{1}{2}\sigma_I^2 I^2 V_{II} + yI\sigma\sigma_I V_{wI} + \frac{c^{1-\gamma}}{1-\gamma} + \delta_D E[V(w,\kappa I) - V(w,I)].\end{aligned}$$

Here, the subscripts of V denote its partial derivatives. Substituting the first-order conditions for consumption c and risky investment y in (16),

$$c = V_w^{-1/\gamma}, \;\; y = -\frac{\theta}{\sigma}\frac{V_w}{V_{ww}} - \frac{I\sigma_I}{\sigma}\frac{V_{wI}}{V_{ww}}, \tag{17}$$

we rewrite (16) as

$$\max\Big\{\mathcal{L}V(w,I), \quad R(w) - V(w,I)\Big\} = 0, \tag{18}$$

where the differential operation $\mathcal{L}$ is given by

$$\mathcal{L}V = -\beta V + (rw + I)V_w + \frac{\gamma}{1-\gamma}V_w^{-1/\gamma} - \frac{1}{2}\Big(\theta V_w + \sigma_I I V_{wI}\Big)^2 \frac{1}{V_{ww}} + \mu_I I V_I + \frac{1}{2}\sigma_I^2 I^2 V_{II} + yI\sigma\sigma_I V_{wI} + \delta_D E[V(w, \kappa I) - V(w, I)].$$

By homogeneity property, we can reduce one dimension by the following transformation:

$$V(w, I) = I^{1-\gamma}\phi(z), \;\; z = \frac{w}{I}.$$

Due to the transformation stated above, the variational inequality (18) can be restated as

$$\max\Big\{\mathcal{L}_1\phi(z), \quad \frac{B^{1-\gamma}\overline{K}^{-\gamma}}{1-\gamma}z^{1-\gamma} - \phi(z)\Big\} = 0, \tag{19}$$

where

$$\mathcal{L}_1\phi(z) = -\beta_2\phi(z) + (\beta_1 z + 1)\phi'(z) + \frac{\gamma}{1-\gamma}\phi'(z)^{1-1/\gamma} - \frac{1}{2}\beta_3^2\frac{\phi'(z)^2}{\phi''(z)} + \delta_D E[\kappa^{1-\gamma}\phi(z/\kappa) - \phi(z)],$$

$$\beta_1 \equiv r - \mu_I + \sigma_I\theta,$$
$$\beta_2 \equiv \beta - \mu_I(1-\gamma) + \frac{1}{2}\gamma(1-\gamma)\sigma_I^2,$$
$$\beta_3 \equiv \gamma\sigma_I - \theta.$$

We now introduce dual variable λ and dual function $G(\lambda)$ defined as

$$\lambda(z) \equiv \phi'(z), \;\; G(\lambda(z)) \equiv \Big(w + \frac{I}{\beta_1}\Big)\Big/ I = z + \frac{1}{\beta_1}, \tag{20}$$

where the first derivative $\phi'(z)$ and the total wealth-to-income ratio are used for the dual variable and the dual function. The following relations then hold:

$$G'(\lambda(z))\lambda'(z) = 1, \;\; G''(\lambda(z))\lambda'(z)^2 + G'(\lambda(z))\lambda''(z) = 0. \tag{21}$$

If we differentiate the both sides of (19) with respect to z, we obtain

$$\max\Big\{\mathcal{L}_2\phi'(z), \quad B^{1-\gamma}\overline{K}^{-\gamma}z^{-\gamma} - \phi'(z)\Big\} = 0, \tag{22}$$

where

$$\begin{aligned}\mathcal{L}_2\phi'(z) = &-\beta_2\phi'(z) + \beta_1\phi'(z) + \beta_1\Big(z + \frac{1}{\beta_1}\Big)\phi''(z) - \phi'(z)^{-1/\gamma}\phi''(z)\\ &-\frac{1}{2}\beta_3^2\frac{2\phi'(z)\phi''(z)^2 - \phi'(z)^2\phi'''(z)}{\phi''(z)^2} + \delta_D E[\kappa^{-\gamma}\phi'(z/\kappa) - \phi'(z)].\end{aligned}$$

The variational inequality (22) can be then restated by (20) and (21) as

$$\max\Big\{\mathcal{L}_3 G(\lambda), \quad B^{1-\gamma}\overline{K}^{-\gamma}z^{-\gamma} - \lambda\Big\} = 0, \tag{23}$$

where

$$\begin{aligned}\mathcal{L}_3 G(\lambda) = -\frac{1}{2}\beta_3^2\lambda^2 G''(\lambda) - \{\beta_3^2 + \beta_2 - \beta_1\}\lambda G'(\lambda) + \beta_1 G(\lambda)\\ - \lambda^{-1/\gamma} + \delta_D E[\kappa^{-\gamma}\lambda(z/\kappa) - \lambda(z)]G'(\lambda).\end{aligned}$$

The work region and retirement region are determined by the so-called critical wealth-to-income ratio $\overline{z}$ associated with $\underline{\lambda}$ by

$$\underline{\lambda} = \phi'(\overline{z}) = B^{1-\gamma}\overline{K}^{-\gamma}\overline{z}^{-\gamma}, \tag{24}$$

so that the variational inequality (23) can be solved by finding a free boundary $\underline{\lambda}$ (or equivalently, an optimal retirement boundary):

$$\mathcal{L}_3 G(\lambda) = 0, \;\; \underline{\lambda} < \lambda < \overline{\lambda}, \tag{25}$$

which is subject to

$$\begin{cases} G(\underline{\lambda}) = \overline{z} + \dfrac{1}{\beta_1} = B^{1/\gamma-1}\overline{K}^{-1}\underline{\lambda}^{-1/\gamma} + \dfrac{1}{\beta_1},\\ G(\overline{\lambda}) = \dfrac{1}{\beta_1},\\ G'(\overline{\lambda}) = 0, \end{cases} \tag{26}$$

where the last two equality results from (10) implying that optimal risky investment y should be zero as wealth approaches zero (Dybvig and Liu, 2010). Also, we know that given the critical wealth level $\overline{w}$ for optimal retirement,

$$V(\overline{w}, I) = R(\overline{w}) = \frac{B^{1-\gamma}\overline{K}^{-\gamma}}{1-\gamma}\overline{w}^{1-\gamma},$$

or equivalently,

$$\phi(\overline{z}) = \frac{B^{1-\gamma}\overline{K}^{-\gamma}}{1-\gamma}\overline{z}^{1-\gamma}. \tag{27}$$

Summarizing, all we need to do from now on is to solve the free boundary problem (25) with (27) by determining $\underline{\lambda}$ associated with optimal retirement and $\overline{\lambda}$ associated with borrowing constraints.

We conjecture a general solution to (25) with (26) and (27) for $\underline{\lambda} < \lambda < \overline{\lambda}$ by

$$G(\lambda) = \frac{1}{\hat{A}+\delta_D}\lambda^{-1/\gamma} + \eta(\lambda)\lambda^{-\alpha_{\delta_D}} + \eta^*(\lambda)\lambda^{-\alpha^*_{\delta_D}}, \tag{28}$$

which is subject to

$$\eta'(\lambda)\lambda^{-\alpha_{\delta_D}} + (\eta^*(\lambda))'\lambda^{-\alpha^*_{\delta_D}} = 0,$$

where

$$\hat{A} = \frac{\gamma-1}{\gamma}\Big(\beta_1 + \frac{\beta_3^2}{2\gamma}\Big) + \frac{\beta_2}{\gamma},$$

and $\alpha_{\delta_D} > 1$ and $-1 < \alpha^*_{\delta_D} < 0$ are the two constants satisfying the following characteristic equation:

$$CE(x; \delta_D) \equiv -\frac{1}{2}\beta_3^2 x(x-1) + (\beta_2 + \delta_D - \beta_1)\alpha + \beta_1 = 0.$$

Substituting the general solution (28) in (25), we obtain that for $\underline{\lambda} < \lambda < \overline{\lambda}$,

$$G(\lambda) = \frac{1}{\hat{A}+\delta_D}\lambda^{-1/\gamma} + C_{\delta_D}\lambda^{-\alpha_{\delta_D}} + C^*_{\delta_D}\lambda^{-\alpha^*_{\delta_D}} + \mathrm{PS}(\lambda; \delta_D), \tag{29}$$

where

$$\mathrm{PS}(\lambda; \delta_D) = \mathrm{PS1}(\lambda; \delta_D) + \mathrm{PS2}(\lambda; \delta_D),$$

$$PS1(\lambda;\delta_D) = \frac{2\delta_D(\alpha_{\delta_D}-1)}{\beta_3^2(\alpha_{\delta_D}-\alpha^*_{\delta_D})}\int_{\underline{\lambda}}^{\lambda}\mu^{\alpha_{\delta_D}-2}E\Big[\kappa^{1-\gamma}\phi\big((G(\mu)-1/\beta_1)/\kappa\big)\Big]d\mu < 0,$$

$$PS2(\lambda;\delta_D) = \frac{2\delta_D(\alpha^*_{\delta_D}-1)}{\beta_3^2(\alpha_{\delta_D}-\alpha^*_{\delta_D})}\int_{\lambda}^{\overline{\lambda}}\mu^{\alpha^*_{\delta_D}-2}E\Big[\kappa^{1-\gamma}\phi\big((G(\mu)-1/\beta_1)/\kappa\big)\Big]d\mu < 0,$$

C_{δ_D} and $C^*_{\delta_D}$ are positive constants to be determined with $\underline{\lambda}$ and $\overline{\lambda}$ according to (26) and (27), which are the so-called value matching and smooth pasting conditions. In particular, (27) can be further used by expressing $\phi(z)$ with dual variable λ and dual function $G(\lambda)$. More specifically, for $0 < z < \overline{z}$, the variational inequality (19) implies $\mathcal{L}_1\phi(z) = 0$, so that

$$\begin{aligned}\phi(z) &= \frac{1}{\beta_2+\delta_D}\Big[\beta_1\Big(z+\frac{1}{\beta_1}\Big)\phi'(z) + \frac{\gamma}{1-\gamma}\phi'(z)^{1-1/\gamma} - \frac{1}{2}\beta_3^2\frac{\phi'(z)^2}{\phi''(z)} + \delta_D E[\kappa^{1-\gamma}\phi(z/\kappa)]\Big] \\ &= \frac{1}{\beta_2+\delta_D}\Big[\beta_1 G(\lambda)\lambda + \frac{\gamma}{1-\gamma}\lambda^{1-1/\gamma} - \frac{1}{2}\beta_3^2\lambda^2 G'(\lambda) + \delta_D E[\kappa^{1-\gamma}\phi(z/\kappa)]\Big].\end{aligned}$$

Hence, when $z = \overline{z}$ (27) shows that

$$\frac{B^{1-\gamma}\overline{K}^{-\gamma}}{1-\gamma}\overline{z}^{1-\gamma} = \frac{1}{\beta_2+\delta_D}\Big[\beta_1 G(\underline{\lambda})\underline{\lambda} + \frac{\gamma}{1-\gamma}\underline{\lambda}^{1-1/\gamma} - \frac{1}{2}\beta_3^2\underline{\lambda}^2 G'(\underline{\lambda})\Big] + \frac{\delta_D}{\beta_2+\delta_D}\frac{B^{1-\gamma}\overline{K}^{-\gamma}}{1-\gamma}\overline{z}^{1-\gamma},$$

accordingly,

$$\beta_2\frac{B^{1-\gamma}\overline{K}^{-\gamma}}{1-\gamma}\overline{z}^{1-\gamma} = \beta_1 G(\underline{\lambda})\underline{\lambda} + \frac{\gamma}{1-\gamma}\underline{\lambda}^{1-1/\gamma} - \frac{1}{2}\beta_3^2\underline{\lambda}^2 G'(\underline{\lambda}).$$

Using (24), we finally obtain

$$\beta_2\frac{B^{1/\gamma-1}\overline{K}^{-1}}{1-\gamma}\underline{\lambda}^{1-1/\gamma} = \beta_1 G(\underline{\lambda})\underline{\lambda} + \frac{\gamma}{1-\gamma}\underline{\lambda}^{1-1/\gamma} - \frac{1}{2}\beta_3^2\underline{\lambda}^2 G'(\underline{\lambda}). \tag{30}$$

Summarizing, the free boundary problem (25) (equivalent to the variational inequality (23)) can be solved by (29) with C_{δ_D}, $C^*_{\delta_D}$, $\underline{\lambda}$, $\overline{\lambda}$ to be determined with (26) and (30).

Deriving the optimal consumption and risky asset allocation strategies is attained by using their first-order conditions given in (17). The first-order conditions can be rewritten by dual variable λ and dual function $G(\lambda)$ as

$$c = V_w^{-1/\gamma} = I\phi'(z)^{-1/\gamma} = I\lambda^{-1/\gamma} \tag{31}$$

$$\begin{aligned}
y &= -\frac{\theta}{\sigma}\frac{V_w}{V_{ww}} - \frac{I\sigma_I}{\sigma}\frac{V_{wI}}{V_{ww}} \\
&= -\frac{\theta}{\sigma}\frac{I^{-\gamma}\phi'(z)}{I^{-\gamma-1}\phi''(z)} - \frac{I\sigma_I}{\sigma}\frac{-\gamma I^{-\gamma-1}\phi'(z) - I^{-\gamma-1}z\phi''(z)}{I^{-\gamma-1}\phi''(z)} \\
&= -\frac{\theta}{\sigma}I\frac{\lambda}{\lambda'} + \frac{I\sigma_I}{\sigma}\Big(\gamma\frac{\lambda}{\lambda'} + z\Big) \\
&= -\frac{\theta}{\sigma}I\lambda G'(\lambda) + \frac{\gamma\sigma_I}{\sigma}I\lambda G'(\lambda) + \frac{\sigma_I}{\sigma}w,
\end{aligned} \tag{32}$$

where the last equality results from (20).

Model 4 includes Model 1, Model 2, and Model 3 as its special cases. Without retirement flexibility ($C_{\delta_D} = 0$), borrowing constraints ($C^*_{\delta_D} = 0$), discrete and jump income shocks ($\delta_D = 0$), dual function $G(\lambda)$ given by (29) reduces to the following: for $\lambda > 0$,

$$G(\lambda) = \frac{1}{\hat{A}}\lambda^{-1/\gamma},$$

thus

$$G'(\lambda) = -\frac{1}{\gamma\hat{A}}\lambda^{-1/\gamma-1}.$$

Replacing $G'(\lambda)$ in (32) with above $G'(\lambda)$ proves Theorem 3.1.

Let us now consider retirement flexibility ($C_{\delta_D} > 0$) only without borrowing constraints ($C^*_{\delta_D} = 0$) and discrete and jump income shocks ($\delta_D = 0$). In this case, dual function $G(\lambda)$ given by (29) reduces to the following: for $\lambda > \underline{\lambda}$,

$$G(\lambda) = \frac{1}{\hat{A}}\lambda^{-1/\gamma} + C_0\lambda^{-\alpha_0}, \tag{33}$$

thus

$$G'(\lambda) = -\frac{1}{\gamma\hat{A}}\lambda^{-1/\gamma-1} - \alpha_0 C_0\lambda^{-\alpha_0-1}. \tag{34}$$

Substituting $G'(\lambda)$ stated above in (32) results in the optimal risky asset allocation (11) of Theorem 3.2.

We then determine C_0 and $\underline{\lambda}$ according to value matching and smooth pasting conditions (26) and (30). Specifically, $G(\underline{\lambda}) = B^{1/\gamma-1}\overline{K}^{-1}\underline{\lambda}^{-1/\gamma} + 1/\beta_1$ given in (26) can be rewritten by using (33) as

$$C_0\underline{\lambda}^{-\alpha_0} = \Big(B^{1/\gamma-1}\overline{K}^{-1} - \frac{1}{\hat{A}}\Big)\underline{\lambda}^{-1/\gamma} + \frac{1}{\beta_1}. \tag{35}$$

Also, (30) can be restated by using (33) and (34) as

$$\beta_2\frac{B^{1/\gamma-1}\overline{K}^{-1}}{1-\gamma}\underline{\lambda}^{1-1/\gamma} = \Big\{\Big(\beta_1 + \frac{\beta_3^2}{2\gamma}\Big)\frac{1}{\hat{A}} + \frac{\gamma}{1-\gamma}\Big\}\underline{\lambda}^{1-1/\gamma} + \Big(\beta_1 + \frac{1}{2}\beta_3^2\alpha_0\Big)C_0\underline{\lambda}^{1-\alpha_0}.$$

Using (35), we thus determine $\underline{\lambda}$ completely as Theorem 3.2 states. This also determines C_0 in (35) as Theorem 3.2 states.

Now we consider both retirement flexibility ($C_{\delta_D} > 0$) and borrowing constraints ($C^*_{\delta_D} > 0$) without discrete and jump income shocks ($\delta_D = 0$). In this case, dual function $G(\lambda)$ given by (29) simplifies to the following: for $\underline{\lambda} < \lambda < \overline{\lambda}$,

$$G(\lambda) = \frac{1}{\hat{A}}\lambda^{-1/\gamma} + C_0\lambda^{-\alpha_0} + C_0^*\lambda^{-\alpha_0^*}, \tag{36}$$

thus

$$G'(\lambda) = -\frac{1}{\gamma\hat{A}}\lambda^{-1/\gamma-1} - \alpha_0C_0\lambda^{-\alpha_0-1} - \alpha_0^*C_0^*\lambda^{-\alpha_0^*-1}. \tag{37}$$

Putting $G'(\lambda)$ stated above into (32) leads to the optimal risky asset allocation (13) of Theorem 3.3.

The constants C_0, C_0^*, $\underline{\lambda}$, $\overline{\lambda}$ are determined according to value matching and smooth pasting conditions (26) and (30), which are rewritten by using (36) and (37) as in Theorem 3.3.

We now consider the most general case (Model 4) which includes retirement flexibility ($C_{\delta_D} > 0$), borrowing constraints ($C^*_{\delta_D} > 0$), and discrete and jump income shocks ($\delta_D > 0$). In this case, dual function $G(\lambda)$ given by (29) is followed by the following its first derivative:

$$
\begin{aligned}
G'(\lambda) = & -\frac{1}{\gamma(\hat{A}+\delta_D)}\lambda^{-1/\gamma-1} - \alpha_{\delta_D}\{C_{\delta_D} - \mathrm{PS1}(\lambda;\delta_D)\}\lambda^{-\alpha_{\delta_D}-1} \\
& - \alpha^*_{\delta_D}\{C^*_{\delta_D} - \mathrm{PS2}(\lambda;\delta_D)\}\lambda^{-\alpha^*_{\delta_D}-1} - \frac{2\delta_D}{\beta_3^2\lambda^2}E[\kappa^{1-\gamma}\phi(z/\kappa)].
\end{aligned} \tag{38}
$$

By replacing $G'(\lambda)$ in (32) with above $G'(\lambda)$ shows the optimal risky asset allocation of Theorem 3.4.

The constants $C_{\delta_D}, C^*_{\delta_D}, \underline{\lambda}, \overline{\lambda}$ are determined according to value matching and smooth pasting conditions (26) and (30).

Numerical Solution

Variational Inequality. To solve the value function (6), we apply the variational inequality approach of Bensoussan and Lions (1982) and Øksendal (2007). For any $w \geq 0$, $I \geq 0$, $z \in \mathbb{R}$,

$$\max_{(c,y)\in\mathcal{A}(w,I,z)} \Big\{\mathcal{L}V(w,I,z), \quad R(w) - V(w,I,z)\Big\} = 0, \tag{39}$$

where the differential operator $\mathcal{L}$ is given by

$$\begin{aligned}\mathcal{L}V =& \frac{c^{1-\gamma}}{1-\gamma} - cV_w + \frac{1}{2}\sigma^2 y^2 V_{ww} + \frac{1}{2}[\sigma_I^2 + (\sigma - \sigma_z)^2]I^2 V_{II} + \frac{1}{2}(\sigma_z^2 + \sigma_I^2)V_{zz} \\ &+ \sigma(\sigma - \sigma_z)yIV_{wI} - \sigma\sigma_z y V_{wz} + [\sigma_I^2 - (\sigma - \sigma_z)\sigma_z]IV_{zI} \\ &+ [y(\mu - r) + rw + I]V_w + [\mu_I - \alpha(z - \overline{z})]IV_I - \alpha(z - \overline{z})V_z \\ &- \beta V + \delta_D\Big(E[V(w, \kappa I, z)] - V(w,I,z)\Big).\end{aligned}$$

Here, the subscripts of V denote its partial derivatives. The first term involving the differential operator $\mathcal{L}$ becomes zero when the investor belongs to the work region, whereas becomes negative when she belongs to the retirement region. The equality states that the investor optimally controls consumption and investment by setting the sum of instantaneous utility value $c^{1-\gamma}/(1-\gamma) - cV_w$ from consumption and instantaneous expected changes of the value function (with respect to changes in wealth w, income I, and the additional state variable z representing cointegration between the stock and labor markets) to zero. The presence of worst-case labor income realizations is captured by the last expectation term involving δ_D in the differential operator $\mathcal{L}$.

We now turn to the second term in (39) that measures the difference between value functions before and after retirement. As long as the difference is negative, i.e., the value function V with an unexercised retirement option exceeds the value function R after retirement, an investor finds it optimal to continue to work, and the retirement option is left unexercised. Once the difference is zero, i.e., the value function V approaches the value function R as the investor accumulates wealth, she finds it optimal to enter retirement, and the retirement option is exercised. Since the work region and the retirement region cannot be overlapped, we should consider the first and second terms in (39) in tandem, allowing the maximum of these two terms to be equated with zero.

Numerical Algorithm. We solve HJB equation (39) numerically. By homogeneity property, we can reduce one dimension by the following transformation:

$$V(w, I, z) = \frac{\bar{K}^{-\gamma}}{1-\gamma}\left(w + \frac{I}{r}\right)^{1-\gamma} e^{(1-\gamma)u(\xi,z)}, \quad \xi = \frac{I/r}{w + I/r} \in [0,1],$$

here $\bar{K} = \frac{\beta}{\gamma} - \frac{(1-\gamma)}{\gamma}\left(r + \frac{\theta^2}{2\gamma}\right)$. After retirement, we know $R(w)$ satisfies $R(w) = \frac{B^{1-\gamma}\bar{K}^{-\gamma}}{1-\gamma} w^{1-\gamma}$. Therefore, the associated HJB equation for new function $u(\xi, z)$ becomes

$$\max_{\bar{y},\bar{c}} \{\mathcal{L}_1 u(\xi, z), \quad \mathcal{R}u(\xi)\} = 0, \tag{40}$$

on $\{(\xi, z) : \xi \in [0,1], z \in \mathbb{R}\}$, where

$$\begin{aligned}
\mathcal{L}_1 u =& \left[\frac{1}{2}\sigma^2\bar{y}^2\xi^2 + \frac{1}{2}(\sigma_I^2 + (\sigma-\sigma_z)^2)\xi^2(1-\xi)^2 - \sigma(\sigma-\sigma_z)\bar{y}\xi^2(1-\xi)\right][u_{\xi\xi} + (1-\gamma)u_\xi^2] \\
&+ \left[\sigma\sigma_z\bar{y}\xi + (\sigma_I^2 - \sigma_z(\sigma-\sigma_z))\xi(1-\xi)\right][u_{\xi z} + (1-\gamma)u_\xi u_z] + \frac{1}{2}(\sigma_I^2 + \sigma_z^2)[u_{zz} + (1-\gamma)u_z^2] \\
&+ \left[\gamma\sigma^2\bar{y}^2 + \gamma\sigma(\sigma-\sigma_z)(2\xi-1)\bar{y} - (\mu-r)\bar{y} - \gamma(\sigma_I^2 + (\sigma-\sigma_z)^2)\xi(1-\xi) + (\mu_I - \alpha(z-\bar{z}))(1-\xi) - r\right]\xi u_\xi \\
&+ \left[-(1-\gamma)\sigma\sigma_z\bar{y} + (1-\gamma)(\sigma_I^2 - \sigma_z(\sigma-\sigma_z))\xi - \alpha(z-\bar{z})\right]u_z \\
&+ (\mu - r - \gamma\sigma(\sigma-\sigma_z)\xi)\bar{y} - \frac{1}{2}\gamma\sigma^2\bar{y}^2 - \frac{1}{2}(\sigma_I^2 + (\sigma-\sigma_z)^2)\gamma\xi^2 + (\mu_I - \alpha(z-\bar{z}))\xi + r - \frac{\beta+\delta_D}{1-\gamma} \\
&+ \delta_D\frac{(1+(\kappa-1)\xi)^{1-\gamma}}{1-\gamma}E\left[e^{(1-\gamma)\left(u\left(\frac{\kappa\xi}{1+(\kappa-1)\xi}\right)-u\right)}\right] + \frac{\bar{K}^\gamma}{1-\gamma}e^{-(1-\gamma)u}\bar{c}^{1-\gamma} - \bar{c}(1-\xi u_\xi), \\
\mathcal{R}u =& \ln(1+(\kappa-1)\xi) + \ln B - u,
\end{aligned}$$

and $\bar{y} = \frac{y}{w+I/r}$, $\bar{c} = \frac{c}{w+I/r}$. At boundary $\xi = 0$, i.e., when $w = \infty$, the HJB equation is degenerated and the solution approximates Merton case. At boundary $\xi = 1$, i.e., when $w = 0$, it is known that the investor could not invest in the stock market anymore.[22] Thus

$$\bar{y}^* = 0, \quad \bar{c}^* = \min\left\{\bar{K}e^{(1-1/\gamma)u(1,z)}(1-u_\xi(1,z))^{-1/\gamma}, r\right\}.$$

For $\xi \in (0,1)$, the optimal investment and consumption in the presence of constrained borrowing and short selling are determined by

$$\bar{y}^* = \min\left\{\max\{h(\xi,z),0\}, 1-\xi\right\}, \quad \bar{c}^* = \bar{K}e^{(1-1/\gamma)u}(1-\xi u_\xi)^{-1/\gamma},$$

where

$$\begin{aligned} h(\xi,z) = -\Big[&\mu - r - \gamma\sigma(\sigma-\sigma_z)\xi + [\gamma\sigma(\sigma-\sigma_z)(2\xi-1)+r-\mu]\xi u_\xi - (1-\gamma)\sigma\sigma_z u_z \\ &+ \sigma\sigma_z\xi[u_{\xi z} + (1-\gamma)u_\xi u_z] - \sigma(\sigma-\sigma_z)\xi^2(1-\xi)[u_{\xi\xi} + (1-\gamma)u_\xi^2]\Big] \\ &\Big/ \Big[\sigma^2\xi^2[u_{\xi\xi} + (1-\gamma)u_\xi^2] + 2\gamma\sigma^2\xi u_\xi - \gamma\sigma^2\Big]. \end{aligned}$$

Moreover, we set the lower bound of solvency domain to $z_{\text{min}} = \bar{z} - 8\sigma_z$ and the upper bound to $z_{\text{max}} = \bar{z} + 8\sigma_z$, and impose boundary condition $V_z\big|_{z=z_{\text{min}}} = V_z\big|_{z=z_{\text{max}}} = 0$, or equivalently, $u_z\big|_{z=z_{\text{min}}} = u_z\big|_{z=z_{\text{max}}} = 0$. We then apply the penalty method of Dai and Zhong (2010) to solve the resulting HJB equation (40). We confirm that our numerical approach is robust to the choice of bounds in z direction.

[22]It is well documented that optimal investment in the stock market should be zero as wealth approaches zero. This condition is exactly same with the borrowing constraint against future labor income (Dybvig and Liu, 2010).

Verification of Value Function and Optimal Policy

Define the optimal policy as a feed-back function of $V(w, I, z)$ by FOC in HJB equation (39):

$$\begin{aligned} c_t^* =& V_w(W_t^*, I_t, Z_t)^{-1/\gamma}, \\ y_t^* =& \max\left\{\min\left\{\frac{\big[\sigma\sigma_z V_{wz} - \sigma(\sigma-\sigma_z)I_t V_{wI} - (\mu - r)V_w\big](W_t^*, I_t, Z_t)}{\sigma^2 V_{ww}(W_t^*, I_t, Z_t)}, 1\right\}, 0\right\}, \\ \tau^* =& \inf\{t \geq 0 : V(W_t^*, I_t, Z_t) \geq R(W_t^*)\}. \end{aligned} \tag{41}$$

where W_t^* follows $dW_t = (rW_t - c_t^* + I_t)dt + y_t^*\sigma(d\mathcal{B}_t^1 + \theta dt)$ with $W_0 = w \geq 0$. Specially, after retirement, the optimal consumption and stock investment follow standard merton line, that is, $c_t^*/W_t^* = \bar{K}$ and $y_t^*/W_t^* = (\mu - r)/(\gamma\sigma^2)$ for $t \geq \tau^*$.

We show that the claimed optimal policies above are actual optimal policies and the solution of HJB equation (39) coincides with the original utility function defined in (5) under regular conditions. The proof of their verification are similar as DL.

Assume $V(w, I, z)$ is a smooth solution of HJB equation (39) and satisfies the transversality condition, i.e.,

$$\lim_{t\to\infty} \mathbb{E}\Big[e^{-\beta t}V(W_t, I_t, Z_t)\Big] = 0, \ \ \forall\, W_t \geq 0, I_t > 0, Z_t \in \mathbb{R},$$

for any admissible controls. Then we want to prove the solution $V(w, I, z)$ is not less than the value function defined in (5) and the equality achieves under optimal strategy defined in (41).

For any admissible strategy $\{c_t, y_t, \tau\}$, let us define

$$M_t = \int_0^t e^{-\beta s}\Big[(1 - R_s)U(c_s)ds + R(W_s)dR_s\Big] + e^{-\beta t}(1 - R_t)V(W_t, I_t, Z_t),$$

where $R_t := \mathbf{1}_{\{t>\tau\}}$. Without loss of generality, we assume $R_0 = 0$. Otherwise if $R_0 = 1$, then we have $V(w, I, z) \geq \mathbb{E}[\int_0^\infty e^{-\beta t}U(Bc_t)dt]$ with equality achieved when $c_t = c_t^*, y_t = y_t^*$, and

$R_t = R_t^*$. By generalized Ito's formula,

$$\begin{aligned}
dM_t =& e^{-\beta t}(1-R_t)U(c_t)dt + e^{-\beta t}R(W_t)dR_t - e^{-\beta t}V(W_t,I_t,Z_t)dR_t \\
&+ e^{-\beta t}(1-R_t)[V_w(W_t,I_t,Z_t)dW_t + V_I(W_t,I_t,Z_t)dI_t + V_z(W_t,I_t,Z_t)dZ_t] \\
&+ e^{-\beta t}(1-R_t)[V_{ww}(W_t,I_t,Z_t)dW_tdW_t + V_II(W_t,I_t,Z_t)dI_tdI_t + V_{zz}(W_t,I_t,Z_t)dZ_tdZ_t] \\
&+ e^{-\beta t}(1-R_t)[V_{wI}(W_t,I_t,Z_t)dW_tdI_t + V_{wz}(W_t,I_t,Z_t)dW_tdZ_t + V_{Iz}(W_t,I_t,Z_t)dI_tdZ_t] \\
&+ (1-R_t)\frac{\partial}{\partial t}\left(e^{-\beta t}V(W_t,I_t,Z_t)\right)dt \\
=& e^{-\beta t}(1-R_t)U(c_t)dt + (1-R_t)\left[e^{-\beta t}(\mathcal{L}V(W_t,I_t,Z_t) - U(c_t))\right]dt \\
&+ e^{-\beta t}R(W_t)dR_t - e^{-\beta t}V(W_t,I_t,Z_t)dR_t \\
&+ e^{-\beta t}(1-R_t)\left[\sigma V_w(W_t,I_t,Z_t)y_t + (\sigma-\sigma_z)V_I(W_t,I_t,Z_t)I_t - \sigma_z V_z(W_t,I_t,Z_t)\right]d\mathcal{B}_t^1 \\
&+ e^{-\beta t}(1-R_t)\left[\sigma_I V_I(W_t,I_t,Z_t)I_t + \sigma_I V_z(W_t,I_t,Z_t)\right]d\mathcal{B}_t^2.
\end{aligned}$$

Define $\mathcal{O}_n := \{(w,I,z) : \frac{1}{2n} \le w \le n, |z| < n, \frac{1}{2n} \le I \le n\}$ and a sequence of stopping time $\theta_n := n \wedge \inf\{t \ge 0 : (W_t,I_t,Z_t) \notin \mathcal{O}_n\}$. We then integrate the above equation from 0 to θ_n:

$$\begin{aligned}
M_{\theta_n} =& M_0 + \int_0^{\tau\wedge\theta_n}(1-R_s)e^{-\beta s}\mathcal{L}V(W_s,I_s,Z_s)ds + \int_{\tau\wedge\theta_n}^{\theta_n}(1-R_s)e^{-\beta s}\mathcal{L}V(W_s,I_s,Z_s)ds \\
&+ \int_0^{\theta_n} e^{-\beta s}\Big[R(W_s) - V(W_s,I_s,Z_s)\Big]dR_s \\
&+ \int_0^{\theta_n} e^{-\beta s}(1-R_s)\Big[\sigma V_w(W_s,I_s,Z_s)y_s + (\sigma-\sigma_z)V_I(W_s,I_s,Z_s)I_s - \sigma_z V_z(W_s,I_s,Z_s)\Big]d\mathcal{B}_s^1 \\
&+ \int_0^{\theta_n} e^{-\beta s}(1-R_s)\Big[\sigma_I V_I(W_s,I_s,Z_s)I_s + \sigma_I V_z(W_s,I_s,Z_s)\Big]d\mathcal{B}_s^2.
\end{aligned}$$

By the form of (39) and the definition of $\{c_t^*, y_t^*, R_t^*\}$ in (41), we obtain that the first integral is always non-positive for any feasible strategy $\{c_t, y_t, R_t\}$ and is equal to zero for the claimed optimal policy $\{c_t^*, y_t^*, R_t^*\}$ in (41). That is because if $(\tau^*\wedge\theta_n) \ge (\tau\wedge\theta_n)$, the solution function V satisfies $\mathcal{L}V \le 0$ by (39) and the equality achieves under claimed optimal strategy $\{c_t^*, y_t^*\}$, and if $(\tau^*\wedge\theta_n) < (\tau\wedge\theta_n)$, we have $V = R(w)$ and $\mathcal{L} = U(c) - c\partial_w + \frac{1}{2}\sigma^2y^2\partial_{ww} + [rw + y(\mu - r)]\partial_w - \beta$ during $[\tau^*\wedge\theta_n, \tau\wedge\theta_n]$ so that $\mathcal{L}V < 0$ as $B > 1$. Therefore, the first non-positive integral equals to zero only when $c_t = c_t^*$, $y_t = y_t^*$, and $R_t = R_t^*$. The second integral equals zero for both

$\theta_n \leq \tau$ and $\theta_n > \tau$ (in this case, $1 - R_t = 0$ during $[\tau, \theta_n]$). The third integral is always non-positive for every feasible policy $\{c_t, y_t, R_t\}$ because $V(W_t, I_t, Z_t) \geq R(W_t)$ and is equal to zero only when $c_t = c_t^*$, $y_t = y_t^*$, and $\tau \geq \tau^*$ as $V(W_t, I_t, Z_t) = R(W_t)$ for $t \geq \tau^*$. The last two stochastic integrals under expectation equals zero as $V_w(W_t, I_t, Z_t)$, $V_z(W_t, I_t, Z_t)$, and $V_I(W_t, I_t, Z_t)$ are bounded when (W_t, I_t, Z_t) is in a bounded domain during $[0, \theta_n]$.

Noticing that $M_0 = V(W_0, I_0, Z_0)$, we then take expectation in above equation to get

$$\begin{aligned} V(W_0, I_0, Z_0) \;\; \geq \;\; & \mathbb{E}\int_0^{\theta_n} e^{-\beta s}\Big[(1-R_s)U(c_s) + R(W_s)dR_s\Big] \\ & +\mathbb{E}\big[e^{-\beta\theta_n}(1-R_{\theta_n})V(W_{\theta_n}, I_{\theta_n}, Z_{\theta_n})\big]. \end{aligned}$$

As analyzed above, the equality above holds only for the claimed optimal strategy $\{c_t^*, y_t^*, R_t^*\}$ defined in (41). As $n \to \infty$, θ_n increases to infinity with probability 1. By the transversality condition of V and dominant convergence theorem, the first expectation above converges to the original utility function $\mathbb{E}[\int_0^\tau e^{-\beta s}U(c_s)ds + e^{-\beta\tau}R(W_\tau)]$ and the second expectation goes to zero. Equality holds for the claimed optimal policy $\{c_t^*, y_t^*, R_t^*\}$ and this completes the proof.

References

Bai, Y., E. C. Chang, J. Wang. 2006. Asset Prices under Short-Sale Constraints. Working Paper.

Basak, S., A. Shapiro, L. Tepla. 2006. Risk Management with Benchmarking. *Management Science.* **52** 542–557.

Baxter, M., U. J. Jermann. 1997. The International Diversification Puzzle is Worse than You Think. *American Economic Review.* **87** 170–180.

Bensoussan, A., B. G. Jang, S. Park. 2016. Unemployment Risks and Optimal Retirement in an Incomplete Market. *Operations Research.* **64** 1015–1032.

Bensoussan, A., J. L. Lions. 1982. *Application of Variational Inequalities in Stochastic Control.* North Holland.

Benzoni, L., P. Collin-Dufresne, R. S. Goldstein. 2007. Portfolio Choice over the Life-Cycle when the Stock and Labor Markets Are Cointegrated. *Journal of Finance.* **62** 2123–2167.

Bewley, T. 1977. The Permanent Income Hypothesis: A Theoretical Formulation. *Journal of Economic Theory.* **16** 252–292.

Bodie, Z., R. C. Merton, W. F. Samuelson. 1992. Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model. *Journal of Economic Dynamics and Control.* **16** 427–449.

Branger, N., M. Muck, F. T. Seifried, S. Weisheit. 2017. Optimal Portfolios when Variances and Covariances can Jump. *Journal of Economic Dynamics and Control.* **85** 59–89.

Calvet, L. E., and P. Sodini. 2014. Twin Picks: Disentangling the Determinants of Risk-Taking in Household Portfolios. *Journal of Finance.* **69** 867–906.

Campbell, J. Y. 1987. Does Saving Anticipate Declining Labor Income? An Alternative Test of the Permanent Income Hypothesis. *Econometrica.* **55** 1249–1273.

Campbell, J. Y., J. Cocco, F. J. Gomes, P. Maenhout. 2001. Investing Retirement Wealth: A Life-Cycle Model. in John Y. Campbell and Martin Feldstein, eds.: *Risk Aspects of Investment-Based Social Security Reform.* (University of Chicago Press, Chicago, IL).

Carroll, C. D. 1992. The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence. *Brookings Papers on Economic Activity.* **2** 61–156.

Carroll, C. D., K. E. Dynan, S. D. Krane. 2003. Unemployment Risk and Precautionary Wealth: Evidence from Households' Balance Sheets. *Review of Economics and Statistics.* **85** 586–604.

Carroll, C. D., M. S. Kimball. 1996. On the Concavity of the Consumption Function. *Econometrica.* **64** 981–992.

Carroll, C. D., A. A. Samwick. 1997. The Nature of Precautionary Wealth. *Journal of Monetary Economics.* **40** 41–71.

Chai, J., W. Horneff, R. Maurer, O. S. Mitchell. 2011. Optimal Portfolio Choice over the Life Cycle with Flexible Work, Endogenous Retirement, and Lifetime Payouts. *Review of Finance.* **0** 1–33.

Cocco, J. F., F. J. Gomes, P. J. Maenhout. 2005. Consumption and Portfolio Choice over the Life Cycle. *Review of Financial Studies.* **18** 491–533.

Cox, J. C., C. Huang. 1989. Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process. *Journal of Economic Theory.* **49** 33–83.

Dai, M., Y. Zhong. 2010. Penalty Methods for Continuous-Time Portfolio Selection with Proportional Transaction Costs. *Journal of Computational Finance.* **13** 1–31.

Davis, S. J., P. Willen. 2013. Occupation-Level Income Shocks and Asset Returns: Their Covariance and Implications for Portfolio Choice. *Quarterly Journal of Finance.* **3** 1–53.

Deaton, A. 1991. Saving and Liquidity Constraints. *Econometrica.* **59** 1221–1248.

Duffie, D., L. G. Epstein. 1992. Stochastic Differential Utility. *Econometrica.* **60** 353–394.

Duffie, D., W. Fleming, M. Soner, T. Zariphopoulou. 1997. Hedging in Incomplete Markets with HARA Utility. *Journal of Economic Dynamics and Control.* **21** 753–782.

Dybvig, P. H., H. Liu. 2010. Lifetime Consumption and Investment: Retirement and Constrained Borrowing. *Journal of Economic Theory.* **145** 885–907.

Epstein, L. G., S. E. Zin. 1989. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica.* **57** 937–969.

Farhi, E., S. Panageas. 2007. Saving and Investing for Early Retirement: A Theoretical Analysis. *Journal of Financial Economics.* **83** 87–121.

Frey, C. B., M. A. Osborne. 2017. The Future of Employment: How Susceptible are Jobs to Computerisation? *Technological Forecasting and Social Change.* **114** 254–580.

Friedman, M. 1957. A Theory of the Consumption Function. Princeton University Press, Princeton.

Garlappi, L., G. Skoulakis. 2010. Solving Consumption and Portfolio Choice Problems: The State Variable Decomposition Method. *Review of Financial Studies.* **23** 3346–3400.

Gomes, F., A. Michaelides. 2005. Optimal Life-Cycle Asset Allocation: Understanding the Empirical Evidence. *Journal of Finance.* **60** 869–904.

Gormley, T., H. Liu, G. Zhou. 2010. Limited Participation and Consumption-Saving Puzzles: A Simple Explanation and the Role of Insurance. *Journal of Financial Economics.* **96** 331–344.

Grossman, G. M., E. Oberfield. 2022. The Elusive Explanation for the Declining Labor Share. *Annual Review of Economics.* **14** 93–124.

Guiso, L., T. Jappelli. 2005. Awareness and Stock Market Participation. *Review of Financial Studies.* **9** 537–567.

Gustman, A. L., T. L. Steinmeier, N. Tabatabai. 2010. What the Stock Market Decline Means for the Financial Security and Retirement Choices of the Near-Retirement Population. *Journal of Economic Perspectives.* **24** 161–182.

Gustman, A. L., T. L. Steinmeier. 2002. Retirement and The Stock Market Bubble. Working Paper.

Guvenen, F., F. Karahan, S. Ozcan, J. Song. 2015. What Do Data on Millions of U.S. Workers Reveal about Life-Cycle Earnings Risk? University of Minnesota. Working Paper.

Heaton, J., D. Lucas. 1997. Market Frictions, Savings Behavior and Portfolio Choice. *Macroeconomic Dynamics.* **1** 76–101.

Heaton, J., D. Lucas. 2000. Portfolio Choice and Asset Prices: The Importance of Entrepreneurial Risk. *Journal of Finance.* **55** 1163–1198.

Hong, H., J. D. Kubik, J. Stein. 2004. Social Interaction and Stock-Market Participation. *Journal of Finance.* **59** 137–163.

Jagannathan, R., N. R. Kocherlakota. 1996. Why Should Older People Invest Less in Stocks than Younger People? *Federal Reserve Bank of Minneapolis Quarterly Review.* **20** 11–23.

Jang, B. G., H. K. Koo, S. Park. 2019. Optimal Consumption and Investment with Insurer Default Risk. *Insurance: Mathematics and Economics.* **88** 44–56.

Jang, B. G., S. Park, Y. Rhee. 2013. Optimal Retirement with Unemployment Risks. *Journal of Banking and Finance.* **37** 3585–3604.

Jang, B. G., S. Park, H. Zhao. 2020. Optimal Retirement with Borrowing Constraints and Forced Unemployment Risk. *Insurance: Mathematics and Economics.* **94** 25–39.

Jin, X., A. X. Zhang. 2012. Decomposition of Optimal Portfolio Weight in a Jump-Diffusion Model and Its Applications. *Review of Financial Studies.* **25** 2877–2919.

Jin, X., D. Luo, X. Zeng. 2017. Dynamic Asset Allocation with Uncertain Jump Risks: A Pathwise Optimization Problem. *Mathematics of Operations Research.* **43** 347–376.

Karatzas, I., J. P. Lehoczky, S. E. Shreve, and G. -L., Xu. 1991. Martingale and Duality Methods for Utility Maximization in an Incomplete Market. *SIAM Journal of Control and Optimization.* **29** 702–730.

Koo, H. K. 1998. Consumption and Portfolio Selection with Labor Income: A Continuous Time Approach. *Mathematical Finance.* **8** 49–65.

Lazear, E. P. 1979. Why is There Mandatory Retirement? Journal of Political Economy. 87, 1261–1284.

Liu, J., F. A. Longstaff, J. Pan. 2003. Dynamic Asset Allocation with Event Risk. *Journal of Finance.* **58** 231–259.

Low, H., C. Meghir, L. Pistaferri. 2010. Wage Risk and Employment Risk over the Life Cycle. *American Economic Review.* **100** 1432–1467.

Lynch, A. W., S. Tan. 2011. Labor Income Dynamics at Business-Cycle Frequencies: Implications for Portfolio Choice. *Journal of Financial Economics.* **101** 333–359.

Mehra, R., E. C. Prescott. 1985. The Equity Premium: A Puzzle. *Journal of Monetary Economics.* **15** 145–161.

Menzly, L., T. Santos, P. Veronesi. 2004. Understanding Predictability. *Journal of Political Economy.* **112** 1–47.

Merton, R. C. 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *Review of Economics and Statistics.* **51** 247–257.

Merton, R. C. 1971. Optimum Consumption and Portfolio Rules in a Continuous-Time Model. *Journal of Economic Theory.* **3** 373–413.

Munk, C., C. Sørensen. 2010. Dynamic Asset Allocation with Stochastic Income and Interest Rates. *Journal of Financial Economics.* **96** 433–462.

Øksendal, B. 2007. *Stochastic Differential Equations: An Introduction with Applications.* Fifth ed. (Springer, Berlin).

Polkovnichenko, V. 2007. Life-Cycle Portfolio Choice with Additive Habit Formation Preferences and Uninsurable Labor Income Risk. *Review of Financial Studies.* **20** 83–124.

Ross, S. A. 1978. A Simple Approach to the Valuation of Risky Streams. *Journal of Business.* **51** 454–475.

Santos, T., P. Veronesi. 2006. Labor Income and Predictable Stock Returns. *Review of Financial Studies.* **19** 1–44.

Viceira, L. M.. 2001. Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *Journal of Finance.* **56** 433–470.

Vissing-Jørgensen, A. 2002. Limited Asset Market Participation and the Elasticity of Intertemporal Substitution. *Journal of Political Economy. 110* 825–853.

Wachter, J. A., M. Yogo. 2010. Why Do Household Portfolio Shares Rise in Wealth?. *Review of Financial Studies.* **23** 3929–3965.

Wang, C., N. Wang, J. Yang. 2016. Optimal Consumption and Savings with Stochastic Income and Recursive Utility. *Journal of Economic Theory.* **165** 292–331.

Weil, P. 1990. Non-Expected Utility in Macroeconomics. *Quarterly Journal of Economics.* **105** 29–42.