DEFAULT HAZARDS AND THE TERM STRUCTURE OF CREDIT SPREADS IN A DUOPOLY

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Abstract

This paper shows how default hazards similar to those suggested by the literature on reduced form credit risk models may arise purely from the strategic behavior of indebted firms operating in a duopoly. Our research advances attempts to reconcile structural and reduced form approaches to modelling credit risk. Firm defaults are generated endogenously by a randomly evolving intensity and short credit spreads are strictly positive. We generalize the model to allow for incomplete information concerning firm types and show how this leads to default intensities that evolve in a path-dependent manner through Bayesian learning.

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1 Introduction

Industry competition can substantially influence the values of claims on firms. Declining performance in one firm may signal the industrial ascendancy of others, while the liquidation of one company may precipitate the acquisition of temporary monopoly power by another. Interactions of this kind are common in oligopolies and give rise to different incentives among debt and equity holders. While there has been much study of how capital structure decisions affect strategic behavior in product markets,[1] few studies have examined how competition affects capital structure decisions and the pricing of securities.

An exception to this is Lambrecht (2001) which investigates the order of bankruptcy, subsequent entry and debt exchange offers within a stochastic continuous-time duopoly model. While Lambrecht's study provides insight on how business failure and renegotiation are related to macro-economic variables as well as firm and industry characteristics, he devotes no attention to the behaviour of default premia in oligopolies. In this paper, we analyse strategic behaviour in a duopoly model and study its impact on credit spreads. Our continuous-time structural model reconciles the two strands of the credit risk pricing literature and provides further theoretical justification for the existence of surprise credit events by implicitly modelling the default intensity.

The recent literature on pricing risky debt has two principal strands. First, several authors have refined and extended the so-called structural models of corporate default first suggested by Merton (1974) and Black and Cox (1976).[2] Second, reduced form models for pricing risky debt have been developed which may be fitted directly to risky bond prices but have no very obvious link to the borrower's financial position.[3] An important difference between structural and reduced-form models is the way in which they assume that default is triggered. In structural models, bankruptcy occurs when the firm's underlying asset value crosses a threshold.[4] By contrast, in reduced

[1]See, for example, Brander and Lewis (1986) and Maksimovic (1988).

[2]See Brennan and Schwartz (1978), Longstaff and Schwartz (1995), Leland (1994), Leland and Toft (1996), Anderson and Sundaresan (1996) and Mella-Barral and Perraudin (1997) amongst other contributions.

[3]See, amongst others, Litterman and Iben (1991), Jarrow, Lando, and Turnbull (1997), Jarrow and Turnbull (1995), and Duffie and Singleton (1999).

[4]If the asset value follows a diffusion process, the probability of default in the next instant of time is either zero (if asset values are a discrete distance from the default threshold) or one (if the threshold is reached).

form models, borrowers may in principle jump into a default state at any time. The likelihood that they will is described by a hazard of default. An advantage of models which permit jumps into default is that they can explain the fact that credit spreads on very short-term bonds appear to be strictly positive. The liquidity spreads found in bond markets mean that it is hard to judge whether this is so for high credit quality short-term debt but it is almost certainly true for lower quality credit exposures.

In our model, we obtain endogenous default intensities, that are functions of firm and industry characteristics. The default intensities are the randomised strategies of a pair of equity holders in a non-cooperative Nash equilibrium. The basic intuition for the bankruptcy game between equity holders is as follows. Default occurs when equity holders decide to cease injecting capital to meet debt service payments, and payoffs are structured so that the last to exit is relatively better off. The trade-off between winning the higher payoff and of waiting inefficiently, cause the equity holders to randomise their default decision through a conditionally Poisson process, as in reduced-form models. Equity holders' randomised default strategies in turn affect debt values and the corresponding credit spreads. The randomised strategies are equivalent to default hazards and so our model resembles other hybrid structural models.

Recently, several other studies have sought to reconcile the two branches of the literature on corporate debt by showing circumstances in which structural models generate default hazards similar to those which arise in reduced-form models. The simplest approach is to include jump components in the diffusion process driving firm assets within a structural model. See Zhou (2001), El Jahel (1999) and Cathcart and El-Jahel (1998).[5] This approach also leads to hybrid models that can explain the empirical fact of positive short credit spreads but which are not entirely compatible with the reduced form literature. Madan and Unal (2000) have refined this approach by assuming that jumps are triggered by cash-shortages in non-interest-rate sensitive components of the firm's assets.

Another approach is to suppose that the firm's asset value within a structural model is imperfectly observed by the market. Again, bankruptcy is triggered when asset values cross a threshold but the bankruptcy event will be a surprise for investors and debt values will jump. This idea has been explored by Duffie and Lando

[5]In contrast, Cathcart and El-Jahel (1998) introduce a conditionally Poisson signalling process. The jumps precipitate default but are not linked to firm asset values in any direct way.

(2000). They show that the first hitting time of an imperfectly-observed Brownian motion behaves like the first jump time of a Poisson process. Hence, such imperfect information generates default hazards similar to those proposed in the reduced-form valuation literature.[6] Finally, Cao and Wei (2001) have considered the behaviour of credit spreads when the indebted firm has short positions in vulnerable options. The presence of these extra corporate liabilities generates positive short term credit spreads.

In most of these approaches, the default intensity generating jumps is either exogenously specified or implied directly by assumptions about effects of different kinds.[7] So the link between default hazards and the firm's capital structure and macro-economic variables is still relatively weak. Our model has the advantage of yielding endogenous default hazards that are a function of the characteristics of the firm as well as reflecting the market environment in which it operates. The endogenous hazards we derive provide intuitions about the causes of random defaults in firms and have a number of appealing features that we discuss.

The structure of the paper is as follows. Section 2 models the hazard rate in a duopoly in which firms issue perpetual debt. Section 3 discusses the structure of these hazards. Section 4 studies the impact of the default hazards on credit spreads and demonstrates the main result of the paper: positive short credit spreads. Section 5 generalises the hazard rate to asymmetric settings with incomplete information and considers some extensions. Section 6 concludes.

[6] Lambrecht and Perraudin (1996) also incorporated incomplete information into a structural model of risky debt, thereby generating "surprise defaults". However, in their model the quantity that is imperfectly observed by investors is the trigger level for bankruptcy (i.e., a random variable) rather than the firm's asset value (i.e., a stochastic process). The structure of Bayesian up-dating that this implies meant that surprise bankruptcies can only occur when the state variable hits new lows and thus the hazard of instantaneous default is either zero or infinite.

[7] Zhou (2001), El Jahel (1999) and Cathcart and El-Jahel (1998) directly specify jump processes that generate defaults. In Duffie and Lando (2000), the conditional distribution of the asset value is exogenous, while in Madan and Unal (2000) the distribution of cash shortages is exogenously given.

2 Default Intensities in a Duopoly

2.1 Equity and Debt Values in a Monopoly

Suppose that all agents are risk-neutral and there is a constant interest rate, r. Consider a firm that enters a product market by investing a fixed start-up amount, K. Equity holders have a maximum sum $J < K$ to invest in the firm. To fund the difference, $K - J$, they issue perpetual debt that pays a continuous coupon, c. Following Mella-Barral and Perraudin (1997), we suppose that the firm's total profit flow is $x_t - w$, where w is a constant continuous flow cost and x_t is a geometric Brownian motion

$$dx_t = \mu x_t dt + \sigma x_t dB_t \tag{1}$$

with constant volatility and drift parameters σ and $\mu < r$.[8] The net income flow to equity holders is, therefore, $x_t - w - c$, while bond holders receive c.

Application of Ito's Lemma and financial market equilibrium with risk-neutral agents imply that the *monopoly*[9] equity value, $\hat{V}(x)$, and debt value, $\hat{D}(x)$, satisfy

$$r\hat{V}(x) = x - w - c + \frac{\sigma^2 x^2}{2}\frac{d^2\hat{V}(x)}{dx^2} + \mu x \frac{d\hat{V}(x)}{dx} \ , \tag{2}$$

$$r\hat{D}(x) = c + \frac{\sigma^2 x^2}{2}\frac{d^2\hat{D}(x)}{dx^2} + \mu x \frac{d\hat{D}(x)}{dx} \ . \tag{3}$$

As x_t tends to infinity, the likelihood of default diminishes and so the equity approaches the expected value of discounted net earnings: $\lim_{x_t\to\infty}(\hat{V}(x_t) - x_t/(r - \mu) + (w+c)/r) = 0$,[10] while the debt value approaches its riskless value, $\lim_{x_t\to\infty}\hat{D}(x_t) = c/r$.

Boundary conditions for low levels of x_t are generated by what happens in the event of default. There is now considerable evidence of deviations from absolute prior-

[8] There is very little difference between constructing a model in which the state variable is an earnings flow or assuming that the state variable is the unlimited liability value of the firm's underlying assets since it is straightforward to show that, given our assumptions, assets are linearly related to earnings.

[9] Throughout this paper we use hatted functions and variables to denote values and parameters relating to the non-strategic, monopoly case.

[10] Specifically:

$$\lim_{x\to\infty}\left(\hat{V}(x_t) - \mathrm{E}_t\left[\int_t^\infty (x_s - w - c)\exp\left[-r\,(s-t)\right]ds\right]\right) = 0.$$

ity in the allocation of firm value between stake holders during bankruptcy, especially in the case of Chapter 11 bankruptcies in the US. The extensive powers given to management in the Chapter 11 process and their ability to delay legal proceedings[11] allow equity holders to extract value in bankruptcy settlements.[12] Consistent with this evidence, we suppose that in the event of bankruptcy equity holders extract a constant value γ_E. Thus, $\hat{V}(\hat{x}_b) = \gamma_E$, at the default trigger, $\hat{x}_b$. Furthermore, we assume that bankruptcy involves dead weight administrative and legal costs equal to a fraction, ϕ, of total value of the firm when it is operated on a pure equity basis, $\hat{W}(x)$ with the possibility of liquidating the firm to obtain a scrapping value of γ. In bankruptcy, debt holders therefore obtain

$$\hat{D}(\hat{x}_b) = (1-\phi)\,\hat{W}(\hat{x}_b) - \gamma_E,$$

Finally, we assume, as in Leland (1994) and Mella-Barral and Perraudin (1997) that there are no net worth covenants on the debt. This implies that bankruptcy occurs when equity holders decide to cease injecting capital. The bankruptcy trigger, $\hat{x}_b$, is therefore determined so as to maximise the equity value. This implies the smooth-pasting (optimality) condition $\hat{V}'(\hat{x}_b) = 0$ where $\hat{x}_b$ $(> \hat{x})$ is the trigger level for bankruptcy.

Standard methods imply that debt and equity values in the monopoly case are as follows.

Proposition 1 *The values of a monopoly firm's equity, $\hat{V}(x_t) = \hat{V}(x_t; w)$, and debt, $\hat{D}(x_t) = \hat{D}(x_t; w)$, prior to bankruptcy are*

$$\hat{V}(x_t; w) = \frac{x_t}{r-\mu} - \frac{w+c}{r} + \left[\gamma_E - \frac{\hat{x}_b}{r-\mu} + \frac{w+c}{r}\right]\left(\frac{x_t}{\hat{x}_b}\right)^{\xi} \quad \text{for} \quad x_t \in [\hat{x}_b, \infty) \tag{4}$$

$$\hat{D}(x_t; w) = \frac{c}{r} - \left[\frac{c}{r} - (1-\phi)\hat{W}(\hat{x}_b) + \gamma_E\right]\left(\frac{x_t}{\hat{x}_b}\right)^{\xi} \quad \text{for} \quad x_t \in [\hat{x}_b, \infty) \tag{5}$$

The trigger point for bankruptcy is

$$\hat{x}_b = \hat{x}_b(w) = \frac{\xi}{\xi - 1}\left(\gamma_E + \frac{w+c}{r}\right)[r-\mu] \tag{6}$$

and $\xi \equiv \left(-(\mu - \sigma^2/2) - \sqrt{(\mu-\sigma^2/2)^2 + 2\sigma^2 r}\right)/\sigma^2$.

[11] See Franks and Torous (1989) and Brown (1989).

[12] Eberhart, Moore, and Roenfeldt (1990) find that on average equity holders receive 7 percent of firm value in Chapter 11.

Lastly, the total firm value, $\hat{W}(x_t)$, is given by:

$$\hat{W}(x_t) = \frac{x_t}{r-\mu} - \frac{w}{r} + \left[\gamma - \frac{\hat{x}}{r-\mu} + \frac{w}{r}\right]\left(\frac{x_t}{\hat{x}}\right)^{\xi} . \tag{7}$$

Here, γ is the liquidation value, $\hat{x}$ is optimally chosen liquidation point:

$$\hat{x} = \frac{\xi}{\xi - 1}\left(\gamma + \frac{w}{r}\right)[r-\mu] \ . \tag{8}$$

2.2 Pure and Mixed Strategy Equilibria in a Duopoly

Now, consider the strategic interaction between two identical levered firms. Suppose that if one firm exits first, the other obtains some monopoly power. The term "monopoly power" should not be interpreted in the literal sense as reorganisation often involves the firm being subsequently run as an impaired pure-equity operation by creditors. In many cases, however, the reorganised firm ends up being liquidated or partially dismantled by creditors (see Franks and Torous (1989)). Specifically, we assume that when one firm exits, the earnings flow variable, x_t, obtained by the remaining firm jumps up by a fixed amount Δ and subsequently evolves according to the same geometric Brownian motion as in equation (1) starting from the new higher level.

The prospect of acquiring monopoly power gives each firm an incentive to outwait its competitor, delaying the decision of equity holders in a financially distressed firm to cease injecting capital. Counter-balancing this incentive, equity holders must inject capital to avoid bankruptcy. The longer the firm waits, the greater the costs incurred. The model therefore resembles a war of attrition.[13]

A study which closely resembles our own in that it focuses on levered firms in a stochastic duopoly model is Lambrecht (2001). In the duopoly he examines, Lambrecht (2001) shows that when firms are identical, there exist two subgame perfect, pure strategy Nash equilibria. These consist of the losing firm exiting first at the trigger which would be optimal for a monopolist. Lambrecht's analysis implies interesting results on the order of firms' departure from industries and relates these to the firms' "fitness" and "fatness", as discussed in the empirical study by Zingales (1998).

[13] The literature on war of attrition games is large. Studies of pure-equity firms operating in a deterministic war of attrition include Ghemawat and Nalebuff (1985) and Fudenberg and Tirole (1986).

Although we shall not focus on them, there *are* asymmetric pure strategy equilibria in our model like those examined by Lambrecht. Under this solution concept, default is triggered when the state variable, x_t, reaches a lower threshold. A striking feature of the pure strategy equilibria however is their extreme asymmetry. Although firms are ex ante identical, one firm extracts the entire "surplus" on offer in the game. Experimental evidence suggests that game-playing agents are often reluctant to accept severely asymmetric allocations. Much of this evidence[14] is in the context of Nash bargaining, but the results have significance for game theory in general.

A second disadvantage with the pure strategy equilibria is that the debt values are greater than or equal to those one would observe in a monopoly.[15] This means that the corresponding default premia are smaller than those in the monopoly case. A common criticism of structural models is the small size of the default premia when they are parameterised in a plausible way.[16]

A significant contribution of this paper is that it introduces a class of randomised strategies and solves for a symmetric equilibrium in which bankruptcy occurs at the first jump time of a point process with rate of jump, λ_t. Since λ_t will turn out to be a function of the contemporaneous levels of the state variables, the random bankruptcy process becomes a conditionally Poisson process. This is important since it means that the pricing expressions in our model resemble reduced-form models for valuing defaultable debt of the kind developed by Duffie and Singleton (1999).

2.3 Claim Values and Default Intensities in a Duopoly

As in the monopoly described above, we consider two identical firms that issue infinite maturity debt. We suppose that the two sets of equity holders precipitate bankruptcy

[14]For example, Weg, Rapoport, and Felsenthal (1990) experimentally test different bargaining outcomes, when players make alternating offers over an infinite horizon with discounting. They reject the hypothesis that players prefer sub-game perfect equilibria (SPE solution) and accept the hypothesis that agents prefer alternative 'focal points', such as a 'split-the-difference' (STD) rule, where the surplus is divided evenly between the two players. Ochs and Roth (1989) find similar results in their experiment.

[15]With identical firms, one firm exits non-strategically while the other reaps the rewards of monopoly power. The debt value of the first firm is the same as the monopoly value and the second firm's debt value is clearly larger.

[16]Jones, Mason, and Rosenfeld (1984) is a standard reference for this problem. Using Merton's structural model, they found that credit spreads were consistently underestimated.

when they decide to cease payments to creditors. Since the firms randomise their default decisions, there will be extra terms in the differential equation representing the probabilities of default by one of the two firms in the duopoly. In a time increment δt, the probability of default with a conditionally Poisson process, λ_t, equals $\lambda_t \delta t$. As the two firms are identical, it is natural to look for a symmetric Nash equilibrium. By financial market equilibrium with risk-neutral agents, the return on safe bonds must equal the net income to equity holders plus the capital gains and the probability-weighted payoffs that arise when one or other firm defaults. Applying the generalised form of Ito's lemma for jump-diffusions, one obtains a differential equation for each of the duopoly equity values, $V(x)$:

$$rV = x - w - c + \mu x \frac{dV}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2 V}{dx^2} + \max_{\lambda \geq 0} \{\lambda [\gamma_E - V]\} + \lambda \left[\hat{V}(x + \Delta) - V\right] \quad (9)$$

The difference between the above Hamilton-Jacobi-Bellman equation and the equation we encountered in the monopoly case, (equation (2)), is the presence of two payoffs, representing the gain to either firm of randomly "losing" or "winning" the game. If a firm loses the game, it defaults first and its equity value jumps by an amount $\gamma_E - V$. If it wins the game, the other exits and the equity of the remaining firm jumps by $\hat{V}(x + \Delta) - V$. Since each set of equity holders only control their own default decision, maximisation operators appear only on the terms that result from this decision (i.e. $\lambda [\gamma_E - V(x)]$).

Two facts influence the equilibrium:

1. Since the equity holders can at any time decide to default receiving γ_E, the absence of arbitrage implies that $V(x) \geq \gamma_E$.

2. To maximise their value, equity holders choose their (non-negative) hazard rate $\lambda(x)$ to maximise $\lambda(x)(\gamma_E - V(x))$, taking the other firm's randomisation, $\lambda(x)$, as given.

These two facts imply that $\lambda(x) = 0$ if $V(x) > \gamma_E$ (as any other positive hazard would leave the term $[\gamma_E - V]$, which is in control of the equity holders, negative) and $\lambda(x) \geq 0$ only if $V(x) = \gamma_E$. By substituting the solution $V(x) = \gamma_E$, into the HJB equation, however, the hazard (in this case the other firm's response) must satisfy:

$$r\gamma_E + w + c - x = \lambda \left[\hat{V}(x + \Delta) - \gamma_E\right] \quad (10)$$

Thus, the equity holders are compensated for their inefficient waiting, where $V(x) = \gamma_E$, by the possibility that the other firm defaults, with the randomisation rate, λ, given in (10).

One may distinguish between an interval over which $\lambda(x) = 0$ and an interval over which $\lambda(x) > 0$. By symmetry, the two intervals will be the same for the two firms. For x less than some level, x^*, $V(x) = \gamma_E$ and $\lambda(x)$ is given by equation (10). For $x \geq x^*$, $V(x) > \gamma_E$ and $\lambda(x) = 0$.

To derive the equity value, we solve equation (9) for $x > x^*$ imposing similar unlimited liability boundary conditions as in the monopoly case and value-matching and smooth-pasting conditions at x^*. This is simple because with a zero hazard the equation is just the monopoly differential equation (2). Effectively, one may think of the equity holders as deciding on the switching point x^*, at which they start randomizing.

As in the case of the equity, the debt values are influenced by the possibility that either firm may default. The value of debt must satisfy an analogous equation to (3), with the addition of two probability-weighted payoffs, corresponding to the impact on bond holders' claim values when their equity holders "win" or "lose" the game. By financial market equilibrium and Ito's Lemma, the value of debt must satisfy the differential equation:

$$rD = c + \mu x \frac{dD}{dx} + \frac{\sigma^2 x^2}{2} \frac{d^2 D}{dx^2} + \lambda \left[(1-\phi)\, \hat{W}(x) - \gamma_E + \hat{D}\,(x + \Delta) - 2D \right] \tag{11}$$

As x_t tends to infinity the risk of default disappears, so the debt must equal its riskless value (i.e. $\lim_{x_t \to \infty} D(x_t) = c/r$). The lower boundary condition is obtained by noting that in the limit as the earnings tend to $x^* - \Delta$ both agents exit with hazards tending to infinity. The reason for this is that the equity holders stand to gain nothing from waiting further in this limit (since $\hat{V}(x^* - \Delta + \Delta) = \gamma_E$ and so the hazard in equation (10) tends to infinity).

We thus arrive at the following important result concerning the duopoly default intensity[17] and claim values:

[17]Throughout this paper we will use the terms 'hazards', 'randomizing strategies', 'strategies' and 'default intensities' interchangeably for $\lambda(x_t)$. It is important to note, however, that $\lambda(x_t)$ is not a standard hazard, as in reduced form models. It is both a default intensity and an "intensity" of a sudden upwards discontinuous jump Δ in the earnings x_t. Both of these effects must be incorporated into other pricing expressions.

Proposition 2 *Under the assumptions of this section, the value of each firm's equity in a complete information, feedback, Nash equilibrium with randomised strategies, prior to bankruptcy of either firm is*

$$V(x) = V(x;w) = \begin{cases} \gamma_E & \text{for } x \in (\hat{x}_b - \Delta, \hat{x}_b] \\ \hat{V}(x) & \text{for } x \in (\hat{x}_b, \infty) \end{cases} . \quad (12)$$

The default hazard rate is

$$\lambda(x) = \lambda(x;w) = \begin{cases} (r\gamma_E + w + c - x)/(\hat{V}(x+\Delta) - \gamma_E) & \text{for } x \in (\hat{x}_b - \Delta, \hat{x}_b], \\ 0 & \text{for } x \in (\hat{x}_b, \infty). \end{cases} \quad (13)$$

The corresponding duopoly debt value is given by the solution to equation (11) with the following boundary conditions:

$$\lim_{x \to \infty} D(x) = c/r,$$

$$\lim_{x \downarrow \hat{x}_b - \Delta} D(x) = \frac{1}{2}\left[(1-\Phi)\left(\hat{W}(\hat{x}_b - \Delta) + \hat{W}(\hat{x}_b)\right)\right] - \gamma_E.$$

The hazards shown in equation (13) have several interesting properties. First, equity holders will not default at a point higher than their non-strategic trigger $\hat{x}_b$. The reason for this is that the equity holders can always obtain the monopoly equity value, by exiting at $\hat{x}_b$. Second, neither agent will default at an income value equal to or lower than $\hat{x}_b - \Delta$. In the limit as $x \downarrow \hat{x}_b - \Delta$ each group of equity holders becomes indifferent between foreclosing first or second since either leads to a post-exit payoff of γ_E.

Figure 1 illustrates the monopoly and duopoly security values and the associated hazard. In all the numerical calculations of this section, the same baseline parameters are used. The short rate is set at 6 percent, while the drift is 0. The continuous flow cost is set at 0.15 and the coupon rate at 0.3. The liquidation value of the firm in the pure-equity case, γ, is set equal to 2, while the deviation from absolute priority was 0.2, which approximately equals 7 percent[18] of $\hat{W}(x)$ over the randomizing interval. Defining $x_b^d \equiv \hat{x}_b - \Delta$, the volatility of the earnings process, σ, is set in such a way that the firm value's volatility over the interval, $(x_b^d, x_b^d + \Delta]$, approximately equals 15 percent. We thus used a earnings volatility of 8 percent. Finally, the monopoly

[18] This is the average value calculated by Eberhart, Moore, and Roenfeldt (1990) in their empirical study of Chapter 11 reorganizations.

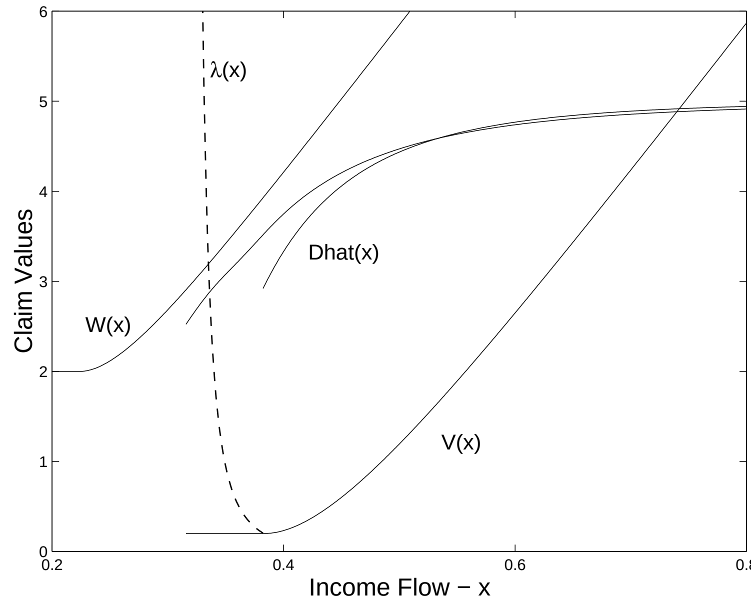

Figure 1: Monopoly and Duopoly Security Values and the Default Hazard.
Base case parameters as described in the text were used for all figures in this section.

jump was set in the base case to a modest level of 0.066 (less than half the flow cost, w). This jump was chosen so that the predictable default point was well above the liquidation point in the pure-equity case. i.e. $x_b^d \gg \hat{x}$.[19] We also set the costs of bankruptcy, ϕ, to 0.2.

Figure 1 shows that the monopoly default trigger is just below an income flow of 0.4. This is the point at which the hazard first becomes non-zero as x_t decreases. The hazard tends to infinity as earnings approach the duopoly default trigger which is clearly well above the smooth-pasting firm liquidation trigger, $\hat{x}$.

[19]Formally, the relation $x_b^d > \hat{x}$, can be accomplished if:

$$\left(\frac{\xi}{\xi - 1}\right)(r - \mu)\left[\gamma_E + \frac{c}{r} - \gamma\right] > \Delta$$

.

3 The Structure of the Default Hazards

3.1 Features of the Hazards

If one examines Figure 1 and equation (13), several features of the default intensities stand out. First, unlike the default hazards that arise in other hybrid structural models, the default intensities in our model are entirely endogenous and reflect the following features of the firm and the environment in which it operates: (i) macroeconomic variables (through the interest rate, drift parameter as well as the volatility); (ii) the capital structure of the firm (through the coupon rate); (iii) the interaction of real investment decisions with debt (through the continuous flow cost w of the financial activity); (iv) shareholder incentives in deviations from the absolute priority rule (through γ_E); (v) oligopoly effects (through the size in monopoly jump, Δ).

Second, the hazards can assume very large values for some levels of the state variable, x_t. Indeed, when earnings approach $x_b^d = \hat{x}_b - \Delta$ the default hazard explodes in that $\lambda_t \to \infty$. In contrast, many reduced-form models, drawing from methods in fixed-income pricing, specify mean-reverting processes for the default intensity. It is very unlikely in such models that the default hazard will ever assume high values.

Third, the hazard rate is non-increasing in the income flow which implies a sensible negative correlation between the firm's financial well-being and the risk of default. Fourth, the recovery rate on the firm's debt in the event of default is random since bankruptcy may occur at any point in a discrete interval of state variable values.

3.2 Comparative Statics of the Default Hazard

The simplicity of the default hazards enables one to derive several comparative statics.

Proposition 3 *Given the default intensity $\lambda(x_t)$ calculated in proposition 2, the following relations apply:*

$$
\begin{array}{llllll}
(i) & \dfrac{\partial \lambda}{\partial x} \leq 0, & (ii) & \dfrac{\partial \lambda}{\partial \sigma} \leq 0, & (iii) & \dfrac{\partial \lambda}{\partial c} \geq 0, \\
(iv) & \dfrac{\partial \lambda}{\partial w} \geq 0, & (v) & \dfrac{\partial \lambda}{\partial \gamma_E} \geq 0, & (vi) & \dfrac{\partial \lambda}{\partial \Delta} \leq 0.
\end{array} \tag{14}
$$

Result (i) shows that the intuitively reasonable relation between short-maturity credit spreads and firm profitability holds in our model. Result (ii) shows that higher volatility reduces the default hazard and consequently short-maturity spreads. The intuition here is that lower volatility reduces the equity value of the firm in the event that it wins the game and hence a larger hazard is required to maintain the two firms in equilibrium.

Results (iii) and (iv) show that increases in total firm costs either through greater debt service costs or greater flow costs are associated with a larger hazard and consequently greater short-maturity credit spreads. The intuition is that higher costs make it less attractive to equity-holders to maintain a firm in operation and so a larger hazard by the competitor firm and greater chance of a jump rise in equity values is required to maintain equilibrium.

Note, moreover, that results (iii) and (iv) confirm the empirical finding of Zingales (1998) that the 'fattest' and 'fittest' firms, respectively, are most likely to survive market shake-outs.[20]

Results (v) and (vi) show the impact on the hazard of increases in γ_E (positive) and of Δ (negative). The intuition is that equity-holders require a greater hazard of exit by their competitor to maintain value equal to γ_E for a given cash-flow $x - c - w$ and require a smaller hazard if the jump in cash flows that occurs in the event of exit by the competitor, Δ, is larger.

4 The Term-Structure of Credit Spreads

4.1 Credit Spreads in a Duopoly

In this section, we study the impact of the default hazards on spreads of different maturities. For simplicity, we suppose that the firms' liabilities consist predominantly of infinite maturity debt like that described above but that they have issued a marginal amount of a pure discount bond. The discount bond issue is assumed to be so small that it does not affect the equilibrium hazard rate so we can concentrate on pricing it while avoiding the complications that arise if the firm has a complex, time-varying capital structure. Duffie and Lando (2000) adopt a similar approach of assuming

[20] Lambrecht (2001) finds analogous results in his model of pure strategy equilibria.

predominantly perpetual debt while examining the pricing of small quantities of finite maturity bonds.

Let $D(t, x_t)$ denote a zero-coupon bond with a terminal maturity T. Suppose that in the event of bankruptcy, holders receive a fraction, $(1-\psi)$, of a riskless bond (i.e. recovery of treasury, see Jarrow and Turnbull (1995)) with the same maturity and contractual cash flow as the original zero-coupon bond. The value of the defaultable bond satisfies the following partial differential equation:

$$\frac{\partial D}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 D}{\partial x^2} + \mu x \frac{\partial D}{\partial x} + \lambda\left[(1-\psi)\, e^{-r[T-t]} + \hat{D}\,(t, x+\Delta) - 2D\right] = rD \qquad (15)$$

with the final boundary condition, $D(T, x_T) = 1$ and the lower predictable boundary condition: $D(t, x_b^d) = (1-\psi)\exp\left[-r\,(T-t)\right]$. As the earnings state variable tends to infinity, the prospect of bankruptcy diminishes and so the bond value tends to that of a riskless bond: $\lim_{x_t\to\infty} D(t, x_t) = \exp\left[-r\,(T-t)\right]$.[21] Should the other firm default first, the firm's income flow will experience a jump Δ and the value of the bond will equal that of a bond in a structural model with no strategic-interaction, $\hat{D}\,(t, x_t)$.[22] The hazard in the differential equation (15) is given by equation (13) in proposition 2.

The credit spread of a zero-coupon bond equals the difference between its yield to maturity and that of a riskless bond:

$$CS(t, x_t) = -\frac{\log\left[D\,(t, x_t)\right]}{T-t} - r.$$

In Figure 2, we compare the credit spreads arising on the zero-coupon bond issued by a firm operating in a duopoly and in a monopoly.[23]

[21]This condition applies since as x_t approaches $\hat{x}_b - \Delta$, it becomes very likely that both firms will exit, the first because it will lose the exit game, and the second because, after the resulting upward jump, x_t will be close to the monopoly-firm exit trigger.

[22]This debt value is given by the solution to the simpler partial differential equation:

$$\frac{\partial \hat{D}}{\partial t} + \frac{\sigma^2 x^2}{2}\frac{\partial^2 \hat{D}}{\partial x^2} + \mu x \frac{\partial \hat{D}}{\partial x} = r\hat{D} \qquad (16)$$

with final and upper boundary conditions similar to those of the duopoly case and the lower boundary condition: $\hat{D}(t, \hat{x}_b) = (1-\psi)\exp\left[-r\,(T-t)\right]$.

[23]We used a fully explicit finite-difference scheme to discretize the PDE (15). In this scheme the income flow spacing, δx, was set such that $x_{max}/\delta x = 200$, where the maximum income flow in the model, $x_{max} = 0.7$. As for the time-step this was set such that $T/\delta t = 4000$, where the maturity $T = 10$. The other parameters we employed were: $r = 0.06$, $\mu = 0.0$, $\sigma = 0.08$, $\gamma = 0.5$, $\Delta = 20 \times \delta x$, $w = 0.08$, $c = 0.3$, $\phi = 0.3$, $\psi = 0.3$, $T = 10$, and $\gamma_E = 0.05$.

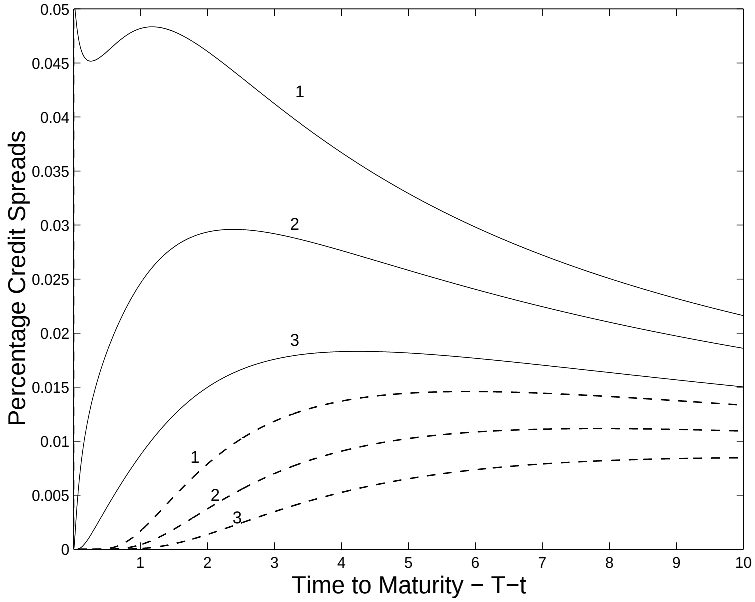

Figure 2: Monopoly and Duopoly Credit Spreads

Dashed lines indicate monopoly credit spreads while solid lines represent duopoly spreads. The different lines show spreads evaluated at different ratios between the state variable x_t and the predictable default points, $\hat{x}_b$ in the monopoly case and $\hat{x}_b - \Delta$ in the duopoly case. The labels '1', '2' and '3' refer to spreads evaluated at the ratios 1.25, 1.30 and 1.35, respectively.

The lines labelled 1 correspond to a case in which the state variable, x_t, lies in the interval $[\hat{x}_b - \Delta, \hat{x}]$. In the duopoly case, equity holders are randomizing in this interval so a surprise bankruptcy is possible and short credit spreads are strictly positive. The lines labelled 1 correspond to cases in which the state variable, x_t exceeds $\hat{x}_b$ so a surprise bankruptcy is not possible in the next moment and so short credit spreads are zero in a duopoly. Since the hazards are time-independent, the lower predictable bankruptcy triggers x_b^d and $\hat{x}_b$ are constant over time. We follow Longstaff and Schwartz (1995) in measuring credit spreads at ratios of the income flow with respect to these two points, $x_t/\hat{x}_b$ and $x_t/(\hat{x}_b - \Delta)$ monopoly and duopoly cases, respectively.[24] Clearly the effect of strategic behaviour is substantial with a strictly positive duopoly default premium at the short end, even for income flows well above the predictable bankruptcy trigger.

The plots in Figure 2 resemble the credit term structures reported by various

[24]Other measures could have been used. For example, we could have compared spreads at equal discrete earnings, δ, from the respective predictable bankruptcy points. The comparison would have been similar, however.

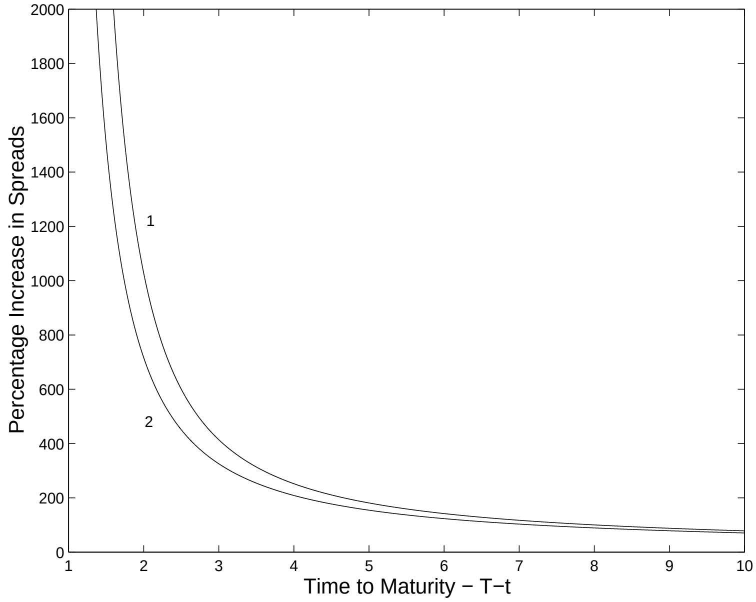

Figure 3: Duopoly to Monopoly Credit Spreads Ratios in Percent
These are taken at different ratios of the income flow from the predictable default point. The labels '1', and '2' refer to income flow ratios 1.30 and 1.35.

empirical studies. For example, see Litterman and Iben (1991), Sarig and Warga (1989) and Fons (1994). Few structural models are able to replicate the downward sloping term structures for low quality debt in their studies. Our results also help to resolve the problems with structural models documented by Jones, Mason, and Rosenfeld (1984) in that, as Figure 2 shows, spreads are higher for all maturities in the duopoly model. For further empirical evidence on the term structure of credit spreads see Duffee (1998) and Helwege and Turner (1999).

To measure the effect of strategic behaviour on the credit spreads, we define "a duopoly percentage increase in credit spreads over the monopoly case":

$$PIC\left(t,x_t\right)=\frac{CS\left(t,x_t\right)-\hat{CS}\left(t,x_t\right)}{\hat{CS}\left(t,x_t\right)}\qquad\times\qquad 100$$

where $\hat{CS}\left(t,x_t\right)=-\log\left[\hat{D}\left(t,x_t\right)\right]/\left(T-t\right)-r$, is the analogous monopoly credit spread. In Figure 3, this quantity is plotted as a function of time to maturity again keeping the ratio between the state variable and the predictable default trigger constant in the monopoly and duopoly cases. Perhaps surprisingly, for maturities in excess of 1 year, the percentage increase in spreads resulting from strategic behaviour tends to increase with credit quality. In other words, the impact of the default hazards is felt throughout a wide range of x_t values, even though the hazards are non-zero only in a limited lower interval, $(x_b^d,\hat{x}_b]$.

5 The Generalised Hazard Rate

5.1 Learning with Incomplete Information

An apparent shortcoming of the analysis of the previous sections is that the Nash equilibrium is a knife-edge case. As is true of mixed strategy equilibria in many other contexts, asymmetries in parameters across agents cause the equilibrium to break down. This would seem to limit the interest of the analysis. However, an equilibrium with multiple types may be sustained if there is incomplete information. One may also view the introduction of incomplete information as desirable as the model is then more realistic. In a duopoly, a significant risk for firms is that their conjectures about their rivals may be incorrect.

In this section, we generalise our model to include incomplete information over firms types. While this adds little to the more important implications of the previous sections, it answers the possible criticism that we are examining a knife-edge case. To be specific, we determine the default hazard when each firm in the duopoly may have flow costs equal to one of two levels, $w_1 < w_2$. Evidently, one could designate other parameters in the model as a source of incomplete information but it is natural to think that information on flow costs will be private to the firm.

We develop a Bayesian model in which equity holders act rationally and filter past events to revise their conjectures about each other's type. In our case, firms acquire new information from the fact that their competitor *has not so far defaulted*. We assume that each firm's prior at date 0 that its competitor is of the more efficient type (i.e., has costs w_1) is $\overline{P}$. For date $t > 0$, we denote the filtered probability that the other firm has costs w_1 as P_t. We show in the Appendix that if firms just observe that the other has not so far defaulted, then in the period up to the other's default, P_t evolves over time according to the Riccati-type equation

$$\frac{dP_t}{dt} = P_t \left(1 - P_t\right) \left[\tilde{\lambda}_2 \left(x_t, P_t\right) - \tilde{\lambda}_1 \left(x_t, P_t\right)\right] \tag{17}$$

where, $\tilde{\lambda}_i$ are the default hazards of firms in this two-type environment. As a conditional expectation, P_t is, of course a martingale. One may show that the upward drift in P_t shown on the right hand side of equation (17) is compensated by the chance that the other firm will default, in which case P_t will jump to zero. Further note that the reason that the evolution of P_t prior to default has no diffusion term is a consequence of the fact that the new information that each firm acquires about its competitor in any instant of time $(t, t+\delta)$ comes not from the level of x_t but from the fact that it does not default in $(t, t+\delta)$ when the hazards $\tilde{\lambda}_2$ and $\tilde{\lambda}_1$ are known and $\tilde{\lambda}_2 > \tilde{\lambda}_1$.

5.2 Default Hazards with Incomplete Information

As in the perpetual debt duopoly model of Section 2, we look for a symmetric Nash equilibrium. The presence of incomplete information introduces a second state variable, P_t, that affects the pricing equations. Ito's lemma and financial market equilibrium imply that the value of equity in a firm, V_i, of type $i \in \{1, 2\}$ satisfies a Hamilton-Jacobi-Bellmann partial differential equation[25]

$$\begin{aligned} rV_i &= \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_i}{\partial x^2} + \mu x \frac{\partial V_i}{\partial x} + P\left(1-P\right)\left[\tilde{\lambda}_2 - \tilde{\lambda}_1\right]\frac{\partial V_i}{\partial P} + x - w_i - c \\ &+ \max_{\tilde{\lambda}_i \geq 0}\left\{\tilde{\lambda}_i\left[\gamma_E - V_i\right]\right\} + \left[P\tilde{\lambda}_1 + \left(1-P\right)\tilde{\lambda}_2\right]\left[\hat{V}\left(x+\Delta; w_i\right) - V_i\right] \end{aligned} \quad (18)$$

As before, the i-th agent maximises terms involving their own hazard rate. The second term on the second line of (18) reflects the probability of winning the game for the i-th agent. Conditional on no defaults occurring up to time t, the opponent will default in the next instant with probability $P_t\tilde{\lambda}_1$ if it is of the more efficient type and with probability $(1-P_t)\tilde{\lambda}_2$, if it is of the less efficient type. The absence of arbitrage opportunities implies that $V_i \geq \gamma_E$, as the equity holders can always default. When the i-th firm's equity value strictly exceeds γ_E, equity holders have no incentive to default so $\tilde{\lambda}_i = 0$. Substituting from the no-arbitrage condition (i.e. $V_i = \gamma_E$) into the HJB partial differential equation reveals that the twin default intensities satisfy:

$$\frac{r\gamma_E + w_i + c - x}{\hat{V}\left(x+\Delta; w_i\right) - \gamma_E} = P\tilde{\lambda}_1 + \left(1-P\right)\tilde{\lambda}_2, \qquad i \in \{1, 2\}. \quad (19)$$

Equation (19) prescribes a linear system satisfied by the hazards. Since $w_1 < w_2$, and $\hat{V}\left(x+\Delta; w_1\right) > \hat{V}\left(x+\Delta; w_2\right)$, the system can only possibly yield solutions to the hazards if one of the hazards is equal to zero. This will be true whenever one of the equity values exceeds γ_E. We show in the Appendix that the value of a firm of the efficient type is always greater than γ_E as long as there is uncertainty about its rival's type (i.e., $P < 1$). Hence, the equilibrium will involve randomisation by less efficient firms (if they are present) in some upper range of the earning process x_t until one exits or P_t equals unity. If $P_t = 1$ and both firms remain, then they behave as in the complete information equilibrium described before.

[25] Debt values satisfy similar equations but we omit the details since the focus of this section is default hazards with incomplete information.

The incomplete information equilibrium is summarised in the following proposition.

Proposition 4 *Under the assumptions of this section, there is a symmetric, feedback, Bayesian Nash equilibrium in which less efficient firms default at the first jump time of conditionally Poisson processes with the default intensity:*

$$\tilde{\lambda}_1 (x_t, P_t) = 0, \qquad P_t \in [\overline{P}, 1), \tag{20}$$

$$\tilde{\lambda}_2 (x_t, P_t) = \lambda (x_t; w_2) / [1 - P_t], \qquad P_t \in [\overline{P}, 1). \tag{21}$$

When $P = 1$, the game reverts to the symmetric equilibrium of proposition 2, (with w replaced by w_1).

In equilibrium, the equity values are

$$V_2 (x_t, P_t) = V (x_t; w_2) \tag{22}$$

and

$$V_1 (x_t, P_t) = V (x_t; w_1) + \mathrm{E}_t \left\{ \int_t^T \left(\hat{V} (x_\tau + \Delta; w_1) - V (x_\tau; w_1) \right) \lambda_2 (x_\tau) \exp \left[- \int_t^\tau (r + \lambda_2 (x_s)) \, ds \right] d\tau \right\} \tag{23}$$

where $\lambda_2 (x_t) = \lambda (x_t; w_2)$ and $T = \inf \{s > 0 | P_s = 1\}$, is the first time that the conditional probability equals unity.

Thus, equilibrium has a 'type-filtering' property, whereby more efficient types delay their randomizing until they are convinced that their opponent is also efficient. Meanwhile, the less efficient firm (if one is present), randomises its default decision with a hazard that increases as the state variable, x_t, declines and as the probability tends to unity. The dramatic effect of learning on the hazard, through P_t, can be seen in equation (21).

The really important point to note here is that the hazards obtained in the incomplete information model of this section is identical to the hazards we obtained in the symmetric firm, complete information model up to a proportional factor that depends only on the conditional type probability, P_t. So for a given P_t, the comparative statics we obtained above and our observations on the form of the hazards remain valid.

Substituting the hazards given in Proposition 4, into the Riccatti equation for the probability, one obtains that the beliefs of the players evolve according to the simpler differential equation

$$\frac{dP_t}{dt} = P_t \lambda_2(x_t) \text{ which implies } P_t = \overline{P} \exp\left[\int_0^t \lambda_2(x_s) ds\right]. \tag{24}$$

The hazard, $\tilde{\lambda}_2$, therefore, equals

$$\tilde{\lambda}_2 = \frac{\lambda_2(x_t)}{1 - \overline{P} \exp\left[\int_0^t \lambda_2(x_s) ds\right]} = -\frac{\partial}{\partial t}\left\{\log\left[\exp\left(-\int_0^t \lambda_2(x_s) ds\right) - \overline{P}\right]\right\}. \tag{25}$$

As one may see from equation (25), the incomplete information default hazards depend on the prior, $\overline{P}$, and the historical time path of the state variable, x_t. An interesting feature of the hazard, as presented in equation (25), is that it is actually a time-derivative of a function of the probability,[26] and thus confirms the important relationship between learning and default events. Duffie and Lando (2000) also obtain default intensities that are first derivatives of a function. In their case the function is the conditional distribution of the firm's assets and the derivative is with respect to the underlying firm value.

5.3 A Numerical Example

If both firms are of type w_1, the time τ at which they conclude that the other is efficient is given implicitly by:

$$\overline{P} \exp\left[\int_0^\tau \lambda_2(x_s)\, ds\right] = 1. \tag{26}$$

Using a Monte Carlo approach, we estimate the expected time, $\mathrm{E}_0(\tau)$, when the parameters are those of the baseline given in Section 2, and assuming that $x_0 = 1/2(\hat{x}_b + \hat{x}_b^d)$ and $\overline{P} = 0.5$. With a time-step of 0.001 years and 80,000 simulations, $\mathrm{E}_0(\tau)$ was estimated to be 5.8 years.

Clearly, the time required to resolve incomplete information can be quite drawn out and indeed longer than the maturity of many corporate bonds. Note that the initial income flow, x_0, is at a value that is halfway over the randomizing interval. Since the firm is assumed to be solvent when it first enters the debt contract, x_0 will be well above the monopoly bankruptcy trigger, and the expected time $\mathrm{E}_t(\tau)$ will, therefore, be even greater.

[26] The argument of the logarithm in equation (25) can be expressed as $(P_t/\overline{P})/(1 - P_t)$.

5.4 Extensions

The incomplete information version of the model outlined in this section can be extended to include a discrete number $n > 2$ of types.[27] The results would be similar in that successively less efficient types would randomise on disjoint intervals of the state variables and we would obtain the same "type-filtering" property described above.

One may note that the assumption that there are no net-worth covenants and, hence, that equity holders decide the timing of bankruptcy is not a prerequisite for obtaining equilibria in which game theoretic default intensities occur. An alternative approach would be to consider a setting in which senior creditors decide when to liquidate the firm, at which point they obtain a constant value, γ_D. The prospect that the firm might become a monopolist whereupon defaultable debt values will jump up may induce the senior bond holders to delay their liquidation decision and, instead, randomise this decision through a conditionally Poisson point process. If the firms are symmetric, and the continuous coupon yield is replaced with a finite maturity principal payment, keeping all other parameters the same as in Section 2, one may show that the symmetric default intensities in such an equilibrium are:

$$\lambda(t,x) = \frac{r\gamma_D}{\hat{D}(t, x+\Delta) - \gamma_D}$$

where $\hat{D}(t,x)$ is the monopoly senior debt value. Thus, endogenous default hazards in firms, arising from strategic behaviour, can exist both in the presence and the absence of protective bond covenants.

So far we have not discussed the incentives that may result after one of the two firms becomes bankrupt. The winner will clearly reap the rewards of temporary monopoly power in the industry and if earnings recover to high levels, one might expect another firm to enter the monopoly, causing the equilibrium to revert to the one described in this section. The industry will thus go through a cycle alternating between duopoly and monopoly market structures. This intuition is important as it means that the default hazards will be present and influence claim values for a significant portion of the time in such an industry.

[27]The main point of this section is to illustrate how asymmetries can be incorporated into the duopoly model. If there were more than 2 types, the results would be similar to those described in this section.

6 Conclusion

This paper has examined the behaviour of credit spreads in a duopoly when the firms' equity holders play a non-cooperative war of attrition game against each other. We show that there are Nash equilibria in which each firm defaults on the first jump time of a conditionally Poisson process, the jump rate of which is a function of the firm's earnings. Asymmetries in firm types may be introduced into the model by including incomplete information.

Using this framework, we demonstrate that surprise defaults may occur even in a complete information structural model of defaultable debt in which the underlying information is generated by a diffusion state variable. The fact that defaults may be a surprise in turn implies that our model can generate strictly positive short-maturity credit spreads for low credit quality bond issuers.

Our analysis advances attempts to reconcile structural models of debt valuation with the reduced form approach. The reduced form approach prices defaultable debt by specifying a hazard that the borrower will jump into default at different levels of a set of state variables. The structural model developed in this paper yields endogenous hazard rates for default for bond issuers that depend on firm-specific parameters and variables describing the firm's profitability and its financial environment.

The hazard rates we obtain have interesting properties. For example, as the state variable for firm profitability approaches certain low levels, the hazards explode to infinity so default takes place for certain. Standard reduced-form models of defaultable debt valuation usually adopt mean-reverting hazard rate specifications similar to those employed in the default-free term structure literature. Our analysis suggests that hazard rate specifications should allow for discontinuous hazard rates that explode to infinity on some sample paths.

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7 Proof of Propositions

7.1 Proof of Proposition 1

The proof of this is standard and is sketched in the text before the proposition with discussion of the boundary conditions on the ordinary differential equations. □

7.2 Proof of Proposition 2

For there to be an equilibrium, the HJB ordinary differential equation, (9), must be satisfied for $V(x_t) \geq \gamma_E$ for all x_t and the relevant maximisation must be achieved. Maximizing: $\lambda\left[\gamma_E - V\right]$ with respect to $\lambda \geq 0$ for $V \geq \gamma_E$ yields the optimal control: $\lambda = 0$ for $V > \gamma_E$ (as any other positive control would leave the term negative), while for $V = \gamma_E$, the equity holders are indifferent to their hazard, as the term is always equal to zero. By substituting, the solution $V(x_t) = \gamma_E$ into the HJB equation, (9), however, the best response control of the other firm is determined: $\lambda = (r\gamma + w + c - x_t)/(\hat{V}(x_t + \Delta) - \gamma_E)$. Thus, the hazards are:

$$\lambda(x_t) = \begin{cases} (r\gamma + w + c - x_t)/(\hat{V}(x_t + \Delta) - \gamma_E), & x \in (x^* - \Delta, x^*) \\ 0, & x \in [x^*, \infty) \end{cases} \tag{27}$$

where x^* denotes the boundary between income flow values for which $V(x_t) = \gamma_E$ and $V(x_t) > \gamma_E$. By symmetry of types, this will be the same for both sets of equity holders. Substitution of the hazards (27) back into the HJB equation over the two regimes, results in the claim value being equal to γ_E for $x \in (x^* - \delta, x^*)$ (as this is the interval over which the hazard (27) is finite) and satisfying the ODE:

$$\frac{\sigma^2 x^2}{2} V''(x) + \mu x V'(x) + x - w - c = rV(x) \tag{28}$$

for $x \in [x^*, \infty)$. The solution to the above ordinary differential equation with smooth-pasting and value-matching conditions at $x_t = x^*(V(x^*) = \gamma_E$ and $V'(x^*) = 0)$ is identical to the non-strategic value, $\hat{V}(x)$, which has a continuous first-derivative at the boundary. The boundary, itself, is also the same as the non-strategic trigger, $x^* = \hat{x}_b$.

As for the debt value, the ODE is given by the one in the monopoly case with the addition of two terms: $\lambda\left[(1-\phi)\hat{W}(x) - \gamma_E - D\right]$ and $\lambda\left[\hat{D}(x+\Delta) - D\right]$ arising from the fact that both sets of agents randomise. This results in the debt value being given by the ODE in the proposition. The first boundary condition is standard and the lower condition stems from the fact that as $x_t \downarrow \hat{x}_b - \Delta$, both firms exit with infinite randomisation. Thus with probability a half the firm exits first at $\hat{x}_b - \Delta$, and with probability a half exits second at $\hat{x}_b - \Delta + \Delta = \hat{x}_b$ in the new monopoly immediately afterwards. □

7.3 Proof of Proposition 3

We start with some important results that are required by the proofs:

Lemma: $\partial \xi / \partial \sigma > 0$.

The parameter ξ is the negative root of the fundamental equation:

$$\xi = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\sigma}{\sigma^2}}. \tag{29}$$

Differentiating this directly with respect to σ yields:

$$\frac{\partial \xi}{\partial \sigma} = \frac{2}{\sigma^3}\left\{\frac{\mu\sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\sigma}{\sigma^2}} + r + \mu\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)}{\sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2\sigma}{\sigma^2}}}\right\}. \tag{30}$$

Note that in equation (30), the magnitude of the square root term in the numerator is always greater than $\mu/\sigma^2 - 1/2$, by Pythagoras. Thus, the both the numerator and denominator are always positive and $\partial \xi / \partial \sigma > 0$.

Lemma: $r\gamma_E + w + c - x > 0$ for all $x \in (x_b^d, \hat{x}_b]$.

It can be readily shown that the term: $r\gamma_E - x_t + w + c$ is always positive, by showing that its value at the maximum x_t (for which the hazard is non- zero. i.e. the smooth-pasting point, $\hat{x}_b$) is positive:

$$r\gamma_E - \hat{x}_b + w + c = [r\gamma_E + w + c]\left(1 - \left(\frac{\xi}{\xi - 1}\right)\left(\frac{r - \mu}{r}\right)\right) \tag{31}$$

The curved bracketed term in (31) is always positive for all values of σ, μ and r as $\xi/(\xi - 1) < 1$ and $(r - \mu)/r \leq 1$. □

Lemma: $(x_t + \Delta/\hat{x}_b)^\xi < 1$ for all $x \in (x_b^d, \hat{x}_b]$.

Over the interval mentioned $(x_t + \Delta/\hat{x}_b)$ is greater than one as the lower bound is $\hat{x}_b - \Delta + \Delta = \hat{x}_b$. By noticing that ξ is negative, the proof of this result is completed. □

Lemma: $\partial \hat{V}(x+\Delta)/\partial \xi > 0$**.**

By directly differentiating the real option value we obtain:

$$\begin{aligned}\frac{\partial \hat{V}\left(x+\Delta\right)}{\partial \xi} &= \log\left(\frac{x+\Delta}{\hat{x}_b}\right)\left[\gamma_E - \frac{\hat{x}_b}{r-\mu} + \frac{w+c}{r}\right]\left(\frac{x+\Delta}{\hat{x}_b}\right)^{\xi}, \\ &= \log\left(\frac{x+\Delta}{\hat{x}_b}\right)\left(\gamma_E + \frac{w+c}{r}\right)\left(\frac{-1}{\xi-1}\right)\left(\frac{x+\Delta}{\hat{x}_b}\right)^{\xi}, \\ &> 0.\end{aligned}$$

where we the second line was calculated using the expression for $\hat{x}_b$. □

For ease of exposition we re-write the hazard:

$$\lambda(x_t) = \frac{r\gamma_E + w + c - x_t}{\hat{V}\left(x_t+\Delta\right) - \gamma_E} \tag{32}$$

Part 1: $\partial\lambda/\partial x \leq 0$

Differentiating the hazard with respect to x, we have:

$$\frac{\partial \lambda(x_t)}{\partial x} = -\frac{\left[\hat{V}(x_t+\Delta) - \gamma_E\right] + \left(r\gamma_E - x_t + w + c\right)\hat{V}'\left(x+\Delta\right)}{\left[\hat{V}\left(x_t+\Delta\right) - \gamma_E\right]^2} \tag{33}$$

where,

$$\hat{V}'\left(x+\Delta\right) = \frac{1}{r-\mu} + \left(\frac{\xi}{x+\Delta}\right)\left[\gamma_E - \frac{\hat{x}_b}{r-\mu} + \frac{w+c}{r}\right]\left(\frac{x+\Delta}{\hat{x}_b}\right)^{\xi}$$

which after some rearrangement can be simplified to:

$$\hat{V}'(x+\Delta) = \frac{1}{r-\mu}\left[1 - \left(\frac{x+\Delta}{\hat{x}_b}\right)^{\xi-1}\right] \tag{34}$$

Clearly the expression in (34) is always positive using the second result above. The first result also implies that the other terms in the numerator of (33) are positive and this completes the proof. □

Part 2: $\partial\lambda/\partial\sigma \leq 0$

Differentiating the hazard with respect to σ, we have:

$$\frac{\partial \lambda(x_t)}{\partial \sigma} = -\underbrace{\frac{r\gamma_E - x + w + c}{\left[\hat{V}\left(x_t+\Delta\right) - \gamma_E\right]^2}}_{>0}\underbrace{\frac{\partial \hat{V}\left(x+\Delta\right)}{\partial \xi}}_{>0}\underbrace{\frac{\partial \xi}{\partial \sigma}}_{>0} < 0 \qquad \square \tag{35}$$

Parts 3 and 4: $\partial\lambda/\partial w \geq 0$ and $\partial\lambda/\partial c \geq 0$:

Differentiating with respect to w we obtain:

$$\frac{\partial\lambda(x_t)}{\partial w} = \frac{\left[\hat{V}(x_t+\Delta)-\gamma_E\right]-(r\gamma_E-x_t+w+c)\left[\frac{1}{r}\left(\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi}-1\right)\right]}{\left[\hat{V}(x_t+\Delta)-\gamma_E\right]^2} \tag{36}$$

Using the second result, the last bracketed term in the numerator of (36) is negative and thus the product of this and $-(r\gamma_E+w+c-x)$ is positive. This completes the proof. □

The proof that $\partial\lambda/\partial c \geq 0$ is almost identical to the one presented here for the cost flow, w.

Part 5: $\partial\lambda/\partial\gamma_E \geq 0$:

Differentiating with respect to γ_E we obtain:

$$\frac{\partial\lambda(x_t)}{\partial\gamma_E} = \frac{r\left[\hat{V}(x_t+\Delta)-\gamma_E\right]-(r\gamma_E-x_t+w+c)\left[\frac{x_t+\Delta}{\hat{x}_b}-1\right]}{\left[\hat{V}(x_t+\Delta)-\gamma_E\right]^2} \tag{37}$$

Using the second result, the last bracketed term in the numerator of (37) is negative and thus the product of this and $-(r\gamma_E+w+c-x)$ is positive. This completes the proof. □

Part 6: $\partial\lambda/\partial\Delta \leq 0$

Differentiating the hazard with respect to Δ:

$$\frac{\partial\lambda(x_t)}{\partial\Delta} = -\frac{(r\gamma_E-x_t+w+c)\left[\frac{1}{r-\mu}+\frac{\xi}{x_t+\Delta}\left[\gamma_E-\frac{\hat{x}_b}{r-\mu}+\frac{w+c}{r}\right]\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi}\right]}{\left[\hat{V}(x_t+\Delta)-\gamma_E\right]^2}$$

$$\frac{\partial\lambda(x_t)}{\partial\Delta} = -\frac{(r\gamma_E-x_t+w+c)\left[\frac{1}{r-\mu}+\frac{\xi}{x_t+\Delta}\left[\gamma_E+\frac{w+c}{r}\right]\left[1-\frac{\xi}{\xi-1}\right]\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi}\right]}{\left[\hat{V}(x_t+\Delta)-\gamma_E\right]^2}$$

$$\frac{\partial\lambda(x_t)}{\partial\Delta} = -\frac{(r\gamma_E-x_t+w+c)\left[\frac{1}{r-\mu}-\frac{1}{x_t+\Delta}\left[\gamma_E+\frac{w+c}{r}\right]\left[\frac{-\xi}{\xi-1}\right]\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi}\right]}{\left[\hat{V}(x_t+\Delta)-\gamma_E\right]^2}$$

$$\frac{\partial\lambda(x_t)}{\partial\Delta} = -\frac{(r\gamma_E-x_t+w+c)\frac{1}{r-\mu}\left[1-\left(\frac{x_t+\Delta}{\hat{x}_b}\right)^{\xi-1}\right]}{\left[\hat{V}(x_t+\Delta)-\gamma_E\right]^2} \tag{38}$$

Analyzing (38), it is evident that the first bracketed term in the numerator is always positive from the first result. The second bracketed term in the numerator of (38) is always also positive from the second result, and this completes the proof. □

7.4 Derivation of Equation (17)

Suppose we observe the sample path of a point process and do not know whether jumps are generated by the jump rate $\tilde{\lambda}_1(x_t)$ or $\tilde{\lambda}_2(x_t)$. Let P_t be the probability that jumps are generated by $\tilde{\lambda}_{1t}$ conditional on observing the past path of x_t and hence of $\tilde{\lambda}_{1t}$ or $\tilde{\lambda}_{2t}$, and suppose that P_t is initially equal to a given prior, $\overline{P}$, i.e., $P_0 = \overline{P}$.

P_t may be up-dated using Bayes' Rule. The analysis may be performed conditional on the time path of x_t so the jump rates, $\tilde{\lambda}_{1t}$ or $\tilde{\lambda}_{2t}$, may be treated as functions of time. By Bayes' Rule

$$\text{Prob}\left\{\tilde{\lambda} = \tilde{\lambda}_1 | \text{no jump by } t+\Delta\right\} = \frac{\text{Prob}\left\{\tilde{\lambda} = \tilde{\lambda}_1 \ \ \textit{and} \ \ \text{no jump by } t+\Delta\right\}}{\text{Prob}\left\{\text{no jump by } t+\Delta\right\}} \ . \tag{39}$$

Writing out the probabilities for a small increment in time, Δ, we get:

$$P_{t+\Delta} = \frac{(1-\tilde{\lambda}_{1t}\Delta)P_t}{(1-\tilde{\lambda}_{1t}\Delta)P_t + (1-\tilde{\lambda}_{2t}\Delta)(1-P_t)} \ , \tag{40}$$

$$\frac{P_{t+\Delta} - P_t}{P_t\Delta} = \frac{(\tilde{\lambda}_{2t} - \tilde{\lambda}_{1t})(1-P_t)}{[(1-\tilde{\lambda}_{1t}\Delta)P_t + (1-\tilde{\lambda}_{2t}\Delta)(1-P_t)]} \ . \tag{41}$$

Taking the limit as $\Delta \downarrow 0$ yields the Riccati equation in (17). □

7.5 Proof of Proposition 4

Solving the Equilibrium

For an equilibrium, the HJB PDE's, must be satisfied for $V_i(x_t, P_t) \geq \gamma_E$ and the maximisations must be achieved. Maximizing: $\tilde{\lambda}_i\,[\gamma_E - V_i]$ with respect to $\tilde{\lambda}_i \geq 0$ for $V_i \geq \gamma_E$ yields the optimal control: $\tilde{\lambda}_i = 0$ for $V_i > \gamma_E$ (as any other positive control would leave the term negative), while for $V_i = \gamma_E$, the equity holders are indifferent to their hazard, as the term is always equal to zero. By substituting, the solutions $V_i(x_t, P_t) = \gamma_E$ into the HJB PDE's, however, the following linked equations must be satisfied:

$$V_1(x_t, P_t) = \gamma_E \qquad \Rightarrow \qquad P_t\tilde{\lambda}_1 + (1-P_t)\,\tilde{\lambda}_2 = \frac{r\gamma_E + w_i + c - x_t}{\hat{V}\left(x_t + \Delta; w_1\right) - \gamma_E} \tag{42}$$

$$V_2(x_t, P_t) = \gamma_E \qquad \Rightarrow \qquad P_t\tilde{\lambda}_1 + (1-P_t)\,\tilde{\lambda}_2 = \frac{r\gamma_E + w_i + c - x_t}{\hat{V}\left(x_t + \Delta; w_2\right) - \gamma_E} \tag{43}$$

Hypothesis: $V_1 > \gamma_E$ **for all** $P_t < 1$

If $V_1 > \gamma_E$ for all $P_t < 1$, then $\tilde{\lambda}_1 = 0$ when $P_t < 1$. The reason for this is that type 1 agents maximise the term, $\tilde{\lambda}_1[\gamma_E - V_1]$, with respect to $\tilde{\lambda}_1$. Since $V_1 > \gamma_E$ the term is always negative unless the hazard is equal to zero.

An important consequence of that fact that $V_1 > \gamma_E$, is that equation (42), no longer applies. Substituting $\tilde{\lambda}_1 = 0$ in (43) then implies that the other hazard, $\tilde{\lambda}_2$, is given by:

$$\tilde{\lambda}_2(x, P) = \frac{1}{1-P}\left[\frac{r\gamma_E + c + w_2 - x}{\hat{V}\left(x + \Delta; w_2\right) - \gamma_E}\right] \tag{44}$$

when $V_2 = \gamma_E$. Substituting these two hazards into the PDE for V_1, (18), then implies that the PDE is:

$$rV_1 = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_1}{\partial x^2} + \mu x\frac{\partial V_1}{\partial x} + P\lambda_2\frac{\partial V_1}{\partial P} + \lambda_2\left[\hat{V}_1\left(x + \Delta\right) - V_1\right] + x - w_1 - c$$

In the case of the less efficient firm, the equity value is either greater than or equal to γ_E. In the former case, $\tilde{\lambda}_2$ is equal to zero and so substituting both zero hazards into the HJB PDE, (18), then implies that the PDE is:

$$rV_2 = \frac{\sigma^2 x^2}{2}\frac{\partial^2 V_2}{\partial x^2} + \mu x\frac{\partial V_2}{\partial x} + x - w_2 - c \tag{45}$$

which is the same differential equation as in the monopoly case, (2). Thus, when $V_2 > \gamma_E$, for some upper interval, (x_2^*, ∞), the equity value satisfies differential equation, (45), while for a lower interval, $V_2 = \gamma_E$. In order to maximise the equity value it must satisfy value-matching and smooth-pasting conditions at x_2^*: $V_2(x_2^*, P) = \gamma_E$, and $\partial V_2(x_2^*, P)/\partial V = 0$. At $x_2^* - \Delta$, the winner's equity value is $V_2(x_2^* - \Delta + \Delta, P) = V_2(x_2^*, P) = \gamma_E$ and so there is no longer an incentive to wait. So the lower interval for which $V_2 = \gamma_E$, is $(x_2^* - \Delta, x_2^*]$. The upper boundary condition is a standard unlimited liability one, and so the equity value is identical to the one in (12):

$$V_2(x, P) = \begin{cases} \hat{V}_2(x), & x \in \left(\hat{x}_b^2\left(w_2\right), \infty\right) \\ \gamma_E, & x \in \left(\hat{x}^b\left(w_2\right) - \Delta, \hat{x}_b^2\left(w_2\right)\right) \end{cases} \tag{46}$$

The only difference here is the smooth-pasting trigger. The hazard, $\tilde{\lambda}_2$, in (44), then applies over the interval, $(x_2^* - \Delta, x_2^*]$ while elsewhere it is equal to zero.

As for the boundary conditions on the more efficient equity value, V_1. First, note that as $P_t \to 1$, the game turns into a game of complete information; and both firms are of type w_1. So the equity value tends towards the one in proposition 2 with $w = w_1$. Thus, $V_1(x_t, 1) = V(x_t; w_1)$, where this latter value is defined in (12). As $x_t \downarrow \hat{x}_b(w_2) - \Delta$, $\tilde{\lambda}_2 \to \infty$. So, once again, the game turns into one of complete information and $\lim_{x \downarrow \hat{x}_b(w_2) - \Delta} V_1(x, P) = V(\hat{x}_b(w_2) - \Delta; w_1)$. Finally, a standard unlimited liability boundary condition applies as $x_t \to \infty$.

Substituting the twin hazards into the Riccati equation, the probabilities evolve according to a much simpler differential equation, whose solution can be written out by inspection:

$$\frac{dP_t}{dt} = P_t \lambda_2(x_t) \quad \Rightarrow \quad P_t = \overline{P} \exp\left[\int_0^t \lambda_2(x_s) ds\right] \tag{47}$$

where $\lambda_2(x_t) = \lambda(x_t; w_2)$, from (13) and $\overline{P} = P_0$. Since the integrand in (47) is non-negative the probability is non-decreasing in time. □

Proof that the Hypothesis: $V_1 > \gamma_E$, for $P_t < 1$ holds

Since P_t is non-decreasing in time (see (47)) define T as the first time that the probability equals unity:

$$1 = P_0 \exp\left[\int_0^T \lambda_2(x_s) \, ds\right] \qquad \Rightarrow \qquad T = \inf\{s > 0 | P_s = 1\} \tag{48}$$

Also, consider the hazard $\tilde{\lambda}_2$, from (44):

$$\tilde{\lambda}_{2t} = \tilde{\lambda}_2(x_t) = \frac{\lambda_2(x_t)}{1 - P_t} = \frac{\lambda_2(x_t)}{1 - P_0 \exp\left[\int_0^t \lambda_2(x_s) \, ds\right]} = \frac{\lambda_2(x_t) \exp\left[-\int_0^t \lambda_2(x_s) \, ds\right]}{\exp\left[-\int_0^t \lambda_2(x_s) \, ds\right] - P_0} \tag{49}$$

Now, consider the value of the firm equity, $V_1(x_t, P_t)$. This may be written down as the sum of (1) the probability of facing a type 1 agent multiplied by the payoff if that is the case, plus (2) the weighted-probability that the opponent is of the less-efficient type. Since the stopping time for the winner's payoff is generated by a point process, $\tilde{\lambda}_2$, this latter term must be conditioned on the fact that default has not taken place up until a given time multiplied by the probability of an exit in the next time increment. The payoff for the former term is simply $V(x_t; w_1) = V_1(x_t)$ from proposition 2. So the equity value of the more efficient firm is given by:

$$V_1(x_t, P_t) \quad = \quad P_t V_1(x_t) + (1 - P_t)\{V_1(x_t) +$$

$$
\begin{aligned}
& \mathrm{E}_t \left(\int_t^T \left(\hat{V}_1 (x_\tau + \Delta) - V_1 (x_\tau) \right) \tilde{\lambda}_{2\tau} \exp \left[- \int_t^\tau \left(r + \tilde{\lambda}_{2s} \right) ds \right] d\tau \right) \Bigg\} \\
= \; & V_1 (x_t) + (1 - P_t) \times \\
& \mathrm{E}_t \left\{ \int_t^T \left(\hat{V}_1 (x_\tau + \Delta) - V_1 (x_\tau) \right) \tilde{\lambda}_{2\tau} \exp \left[- \int_t^\tau \left(r + \tilde{\lambda}_{2s} \right) ds \right] d\tau \right\} \quad (50)
\end{aligned}
$$

where $\hat{V}_1(x_\tau+\Delta) = \hat{V}(x_\tau+\Delta; w_1)$ and $\tilde{\lambda}_{2\tau} = \tilde{\lambda}_2(x_\tau)$. The term $\tilde{\lambda}_{2\tau} \exp[-\int_t^\tau (r+\tilde{\lambda}_{2s})ds]$ can be simplified, using (49):

$$
\begin{aligned}
\tilde{\lambda}_{2\tau} \exp \left[- \int_t^\tau \left(r + \tilde{\lambda}_{2s} \right) ds \right] &= \left(\frac{\lambda_2 (x_\tau) \exp [-r (\tau - t)]}{1 - P_t \exp [\int_t^\tau \lambda_2 (x_s) \, ds]} \right) \times \\
& \quad \exp \left[\int_t^\tau \frac{-\lambda_2 (x_s) \exp [- \int_t^s \lambda_2 (x_\nu) \, d\nu]}{\exp [- \int_t^s \lambda_2 (x_\nu) \, d\nu] - P_t} ds \right] \\
&= \left(\frac{\lambda_2 (x_\tau) \exp [-r (\tau - t)]}{1 - P_t \exp [\int_t^\tau \lambda_2 (x_s) \, ds]} \right) \times \\
& \quad \exp \left\{ \left[\log \left(\exp \left[- \int_t^s \lambda_2 (x_\nu) \, d\nu \right] - P_t \right) \right]_{s=t}^{s=\tau} \right\} \\
&= \left(\frac{\lambda_2 (x_\tau) \exp [-r (\tau - t)]}{1 - P_t \exp [\int_t^\tau \lambda_2 (x_s) \, ds]} \right) \times \\
& \quad \left(\frac{\exp [- \int_t^\tau \lambda_2 (x_\nu) \, d\nu] - P_t}{1 - P_t} \right) \\
&= \left(\frac{\lambda_2 (x_\tau) \exp [-r (\tau - t)]}{1 - P_t} \right) \exp \left[- \int_t^\tau \lambda_2 (x_\nu) \, d\nu \right]
\end{aligned}
$$

Substituting this back into the expression for $V_1(x_t, P_t)$ we obtain:

$$
\begin{aligned}
V_1 (x_t, P_t) \; = \; & V_1 (x_t) + \quad (51) \\
& \mathrm{E}_t \left\{ \int_t^T \left(\hat{V}_1 (x_\tau + \Delta) - V_1 (x_\tau) \right) \lambda_2 (x_\tau) \exp \left[- \int_t^\tau (r + \lambda_2 (x_s)) \, ds \right] d\tau \right\}
\end{aligned}
$$

Since P can only go up, for any time path of x_τ starting at x_t, $T - t$ cannot be higher if P_t is larger. This follows obviously from the fact that $P_\tau = P_t \exp[\int_0^\tau \lambda_2(x_s) ds]$. Given this sample-path by sample-path result, the integral must be smaller when P_t is larger. Thus, $\partial V_1 / \partial P < 0$, and the more efficient equity value is strictly decreasing in P_t. Since the final condition on the equity value is greater than or equal to γ_E (i.e. $V_1(x, 1) \geq \gamma_E$), we have that $V_1 > \gamma_E$ for all $P_t \in [P_0, 1)$. Thus the hypothesis required by the proof is verified and this completes the proof. □