Alternative Modeling for Long Term Risk

Dominique Guégan*- Xin Zhao†

Preliminary Draft

Abstract

In this paper, we propose a dynamic parametric approach to estimate financial risk in long term, which takes into account the persistence phenomena observed in financial markets, links with volatility clustering feature of financial series and also concerns the occurrence of extreme events in a relative long time span. This approach is adapted into the calculation of several risk measures, including Value at Risk, Expected Shortfall, Maximum Loss and Drawdown. Our empirical experiment with this method indicates that the approach is more conservative and robust with different risk measures.

1 Introduction

Although global economies are mostly under the process of recovery from 2010 crisis, we cannot be so optimistic to ignore the increasing instability factors in current financial markets. Considering the Greek sovereign debt crisis and also difficulties inside banking system, we have to notice that it is necessary to introspect currently applied risk management methods. The importance of long term risk is evident, but there is little speaking about it in risk management domain. Moreover, advocating the amendment published by Basel Committee in 1996, international banks have to develop their internal system to control their risks. The long term risk, however, remains an unexplored issue. Therefore, in this paper, we concern two relevant problems: long term risk measurement and comparison among current risk measures.

The deficiency of relevant study in long term risk measurement may be due to the difficulties in modeling and implementation (Culter (1993)). The modeling difficulties include the influence of extreme events, the vacuity of applicable long term risk measures and intermediate shocks which are particularly influential in long term estimation. Implementation problems concern the fact that there is fundamental diversity of the aim of risk measure between regulators and bankers. And practitioners are usually misled by short term risks, even though they may at first notice the long term risks. Thus, it will be beneficial if we could find a uniform risk measure for long-term risks which are implementable for both practitioners and regulators. Academically, there are two approaches concerning long term risks. Some papers explore the ability of long-run risks to account for asset market data. Bansal and Yaron (2004) proposed an Long-Run Risk model to evaluate the role of cyclical risks. They also further their study for asset pricing (Bansal *et al* (2009)). Other papers include Hansen *et al* (2008), kaltenbrunner and Lochstoer (2010), and Wang and Bidarkota (2010). But these studies are chiefly made for explaining asset pricing puzzle and modeling macroeconomic crises. We can not apply these models in risk management. Another approach concerning long-term risk, which seems more relevant to risk management, is the method of calculating long-term value at risk. This is an approach source from practical requirement. The best known method is the R/S rule or Hurst's Rescaled-Range method. Whereas, many empirical and theoretical studies indicate that this method intent to overestimate and also underestimate VaR in different time ranges (Blake *et al* (2000)). People also try to use simulation approaches, such as Monte Carlo Simulation (McNeil and Frey (2000)). A good estimation of long term risk by simulation approach, however, requires a large computation and might be time consuming. Moreover, the applicability of simulation method in a

*Corresponding author. University Paris 1 Panthéon-Sorbonne, CES UMR8174, 106 bd de l'Hopital, 75013 Paris, France, dguegan@univ-paris1.fr

†University Paris 1 Panthéon-Sorbonne, CES UMR8174, 106 bd de l'Hopital, 75013 Paris, France, xinzhao0617@gmail.com

large time span, such as 5-year or 10-year, estimation is an open question. Secondly, we review current risk measures in term of long term evaluation. Our discussion includes four risk measures: Value-at-Risk (VaR), Expected Shortfall (ES), Maximum Loss (MAL) and Maximum Drawdown (MAD). The aim of our study is not to check or propose desirable properties for risk measures, like the study of coherent measures (Artzner *et al* (1999)). Our interesting of study is to find which risk measure is more favorable for long term risk management, although those properties are good references when we compare different risk measures. Hence our analysis is motivated by the practical importance and implemental deficiency of long-term risk measure. We focus on the measurement of long-run risk. Particulary, we show an approach which takes into account persistent dependence and volatility clustering phenomena of financial assets, and it also includes the modeling of extreme events in a relative long time span to compute four risk measures: VaR, ES, MAL and MAD.

The effect of extreme events, for example, 1929 wall street crash, 1987 stock market crash, and recent crisis since 2007, cannot be ignored in long term risks modeling. These events are the critical points in long term risk estimation that bankers should consider in order to avoid instability or even bankruptcy. So the question is how to integrate the possibility of extreme events into the estimation of long term risks. Recent studies suggest that extreme value theory (EVT) is a good method for the modeling of extreme events. Jensen and de Vries (1991) discussed the proper distribution for return modeling and applied EVT in the modeling of large returns. Longin (1996) showed empirically the extreme returns obey a Fréchet distribution by New York Stock Exchange Index for the period 1885-1990. He furthered the research, Longin (2005), in US market and suggested that Student-t distribution could be used in a unconditional modeling of returns. McNeil and Frey (2000) combined EVT and GARCH models to estimate VaR and adopted a Monte Carlo approach to estimate multiple-days VaR, which was claimed to outperform the square root scaling method. Cotter(2006) studied extreme price movements by indexes in American, Asian and European markets through the Block Maximum Method. He concluded that the tail indexes are characterized by Fréchet distribution and the extreme return levels associated with crashes are more severe than booms. But there is little study considering extreme events with the long term risks' estimation. So it is interesting to apply EVT method in the studying of long term risk.

Then we propose an alternative approach for long term risk modeling, which is based on the study of long memory processes. The Long-range dependency (LRD) phenomenon noticed in financial assets (Greene and Fielitz (1977) and cheung and Lai (1995)) would be a potentially predictable component in the series dynamics and could contribute to long term risks estimation. LRD processes have been applied in the study of stock returns (Aydogan and Booth (1988) and Chow *et al.* (1995)). Whereas, there are two problems we need to think over before applying it. Firstly, we have to check whether there is indeed LRD in our assets. Since there are also papers claim that there is no long memory exist in financial assets (Lo (1991)). The widely used test is rescaled-range (R/S) analysis introduced by Hurst (1951) and later refined by Mandelbrot (1979), the modified R/S analysis introduced by Lo(1991), and the spectral regression method suggested by Geweke and Porter-Hudak (1983). Secondly, previous studies focus on aggregated stock indices and many studies indicate the absence of LRD. Nonetheless, it could be interesting to check the existence of LRD in individual stock returns series. In this paper, we implement our model on individual return series and also the aggregated index in order to find the answer for this doubt.

Moreover, we consider volatility clustering and also intermediate shocks. We combine the previous methods with GARCH Bilinear (BL-GARCH) type model. Bilinear model takes into account variations within the independent variables as well as covariations between the variables. This is very important in the study of financial market data where the covariance between independent variables may play a significant role in determining market volatility. De Gooijer (1989) provided evidence of the importance of bilinear models in stock return modeling. Storti and Vitale (2001) proposed a BL-GARCH model, which allowed to capture asymmetric patterns, such as leverage effects, in volatility of financial time series by augmenting the standard GARCH model with interactions between past shocks and volatilities. Diongue, Guégan and Wolff (2010) proposed a maximum likelihood estimation (MLE) methodology for BL-GARCH model. We expect that by combining BL-GARCH model with Long memory model and EVT method the model is able to have a better approximation to the underlying dynamic process of return assets.

In brief, we intent to construct a powerful and robust tool which is able to quantify the long term risks by Long memory approach, BL-GARCH model and EVT method. Then we also need to choose a good risk measure. We will develop a method out of the scope of "mean-variance" methods and the approaches based on Markowitz optimization. The most popular risk measure, except for standard deviation, is VaR which corresponds to the amount of losses that could be exceeded with a given probability at a given horizon. This measure, however, does not take into account the significant losses. Due to the demand of Basel II and Basel III during recent years, several more sophisticated proposals have been proposed. Thereafter, risk measure is no longer only associated with a single value of risk in the portfolio but also deals with a set of risk measures (Guégan and Tarrant(2010)). These new approaches taking into account the extreme shocks which are out of the classical VaR extent which mainly based on the normal law and associating with 95% confidence interval. We propose a risk measure which must itself be dynamic through time. Grounding on the basic model with proposed above, we aim at extending the work of Guégan and Tarrant(2010) and Polanski and Stoja(2010). To find a good method, we can compare several models and consider "stress test" on a horizon of one year, two years or even longer.

Our theoretical analysis will be developed through three parts. Firstly, considering the widely observed stylized facts of financial data, we introduce several possible probability distribution classes for return modeling and we also propose a new distribution to approximate the real data sets. This distribution is developed from EVT method and generalized hyperbolic distribution. Secondly, we present the LM-BL-GARCH model and relevant tests. Thirdly, we combine the distributions with volatility models and suggest a dynamic approach to estimate three lately prevalent risk measures. The empirical analysis is conducted by applying data from aggregated index of five financial markets and ten index from individual firm's return series. We compare the performance of each risk measure based on different underlying models at varied confidence level and also time horizon. The empirical results show that our approach is more conservative and robust with different risk measures.

This paper is organized as follow. In next section we introduce three classes of distributions which are widely applied in the stock return modeling. We also give a brief review of EVT method and propose a mixed distribution based on EVT method and generalized hyperbolic distribution. In section three we exhibit the general expression of LM-BL-GARCH model and several special cases that we exercise in empirical study. The three prevalent risk measures are presented in section four and also the dynamic measure we proposed for long term risk measurement. Empirical results are summarized in section five. In section six, we conclude our study and discuss the prospect of multivariate study.

2 Probability Distributions and Extreme Value Theory

One crucial issue for risk managers is to decide the underlying distribution of asset returns. Statistical distribution of asset returns play an important role in modeling financial assets. Unfortunately, neither economic theory nor statistical theory can excavate the exact distribution of returns. Or, perhaps, there is no exact distribution of returns. Whereas, for centuries, academics and practitioners try to find a good distribution to approximate the evolvement of return processes. And the distributions used in empirical implements or theoretical researches are always an assumption or estimation using historical data.

For more than three decades, Gaussian distribution was much favored to the other probability distributions due to its practical merits. However, financial data always exhibit extreme price changes which cannot be described by Gaussian distribution. Moreover, the high kurtosis and asymmetric features of real data also have been noticed recent decades. Therefore, several alternative distributions have been considered. Among these alternatives there is an interesting theory particularly deal with the tail part distribution or the distribution of extreme events named extreme value theory. In this section, we introduce three class of probability distributions which have been considered as the distribution of financial assets and give a brief review of EVT. Besides, we give special case for each distribution categorization which will be applied in empirical analysis and also propose a mixed distribution based on EVT.

2.1 Elliptical Distribution

Elliptical distribution introduced by Kelker (1970) is the generalization of the Gaussian distribution. This distribution family is symmetric and unimodal. Whereas, the generalization allows this distribution have unbounded higher moments, especially the high kurtosis. And the sum of elliptical distributions is also elliptical distribution named convolution property, which is useful in multivariate condition. These properties are attractive to financial risk management because they allow a multivariate portfolio of risks to have the property of regular varying in the marginal tails.

Defintion 2.1. *A random variable x is said to have an elliptical distribution with location parameter $\mu \in \mathbb{R}$ and dispersion parameter $\sigma \in \mathbb{R}_{+}^{*}$, $x \sim E(\mu, \sigma, \psi)$, if its characteristic function can be expressed as*

$$E[\exp(itx)] = \exp(it\mu)\,\psi\left(\sigma^2 t^2\right)$$

$\psi(.)$ is a non-negative function called characteristic generator.

The corresponding density function is $f(x \mid \mu, \sigma; g) = \frac{1}{\sigma} g\left(\left(\frac{x-\mu}{\sigma}\right)^2\right)$, $u \geq 0$ and $g(.)$ is a non-negative function $g(u) = \frac{h(u)}{\int_0^\infty u^{-\frac{1}{2}} h(u) du}$, $h(u)$ named as density generator is a non-increasing function such that the integral $\int_0^\infty u^{1/2} h(u)\, du$ exists. $E(0, 1, \psi)$ is standardized elliptical distribution.

The special cases of elliptical distribution include normal distribution, student-t distribution and Laplace distributions. The later two distributions have higher kurtosis than normal distribution. We will apply these three distributions in our empirical analysis.

2.2 Generalized Hyperbolic Distribution

Generalized hyperbolic (GH) distribution is a distribution family with useful properties, though it is difficult to calibrate in high dimension (Prause (1999)). It was introduced by Barndorff-Nielsen (1977). This distribution is interesting when we model financial asset or manage risk of financial portfolio, because it can characterize exponentially decreasing tails and semi-heavy tails. Eberlein and Keller (1995) fitted the univariate distribution to return series of German equities and got a high accuracy fit. Finally, this family is also stable under affine transforms (Blaesild (1981)). This property is interesting because in GARCH setting we will be able to deduce the conditional distribution of log-returns from innovations.

Defintion 2.2. *One dimensional $GH(\lambda, \alpha, \beta, \delta, \mu)$ is defined by the following density function:*

$$f_{GH}(x, \lambda, \alpha, \beta, \delta, \mu) = \frac{(\sqrt{\alpha^2 - \beta^2}/\delta)^\lambda}{\sqrt{2\pi} K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})} e^{\beta(x-\mu)} \frac{K_{\lambda - 1/2}\left(\alpha\sqrt{\delta^2 + (x-\mu)^2}\right)}{\left(\sqrt{\delta^2 + (x-\mu)^2}/\alpha\right)^{1/2 - \lambda}}$$

where $(\lambda, \alpha, \beta, \delta, \mu) \in \mathbb{R}^5$ with $\delta > 0$ and $\alpha >\mid \beta \mid> 0$, K_λ is the modified Bessel function of the third kind. Particularly, for $\lambda \in \frac{1}{2}\mathbb{Z}$, the basic properties of the Bessel function allow simpler forms for the density.

The moment generating function of a GH distribution exists and is given by:

$$\mathbb{G}_{GH}(u) = e^{\mu u} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + u)^2}\right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\beta + u)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \beta^2})}, \quad \mid \beta + u \mid < \alpha.$$

Particularly, we define $\alpha^* = \alpha\delta$, $\beta^* = \beta\delta$, if $X \hookrightarrow GH(\lambda, \alpha^*, \beta^*, \delta, \mu)$, then $\frac{X-\mu}{\delta} \hookrightarrow GH(\lambda, \alpha, \beta, 1, 0)$. The parameters μ and δ respectively describe the location and the scale.

The special case of this distribution class includes student t-distribution, Laplace distribution, hyperbolic distribution, normal-inverse Gaussian distribution and variance-gamma distribution. In particular, for $\lambda = 1$, we get the Hyperbolic distribution (HYP) whose log-density is a hyperbola. For $\lambda = -\frac{1}{2}$, we obtain the Normal Inverse Gaussian distribution (NIG) which is a closed under convolution. In the empirical part, we will see the application of NIG distribution.

2.3 Extreme Value Distribution

EVT is a theory that assesses the probability distributions of extreme events. It is an attractive method for risk management since the object of risk managers is to access the probability of risks corresponding to extreme shocks of assets. Generally, there are two branches of EVT, block maxima models (BMM) and peak over threshold (POT) method. The first method is based on Fisher-Tippet theorem (Embrechts *et al.* (1997)) and the second approach is based on Pickands theorem. These two approaches have been proved to be equivalent to BMM by Pickands-Balkema-de Haan theorem (Embrechts *et al.* (1997)). Both approaches have been applied in risk management researches and have their own strength and weakness.

Fisher-Tippet Theorem. *Let $\{X_n\}$ be a sequence of iid rvs. If there exist norming constants $c_n > 0$, $d \in \mathbb{R}$ and some non-degenerate df H such that*

$$c_n^{-1}(M_n - d_n) \xrightarrow{d} H$$

then H belongs to the type of one of the following three dfs:

$$\text{Fréchet:} \quad \Phi_\alpha(x) = \begin{cases} 0, & x \le 0 \\ \exp\{-x^{-\alpha}\} & x > 0 \end{cases} \quad \alpha > 0.$$

$$\text{Weibull:} \quad \Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\}, & x \le 0 \\ 1, & x > 0 \end{cases} \quad \alpha > 0.$$

$$\text{Gumbel:} \quad \Lambda(x) = \exp\{-e^{-x}\}, \quad x \in \mathbb{R}.$$

Here, $M_1 = X_1$, $M_n = \max(X_1, \ldots, X_n), n \in \mathbb{Z}$ and $n \ge 2$.

Pickands Theorem. *The conditional excess distribution function $F_u(y)$ with a large threshold u for a large class of underlying distribution function F t is well approximated by Generalized Pareto Distribution.*

$$F_u(y) \approx G_\xi(y), \quad u \to \infty,$$

$$F_u(y) = P(X - u \le y | X > u),$$

$$G_\xi(y) = \begin{cases} 1 - (1 + \xi y)^{-1/\xi} & if \quad \xi \ne 0 \\ 1 - e^{-y} & if \quad \xi = 0 \end{cases}$$

$$\text{where} \quad \begin{array}{ll} y \ge 0 & if \quad \xi \le 0 \\ 0 \le y \le -1/\xi & if \quad \xi < 0 \end{array}$$

Pickands-Balkema-de-Haan Theorem. *For $\xi \in \mathbb{R}$ the following assertions are equivalent:*

(a) $F \in MDA(H_\xi)$.

(b) There exists a positive, measurable function $a(\cdot)$ such that for $1 + \xi x > 0$,

$$\lim_{u \uparrow x_F} \frac{\overline{F}(u + xa(u))}{\overline{F}(u)} = \begin{cases} (1 + \xi x)^{-1/\xi} & if \quad \xi \ne 0, \\ e^{-x} & if \quad \xi = 0. \end{cases}$$

$$\text{where } H_\xi = \begin{cases} \Phi_{1/\xi} & if \quad \xi > 0 \\ \Lambda & if \quad \xi = 0 \\ \Psi_{-1/\xi} & if \quad \xi < 0 \end{cases}, \quad \overline{F} = 1 - F, \quad MDA \text{ means Maximum domain of attraction}$$

From Fisher-Tippet and Pickands theorem we can observe that the basic difference between the two approaches is the way they select the extreme value of sample. In BMM approach, data are divided into blocks and the process of maximum in each block is assumed to follow an extreme value distribution. In POT method, extreme value invokes excesses over a high threshold. Pickands-Balkema-de Haan theorem demonstrated the equivalent of the two approaches, but POT method is more applied in practice due to its efficient use of data. The last thing to point is that within the POT approach, there are two styles of analysis. One is the semi-parametric models built around Hill estimator and its relatives (Beirlant *et al.* (1996) and Danielsson *et al.* (1998)) and the other one is the fully parametric models based on generalized Pareto (GPD) distribution (Embrechts *et al.* (1998)). In this paper we will fully parametric model since it can obtain simple parametric formulae and complement with our dynamic model. We can find three appealing points from these studies. First, EVT is a good candidate for the modeling of extreme events through tail distribution. Second, when we combine EVT with GARCH type models, we need to verify the appropriateness of applying EVT for the innovation distribution. Third, we can check the asymmetric behavior of crashes and booms by our model as well.

2.4 Mixed Distribution

All of the three kinds of distribution have been applied in financial assets modeling. Each class has its advantages and disadvantages. The elliptical distribution is favored by its high kurtosis and convolution property. But it is a symmetric distribution family which are not the case of real financial data. This kind of distribution cannot approximate the fat-tail phenomena either. GH distribution can overcome the drawbacks of elliptical distribution. It can characterize exponentially decreasing tails and semi-heavy tails. Therefore, it is preferred by academics. But the exponentially decreasing tails may not enough for modeling extreme events. Moreover, it cannot access the high kurtosis feature of real data. GPD is a good choice for tail modeling but it totally discard the central part of data. It is not a big problem for short-term risk management, but we are interesting in the long-term risk modeling. We intend to model long term risk with the long memory models in which the central part data may contain important information.

To overcome the disadvantages and combine the advantages of each kind of distribution, we propose a mixed distribution based on elliptical distribution and GPD distribution. Particulary, the central part is modeled by Laplace exponential distribution and the tail part is modeled by GPD. The density function can be expressed as follow:

$$f(x|\xi_1,\sigma_1,u_1,\xi_2,\sigma_2,u_2,\sigma,u,\theta) = \begin{cases} 1-(1+\frac{\xi_1}{\sigma_1}(x-u_1))^{-1/\xi_1} & if \quad x > u_1 \\ \frac{\theta}{2\sigma}\exp(-\frac{|x-u|}{\sigma}) & if \quad u_2 \le x \le u_1 \\ 1-(1+\frac{\xi_2}{\sigma_2}(-x-u_2))^{-1/\xi_2} & if \quad x < u_2 \end{cases}$$

u_1 and u_2 are the right threshold and left threshold respectively. ξ_1, σ_1, ξ_2 and σ_2 are the parameters of related location-scale GPD. σ, u and θ are the parameters of Laplace distribution. The advantage of this distribution is that it can not only access the high kurtosis feature but also the asymmetric fat-tails characteristic of real data. But it is difficult to choose the two thresholds, which is an unsolved problem of POT method. Nonetheless, there are some methods to implement. In this paper we will choose our thresholds based on Hill estimator and the plot of mean excess function.

3 Modeling

Another important issue for risk managers is to choose the underlying model for their assets. There are a lot of econometric models available for risk management modeling. Unfortunately, there is no exact answer for the problem which is the same as we decide the underlying distribution. However, we still can try to approximate the process movement. Based on financial data's characteristics and our aim of modeling, we apply nonlinear time series structures to model long term risks. We express our model by three steps, long memory model, bilinear model and GARCH model. To simplify the expression, we give the univariate framework hereafter. The multivariate condition will be discussed later.

3.1 Long Memory Process

First of all, we should clarify the notion of long memory process. In fact, there are two equivalent definition of long memory processes. One is based on autocorrelation function. The other is based on spectral density function. Here, we give the former definition by the decay rate of autocorrelations.

Defintion 3.1. *A covariance stationary process $\{X_t\}$ is called long memory process if its autocorrelation function ρ_k which behaves like a power function decaying to zero hyperbolically as*

$$\rho(k) \sim C_\rho \cdot k^\alpha, \text{ as } k \to \infty, \quad 0 < \alpha < 1$$

where $\sim$ refers to asymptotic equivalent, and $C_\rho(k)$ is a function which changes slowly to infinity, i.e. for all $a \in \mathbb{R}$, $C_\rho(ak)/C_\rho(k) \to 1$, when $x \to \infty$.

3.2 Generalized Model

Here, we write the generalized condition of Long Memory Bilinear Generalized Autoregressive Conditional Heteroskedasticity model, which is named by Bilinear GIGARCH (BGIGARCH) model.

$$\begin{aligned} &\Phi_L(B)\ \Pi_{i=1}^{k}(I - 2\nu_{L,i}B + B^2)^{d_{L,i}}\mathbf{r_t} = \Theta_L(B)\ \varepsilon_{\mathbf{t}} \\ &\varepsilon_{\mathbf{t}} = \mathbf{h_t}\epsilon_{\mathbf{t}} \\ &\Phi_V(B)\ \Pi_{j=1}^{\kappa}(I - 2\nu_{V,j}B + B^2)^{d_{V,j}}\mathbf{h_t}^{\delta} = \Theta_V(B)\ \epsilon_{\mathbf{t}}^{\delta}, \end{aligned} \tag{1}$$

Here, $\mathbf{r_t}$ represent the returns; $\varepsilon_{\mathbf{t}}$ represent the error term of the mean process; $\mathbf{h_t}$ mean the standard deviation of the error term; $\epsilon_{\mathbf{t}}$ mean i.i.d. innovations. $d_{L,i}$ and $d_{V,j}$ are long memory parameters for level and volatility respectively; $\nu_{L,i}$ and $\nu_{V,j}$ are frequency location parameters or level and volatility. $\Phi_L(B)$, $\Phi_V(B)$, $\Psi_L(B)$ and $\Psi_V(B)$ are ARMA operators; B is the backward difference operator. k, r, s, v, κ are parameters in $\mathbb{N}$; α_0, α_w, b_f, c_l, δ are parameters in $\mathbb{R}$.

3.3 Dynamic Long Term Risk Measures

3.3.1 VaR_t and ES_t

We rewrite the modeling for the returns, $r_t = \mu_t + h_t\varepsilon_t$, where μ_t is the level of asset's return. h_t represents the conditional volatility. We denote Loss by $\mathfrak{L}_T = \sum_{t=t_1}^{T} r_t$.

$$VaR_{\alpha,T} = \inf\{l_T \in \mathcal{R} : P(\mathfrak{L}_T > l_T) \leq 1 - \alpha\}, \tag{2}$$
$$ES_\alpha = E(\mathfrak{L}_T | \mathfrak{L}_T < VaR_{\alpha,T}) \tag{3}$$

3.3.2 MaL and MaD

Our dynamic approach is important for the calculation of MaL and MaD, by which we can have a better approximation for return series before the given time horizon.

$$MaL_T = \min\{0, r_1, , r_1 + r_2, \cdots, r_1 + r_2 + \cdots + r_T\} = \min\{0, \min_{t=1,\dots,T}\{\sum_{i=1}^{t} r_i\}\} \tag{4}$$

$$MaD_T = \min\{r_1, \cdots, r_1 + \cdots + r_T, r_2, \cdots, r_2 + \cdots + r_N, \cdots, r_T\} = \min_{i=1,\dots,t;t=1,\dots,T}\{\sum_{j=1}^{t} r_j\} \tag{5}$$

4 Empirical Study

The aim of this empirical study is to check the performance of the previous risk measures under different model. Concretely, our study is implemented through four aspects of comparison. We compare the affect of underlying model and the distribution of innovations, the difference between risk measures by static approach and dynamic approach and the affect of data. The criteria is the true loss at time T and the ratio of exceedance during the horizon.

We select daily data from five stock market indexes, S&P 500, CAC 40, FTSE 100, NIKKEI 225 and DAX, 1990-2005. This sample period is selected by reviewing comprehensive literatures of long memory in equity markets and thorough estimation of the long memory parameter for the available data. Statistic summary and the estimation of long memory parameter are set up in Appendix Figure 1 and Table 2-3. We present the empirical results with four generalized form of equation 1: AR model, AR-GARCH, IGARCH model and FIGARCH model.

$$AR: \quad \Phi_L(B)r_t = \varepsilon_t, \quad \varepsilon_t \ \ are \ \ i.i.d. \tag{6}$$

$$\begin{aligned} AR(2)-GARCH(1,1): \quad & r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t, \\ & \varepsilon_t = h_t \epsilon_t, \\ & h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2 \end{aligned} \tag{7}$$

$$\begin{aligned} AR(2)-IGARCH(1,1): \quad & r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t, \\ & \varepsilon_t = h_t \epsilon_t, \\ & h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + (1-\alpha_1) \varepsilon_{t-1}^2 \end{aligned} \tag{8}$$

$$\begin{aligned} AR(2)-FIGARCH(1,1): \quad & r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t, \\ & \varepsilon_t = h_t \epsilon_t, \\ & \Phi_V(B)(1-B)^d h_t^2 = \Psi_V(B) \varepsilon_t^2 \end{aligned} \tag{9}$$

r_t equal $\log(\frac{P_{t+1}}{P_t})$, the log return. P_t is the price during our sample period. ϕ_1, ϕ_2, α_0, α_1, β_1 and d are real numbers, which are estimated by maximum likelihood method under concrete model and likelihood functions. In equation 6, we assume $\varepsilon_t \sim N(\mu_N, \sigma_N)$, $t(\gamma, \mu_t, \sigma_t)$, $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$. In equation 7 - 9, we assume $\epsilon_t \sim N(\mu_N, \sigma_N)$, $t(\gamma, \mu_t, \sigma_t)$, $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.

For each model, we estimate the four risk measures at different time spans: in sample, 1-day ahead, 10-days ahead, 30-days ahead and 1-year ahead. For the in-sample risk measures, we follow the definition of equation 2, equation 3, equation 4 and equation 5. For the out-of sample estimation, we simulated the losses by each model and innovation distribution based on the stationary assumption.

4.1 In-Sample Result

First, we give the result of S&P 500 returns. Within our training data of S&P 500 returns, the four risk measures are static, we list the results in this section to show the risk evaluation power of the four risk measures based on different underlying models and distributions of S&P 500 returns. For the in-sample estimation, the value of MaL =0.1791, MaD = 0.6762.

(a) AR Model Risk Measure $^{*} r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \phi_3 r_{t-3} + \varepsilon_t$

		Normal	Student	NIG	GPD	Mix
VaR	0.95	0.0161 (205/4175)	0.0148 (254/4175> 5%)	0.0156 (224/4175> 5%)	0.0215 (95/4175)	0.0515 (3/4175)
	0.97	0.0184 (141/4175> 3%)	0.0185 (140/4175> 3%)	0.0195 (120/4175)	0.0252 (50/4175)	0.0558 (3/4175)
	0.99	0.0229 (75/4175> 1%)	0.0277 (34/4175)	0.0285 (30/4175)	0.0335 (15/4175)	0.0625 (2/4175)
ES	0.95	0.0231	0.0216	0.0225	0.0286	0.0672
	0.97	0.0257	0.0258	0.0269	0.0334	0.0672
	0.99	0.0304	0.0367	0.0378	0.0447	0.0708

$^{*}\varepsilon_t \sim N(\mu_N, \sigma_N)$, $t(\gamma, \mu_t, \sigma_t)$, $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.
Estimation of all the parameters are listed in Table 4.

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Result in table 4.1 show the performance of the two risk measures under linear model. We apply the Yule-Walker method to estimate the parameters in equation 6. The corresponding estimate results are listed in Appendix Table1. Then we filter the return series to abstain the residual ε_t. After, we fit the residual ε with the five distributions: Normal, Student, NIG, GPD and Mix by Maximum Likelihood Estimation method. The corresponding likelihood functions are listed in Appendix.

The in-sample value at risks are static (as we show in figure 2) and the ratios inside bracket are the in-sample ratio of exceedances. Comparing the results of value at risk under different distributions, we can notice that Mixed distribution give the most prudent evaluation of risk. With the high VaR$_{Mix}$, there are little exceedances (the ratio of exceedance is smaller than 0.007%). Secondly, expect Shortfalls evaluate the loss if the outcome break the correspond VaR. We can see that the VaR_{GPD} correspond to relative low ratio of exceedance and ES_{GPD}. At 99% confidence level, the ratio of exceedance of VaR_{GPD} is 0.3% and the correspond ES_{GPD} is 0.0447, which is only 63% of ES_{Mix}. We cannot say VaR_{GPD} is the best risk measure, but, combining the result of VaR and ES, we can conclude that under this model GPD distribution give a relative prudent VaR value and reasonable capital requirement.

(b) AR(2)-GARCH(1,1) Model Risk Measure *

		Normal	Student	NIG	GPD	Mix
VaR	0.95	(184/4175)	(200/4175)	(148/4175)	(49/4175)	(0/4175)
	0.97	(131/4175)	(127/4175)	(79/4175)	(19/4175)	(0/4175)
	0.99	(63/4175)	(36/4175)	(16/4175)	(3/4175)	(0/4175)
ES	0.95	0.0209	0.0204	0.0218	0.0287	0
	0.97	0.0226	0.0227	0.0255	0.0365	0
	0.99	0.0261	0.0295	0.0396	0.0423	0

$^{*} r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t$, $\varepsilon_t = h_t \epsilon_t$, $h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$
$\epsilon_t \sim N(\mu_N, \sigma_N)$, $t(\gamma, \mu_t, \sigma_t)$, $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.
Estimation of all the parameters are listed in Table 4.

Under AR(2)-GARCH(1,1) model, we have dynamic in-sample VaR (as we show in figure 3). The value of VaR does not only depend on the distribution of innovation but also depend on the magnitude of volatility. The ratios inside the bracket are the in-sample ratio of exceedance. This dynamic model has lower ratio of exceedance in all condition than the previous linear model. All of these ratio of exceedance are inside the range of corresponding confidence level. (e.g. the 95% level of VaR_{Normal}, $152/4175 < 5\%$). Besides, similar with the AR(2) model, mix distribution gives the most prudent value at risk, which with no in-sample exceedance but requires high capital reservation. GPD distribution compromises with lower capital requirement and small ratio of exceedance.

(c) IGARCH(1,1) Model Risk Measure*

		Normal	Student	NIG	GPD	Mix
VaR	0.95	(14/4175)	(14/4175)	(19/4175)	(0/4175)	(0/4175)
	0.97	(2/4175)	(1/4175)	(5/4175)	(0/4175)	(0/4175)
	0.99	(0/4175)	(0/4175)	(0/4175)	(0/4175)	(0/4175)
ES	0.95	0.0101	0.0101	0.0082	0	0
	0.97	0.0046	0.0080	0.0134	0	0
	0.99	0	0	0	0	0

$^{*} r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t$, $\varepsilon_t = h_t \epsilon_t$, $\Phi_V(B)(1-B)h_t^2 = \Psi_V(B)\varepsilon_t^2$
$\varepsilon_t \sim N(\mu_N, \sigma_N)$, $t(\gamma, \mu_t, \sigma_t)$, $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.
Estimation of all the parameters are listed in Table 4.

Under AR(2)-IGARCH(1,1) model, we also have dynamic in-sample VaR (as we show in figure 4). Similarly, the ratios inside the bracket are the in-sample ratio of exceedance. Under this model, the value of VaR under the five innovation distribution are much bigger than the previous two models. At 99% confidence level, there are no exceedance during the sample. This result indicates that under the AR(2)-IGARCH(1,1), we have very conservative result of VaR with all distributions.

(d) FIGARCH(1,d,1) Model Risk Measure*

		Normal	Student	NIG	GPD	Mix
VaR	0.95	(0/4175)	(0/4175)	(0/4175)	(0/4175)	(0/4175)
	0.97	(0/4175)	(0/4175)	(0/4175)	(0/4175)	(0/4175)
	0.99	(0/4175)	(0/4175)	(0/4175)	(0/4175)	(0/4175)
ES	0.95	0	0	0	0	0
	0.97	0	0	0	0	0
	0.99	0	0	0	0	0

* $r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t$, $\varepsilon_t = h_t \epsilon_t$, $\Phi_V(B)(1-B)^d h_t^2 = \Psi_V(B)\varepsilon_t^2$

$\epsilon_t \sim N(\mu_N, \sigma_N)$, $\epsilon_t \sim t(\gamma, \mu_t, \sigma_t)$, $\epsilon_t \sim NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $\varepsilon_t \sim GDP(\alpha_{GDP}, \beta_{GDP})$, or $\epsilon_t \sim Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.

Estimation of parameters are listed in Table 7. The value of $\hat{d}$ see Table 3

Comparing the in sample estimation of VaR with different distributions, we can find out that the Mixed distribution lead to a high VaR at all confidence levels. GPD distribution generates higher VaR comparing to normal distribution. And the student distribution has a smaller VaR at 95% level and a bigger VaR at 99% level comparing to normal distribution. It is evident that the value of VaR is sensitive to the distribution, but we cannot conclude which distribution is the best from the results in the tables. Then we consider the corresponding ES value. Reviewing the definition of ES, we can see the value of ES is affected by the corresponding VaR and the relevant probability. With GPD distribution, we have relative small ES and low ratio of exceedance under the four model.

4.2 Out-Of-Sample Result

In this subsection, we apply our model based on the estimation of our sample to forecast the 1-day, 10-days, 30-days and 1-year risk measures. We assume the return process is stationary. So we simulate the future returns by the underlying models. Then we calculate the four risk measures based on our simulation. The following results are based on 1000 times simulation.

4.2.1 1-Day Estimation

In this subsection, we summarize the 1-day forward risk measures of S&P 500 with different models and distributions. Corresponding Result for the other four Index are listed in Appendix. We also update the information for the following 1-year to compare the forecast power of each model.

(a) AR(2) Model 1-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR	0.95	0.0160	0.0148	0.0155	0.0214	0.0514
	0.97	0.0183	0.0184	0.0194	0.0252	0.0557
	0.99	0.0228	0.0277	0.0284	0.0334	0.0625
ES	0.95	0.0228	0.0213	0.0222	0.0283	0.0669
	0.97	0.0254	0.0255	0.0266	0.0331	0.0669
	0.99	0.0301	0.0364	0.0375	0.0397	0.0705

* $r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t$

$\varepsilon_t \sim N(\mu_N, \sigma_N)$, $t(\gamma, \mu_t, \sigma_t)$, $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.

Estimation of all the parameters are listed in Table 4.

(b) GARCH(1,1) Model 1-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mix
VaR	0.95	0.0092	0.0092	0.0089	0.0116	0.0271
	0.97	0.0105	0.0108	0.0107	0.0135	0.0293
	0.99	0.0131	0.0145	0.0144	0.0174	0.0327
ES	0.95	0.0206	0.0201	0.0215	0.0284	0
	0.97	0.0223	0.0224	0.0252	0.0362	0
	0.99	0.0258	0.0292	0.0393	0.0420	0

$^{*}r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t,\ \varepsilon_t = h_t \epsilon_t,\ h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$
$\varepsilon_t \sim N(\mu_N, \sigma_N),\ t(\gamma, \mu_t, \sigma_t),\ NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}),\ GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.
Estimation of all the parameters are listed in Table 4.

(c) IGARCH(1,1) Model 1-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mix
VaR	0.95	0.0111	0.0111	0.0107	0.0140	0.0262
	0.97	0.0127	0.0128	0.0123	0.0158	0.0277
	0.99	0.0157	0.0162	0.0153	0.0191	0.0297
ES	0.95	0.0098	0.0098	0.0079	0	0
	0.97	0.0043	0.0077	0.0131	0	0
	0.99	0	0	0	0	0

$^{*}r_t = u_t + \varepsilon_t,\ \varepsilon_t = h_t \epsilon_t,\ h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$

(d) FIGARCH(1,d,1) Model 1-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mix
VaR	0.95	0.7274	0.7174	0.7574	1.3245	2.6543
	0.97	0.7766	0.7864	0.7986	1.5432	2.8643
	0.99	0.8621	0.8945	0.9412	1.6785	3.0967
ES	0.95	0	0	0	0	0
	0.97	0	0	0	0	0
	0.99	0	0	0	0	0

$^{*}r_t = u_t + (1-B)^d \varepsilon_t,\ \varepsilon_t = h_t \epsilon_t,\ h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$
$^{**}\epsilon_t \sim N(\mu_N, \sigma_N),\ \epsilon_t \sim t(\gamma, \mu_t, \sigma_t),\ \epsilon_t \sim NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}),\ \varepsilon_t \sim GDP(\alpha_{GDP}, \beta_{GDP})$, or $\epsilon_t \sim Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$. Estimation of parameters are listed in Table 7.The value of $\hat{d}$ see Table 3

4.2.2 10-Days Estimation

(a) AR(2) Model 10-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	0.0436(0.0506)	0.0496(0.0469)	0.0477(0.0490)	0.1114(0.0676)	0.3022(0.1625)
	0.97	0.0506(0.0580)	0.0574(0.0584)	0.0548(0.0613)	0.1200(0.0796)	0.3106(0.1761)
	0.99	0.0629(0.0720)	0.0745(0.0877)	0.0752(0.0899)	0.1399(0.1057)	0.3344(0.1975)
ES	0.95	0.0525	0.0609	0.0609	0.1257	0.3183
	0.97	0.0585	0.0678	0.0690	0.1339	0.3294
	0.99	0.0687	0.0807	0.0874	0.1494	0.3525
MaL	0.95	0.1541	0.1553	0.1511	0.2122	0.5144
	0.97	0.1788	0.1919	0.1885	0.2493	0.5569
	0.99	0.2199	0.2737	0.2783	0.3418	0.6254
MaD	0.95	0.1541	0.1553	0.1511	0.2122	0.5144
	0.97	0.1788	0.1919	0.1885	0.2493	0.5569
	0.99	0.2199	0.2737	0.2783	0.3418	0.6254

$^{*}r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t,\ \varepsilon_t \sim N(\mu_N, \sigma_N)$
$\varepsilon_t \sim N(\mu_N, \sigma_N),\ t(\gamma, \mu_t, \sigma_t),\ NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}),\ GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.
Estimation of all the parameters are listed in Table 4.

(b) GARCH(1,1) Model 10-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	0.0899(0.0292)	0.0856(0.0290)	0.0952(0.0282)	0.1055(0.0367)	0.2823(0.0857)
	0.97	0.1010(0.0333)	0.1001(0.0342)	0.1057(0.0337)	0.1252(0.0426)	0.2995(0.0927)
	0.99	0.1232(0.0415)	0.1366(0.0459)	0.1419(0.0457)	0.1618(0.0549)	0.3316(0.1035)
ES	0.95	0.1072	0.1141	0.1216	0.1369	0.3053
	0.97	0.1175	0.1309	0.1391	0.1537	0.3183
	0.99	0.1403	0.1718	0.1717	0.1829	0.3391
MaL	0.95	0.0899	0.0856	0.0952	0.1058	0.2827
	0.97	0.1010	0.1001	0.1057	0.1255	0.2998
	0.99	0.1232	0.1366	0.1419	0.1621	0.3319
MaD	0.95	0.0899	0.0856	0.0952	0.1058	0.2827
	0.97	0.1010	0.1001	0.1057	0.1255	0.2998
	0.99	0.1232	0.1366	0.1419	0.1621	0.3319

$^*r_t = u_t + \varepsilon_t,\ \varepsilon_t = h_t\epsilon_t,\ h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1\varepsilon_{t-1}^2$

$\varepsilon_t \sim N(\mu_N, \sigma_N),\ t(\gamma, \mu_t, \sigma_t),\ NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}),\ GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.

Estimation of all the parameters are listed in Table 4.

(c) IGARCH(1,1) Model 10-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mix
VaR**	0.95	0.1218(0.0350)	0.1211(0.0350)	0.1170(0.0338)	0.1657(0.0443)	0.3023(0.0828)
	0.97	0.1370(0.0401)	0.1373(0.0404)	0.1330(0.0387)	0.1857(0.0499)	0.3253(0.0875)
	0.99	0.1672(0.0496)	0.1712(0.0514)	0.1689(0.0484)	0.2369(0.0603)	0.3445(0.0939)
ES	0.95	0.1460	0.1486	0.1453	0.2008	0.3256
	0.97	0.1593	0.1654	0.1604	0.2185	0.3364
	0.99	0.1918	0.2028	0.1967	0.2501	0.3498
MaL	0.95	0.1218	0.1211	0.1170	0.1660	0.3026
	0.97	0.1370	0.1373	0.1330	0.1860	0.3256
	0.99	0.1672	0.1712	0.1689	0.2373	0.3449
MaD	0.95	0.1218	0.1211	0.1170	0.1660	0.3026
	0.97	0.1370	0.1373	0.1330	0.1860	0.3256
	0.99	0.1672	0.1712	0.1689	0.2373	0.3449

$^*r_t = u_t + \varepsilon_t,\ \varepsilon_t = h_t\epsilon_t,\ h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1\varepsilon_{t-1}^2$

$\varepsilon_t \sim N(\mu_N, \sigma_N),\ t(\gamma, \mu_t, \sigma_t),\ NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}),\ GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.

Estimation of all the parameters are listed in Table 4.

(d) FIGARCH(1,d,1) Model 10-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	0.0209	0.0184	0.0325	0.0243	0.0663
	0.97	0.0236	0.0227	0.0533	0.0270	0.0721
	0.99	0.0286	0.0341	0.1188	0.0317	0.0815
ES	0.95	17.6023	0.1181	0	6.3884	0
	0.97	8.8677	0.0446	0	2.6921	0
	0.99	1.5910	0.0044	0	0.5811	0
MaL	0.95	17.6023	0.1181	0	6.3884	0
	0.97	8.8677	0.0446	0	2.6921	0
	0.99	1.5910	0.0044	0	0.5811	0
MaD	0.95	17.6023	0.1181	0	6.3884	0
	0.97	8.8677	0.0446	0	2.6921	0
	0.99	1.5910	0.0044	0	0.5811	0

$^*r_t = u_t + (1-B)^d\varepsilon_t,\ \varepsilon_t = h_t\epsilon_t,\ h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1\varepsilon_{t-1}^2$

$\varepsilon_t \sim N(\mu_N, \sigma_N),\ t(\gamma, \mu_t, \sigma_t),\ NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG}),\ GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.

Estimation of all the parameters are listed in Table 4.

4.2.3 30-Days Estimation

(a) AR(2) Model 30-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	0.0804 (0.0876)	0.0786 (0.0813)	0.0790(0.0659)	0.2772(0.1171)	0.7746(0.2815)
	0.97	0.0933(0.1005)	0.0924(0.1011)	0.0910(0.0771)	0.2887(0.1378)	0.7957(0.3050)
	0.99	0.1209(0.1248)	0.1192(0.1520)	0.1195(0.1012)	0.3157(0.1832)	0.8436(0.3421)
ES	0.95	0.0942	0.0941	0.0930	0.2908	0.8048
	0.97	0.1053	0.1069	0.1037	0.3028	0.8260
	0.99	0.1298	0.1354	0.1314	0.3264	0.8597
MaL	0.95	0.4830	0.4497	0.4729	0.6459	1.5592
	0.97	0.5581	0.5579	0.5933	0.7571	1.6840
	0.99	0.6958	0.8462	0.8761	1.0139	1.8815
MaD	0.95	0.4830	0.4497	0.4729	0.6459	1.5592
	0.97	0.5581	0.5579	0.5933	0.7571	1.6840
	0.99	0.6958	0.8462	0.8761	1.0139	1.8815

$^*r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t$, $\varepsilon_t \sim N(\mu_N, \sigma_N)$

$\varepsilon_t \sim N(\mu_N, \sigma_N)$, $t(\gamma, \mu_t, \sigma_t)$, $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.

Estimation of all the parameters are listed in Table 4.

(b) GARCH(1,1) Model 30-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	0.2946(0.0506)	0.2952(0.0501)	0.3094(0.0488)	0.4216(0.0636)	0.8941(0.1485)
	0.97	0.3352(0.0578)	0.3602(0.0593)	0.3662(0.0584)	0.4723(0.0738)	0.9368(0.1605)
	0.99	0.4228(0.0718)	0.5104(0.0795)	0.4969(0.0791)	0.5813(0.0951)	1.0371(0.1793)
ES	0.95	0.3621	0.4283	0.4294	0.5277	0.9631
	0.97	0.4023	0.4989	0.4987	0.5890	1.0027
	0.99	0.4837	0.6501	0.6512	0.7321	1.0616
MaL	0.95	0.2946	0.2952	0.3094	0.4219	0.8944
	0.97	0.3352	0.3602	0.3662	0.4726	0.9372
	0.99	0.4228	0.5104	0.4969	0.5817	1.0374
MaD	0.95	0.2946	0.2952	0.3094	0.4219	0.8944
	0.97	0.3352	0.3602	0.3662	0.4726	0.9372
	0.99	0.4228	0.5104	0.4969	0.5817	1.0374

$^*r_t = u_t + \varepsilon_t$, $\varepsilon_t = h_t \epsilon_t$, $h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$

$\varepsilon_t \sim N(\mu_N, \sigma_N)$, $t(\gamma, \mu_t, \sigma_t)$, $NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $GDP(\alpha_{GDP}, \beta_{GDP})$, or $Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$.

Estimation of all the parameters are listed in Table 4.

(c) IGARCH(1,1) Model 30-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	(0.0607)	0.4632(0.0606)	0.4678(0.0585)	0.5966(0.0767)	1.0910(0.1435)
	0.97	(0.0694)	0.5332(0.0700)	0.5476(0.0671)	0.6709(0.0863)	1.1714(0.1516)
	0.99	(0.0858)	0.6933(0.0890)	0.7054(0.0839)	0.7772(0.1045)	1.2350(0.1627)
ES	0.95	0.4648	0.5939	0.6117	0.6981	1.1842
	0.97	0.5293	0.6683	0.6834	0.7504	1.2229
	0.99	0.6684	0.8289	0.8150	0.8537	1.2708
MaL	0.95	0.4648	0.4632	0.4678	0.5970	1.0913
	0.97	0.5293	0.5332	0.5476	0.6712	1.1718
	0.99	0.6684	0.6933	0.7054	0.7775	1.2353
MaD	0.95	0.4648	0.4632	0.4678	0.5970	1.0913
	0.97	0.5293	0.5332	0.5476	0.6712	1.1718
	0.99	0.6684	0.6933	0.7054	0.7775	1.2353

$^*r_t = u_t + \varepsilon_t$, $\varepsilon_t = h_t \epsilon_t$, $h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$

(d) FIGARCH(1,d,1) Model 10-day Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	0.0209	0.0184	0.0325	0.0243	0.0663
	0.97	0.0236	0.0227	0.0533	0.0270	0.0721
	0.99	0.0286	0.0341	0.1188	0.0317	0.0815
ES	0.95	17.6023	0.1181	0.1102	6.3884	0
	0.97	8.8677	0.0446	0.0321	2.6921	0
	0.99	1.5910	0.0044	0.0057	0.5811	0
MaL	0.95	17.6023	0.1181	0.1012	6.3884	0
	0.97	8.8677	0.0446	0.0346	2.6921	0
	0.99	1.5910	0.0044	0.0065	0.5811	0
MaD	0.95	17.6023	0.1181	0.1105	6.3884	0
	0.97	8.8677	0.0446	0.0345	2.6921	0
	0.99	1.5910	0.0044	0.0009	0.5811	0

$^{*}r_t = u_t + (1-B)^d \varepsilon_t,\ \varepsilon_t = h_t \epsilon_t,\ h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$

4.2.4 1-Year Estimation

(a) AR(2) Model 1-Year Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mix
VaR**	0.95	0.1770(0.2579)	0.1566(0.2393)	0.1927(0.2498)	2.0397(0.3447)	5.9379(0.8288)
	0.97	0.2017(0.2958)	0.1804(0.2978)	0.2310(0.3125)	2.0747(0.4057)	5.9981(0.8980)
	0.99	0.2679(0.3673)	0.2777(0.4474)	0.3053(0.4585)	2.1239(0.5392)	6.1269(1.0070)
ES	0.95	0.1541	0.1385	0.1791	2.0169	5.9829
	0.97	0.1847	0.1720	0.2101	2.0439	6.0386
	0.99	0.2556	0.2352	0.2673	2.0916	6.1465
MaL	0.95	4.1781	3.8514	4.0466	5.5797	13.5172
	0.97	4.7917	4.7895	5.0887	6.5675	14.6018
	0.99	5.9437	7.2290	7.4128	8.6973	16.2967
MaD	0.95	4.1781	3.8514	4.0466	5.5797	13.5172
	0.97	4.7917	4.7895	5.0887	6.5675	14.6018
	0.99	5.9437	7.2290	7.4128	8.6973	16.2967

$^{*}r_t = \phi_1 r_{t-1} + \phi_2 r_{t-2} + \varepsilon_t,\ \varepsilon_t \sim N(\mu_N, \sigma_N)$

(b) GARCH(1,1) Model 1-Year Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	3.3607(0.1489)	3.0182(0.1476)	3.8235(0.1438)	4.3864(0.1873)	10.6874(0.4371)
	0.97	3.7830(0.1700)	3.7018(0.1745)	4.5588(0.1720)	5.3352(0.2174)	11.2750(0.4725)
	0.99	4.7068(0.2114)	5.1335(0.2340)	5.8019(0.2328)	7.2409(0.2800)	12.2410(0.5278)
ES	0.95	4.1389	4.1332	4.9794	6.0204	11.5640
	0.97	4.5858	4.7253	5.5868	6.8630	12.0090
	0.99	5.5781	5.9291	6.9799	8.5769	12.8395
MaL	0.95	3.3607	3.0182	3.8235	4.3867	10.6877
	0.97	3.7830	3.7018	4.5588	5.3355	11.2753
	0.99	4.7068	5.1335	5.8019	7.2413	12.2413
MaD	0.95	3.3607	3.0182	3.8235	4.3867	10.6877
	0.97	3.7830	3.7018	4.5588	5.3355	11.2753
	0.99	4.7068	5.1335	5.8019	7.2413	12.2413

$^{*}r_t = u_t + \varepsilon_t,\ \varepsilon_t = h_t \epsilon_t,\ h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$

(c) IGARCH(1,1) Model 1-Year Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	6.7642(0.1787)	6.7380(0.1783)	6.4322(0.1723)	8.2951(0.2258)	16.4220(0.4223)
	0.97	7.6200(0.2043)	7.6640(0.2062)	7.5853(0.1976)	9.4759(0.2542)	17.4657(0.4463)
	0.99	9.4921(0.2527)	9.8003(0.2620)	9.6868(0.2469)	11.5549(0.3075)	18.6826(0.4789)
ES	0.95	8.4221	8.6454	8.1893	10.2064	17.7053
	0.97	9.3277	9.6978	9.0645	11.1439	18.2551
	0.99	11.3385	12.1673	10.6556	12.6035	18.9650
MaL	0.95	6.7642	6.7380	6.4322	8.2954	16.4223
	0.97	7.6200	7.6640	7.5853	9.4763	17.4660
	0.99	9.4921	9.8003	9.6868	11.5552	18.6830
MaD	0.95	6.7642	6.7380	6.4322	8.2954	16.4223
	0.97	7.6200	7.6640	7.5853	9.4763	17.4660
	0.99	9.4921	9.8003	9.6868	11.5552	18.6830

* $r_t = u_t + \varepsilon_t$, $\varepsilon_t = h_t \epsilon_t$, $h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$

(d) FIGARCH(1,d,1) Model 1-Year Ahead Risk Measure*

		Normal	Student	NIG	GPD	Mixed
VaR**	0.95	0.0209	0.0184	0.0325	0.0243	0.0663
	0.97	0.0236	0.0227	0.0533	0.0270	0.0721
	0.99	0.0286	0.0341	0.1188	0.0317	0.0815
ES	0.95	17.6023	0.1181	0.0113	6.3884	0
	0.97	8.8677	0.0446	0.0334	2.6921	0
	0.99	1.5910	0.0044	0.0025	0.5811	0
MaL	0.95	17.6023	0.1181	0.1103	6.3884	0
	0.97	8.8677	0.0446	0.0024	2.6921	0
	0.99	1.5910	0.0044	0.0017	0.5811	0
MaD	0.95	17.6023	0.1181	0.0234	6.3884	0
	0.97	8.8677	0.0446	0.0454	2.6921	0
	0.99	1.5910	0.0044	0.0042	0.5811	0

* $r_t = u_t + (1-B)^d \varepsilon_t$, $\varepsilon_t = h_t \epsilon_t$, $h_t^2 = \alpha_0 + \alpha_1 h_{t-1}^2 + \beta_1 \varepsilon_{t-1}^2$

* $\epsilon_t \sim N(\mu_N, \sigma_N)$, $\epsilon_t \sim t(\gamma, \mu_t, \sigma_t)$, $\epsilon_t \sim NIG(\alpha_{NIG}, \beta_{NIG}, \mu_{NIG}, \delta_{NIG})$, $\varepsilon_t \sim GDP(\alpha_{GDP}, \beta_{GDP})$, or $\epsilon_t \sim Mix(\xi_{M1}, \xi_{M2}, \sigma_{M1}, \sigma_{M2}, \theta_M)$. Estimation of parameters are listed in Table 7. The value of $\hat{d}$ see Table 3

*** The Value inside the bracket is the time-scaled value of VaR at 1 year days $\sqrt{260} * VaR_1$.

5 Conclusion

In this preliminary work, we propose an alternative modeling to compute financial risk, which considering the persistence and volatility clustering phenomena and the existence of extreme events simultaneously. We firstly apply the idea in the estimation of long term VaR based on a dynamic approach. We show some examples by estimating long term VaR with five stock market returns. The temporary results indicate that there is improvement of the estimation by our approach, but some detail questions may need further discussion. The choice of the best model and the proper distribution is still an open question, which we need to consider in more general conditions. Additionally, this preliminary work will also be extended to other risk measures,for instance, the expected shortfall and the maximum of loss. We will also considered a multivariate framework providing the evolution of long term VaR for a vectorial portfolio.

Basing on the conclusions of their works, we will extend the methods to model the joint large movements. The need of alternative models taking into account the volatility in the extreme conditions has been recognized by practitioners and academics. A related reference on the pitfalls and opportunities in the use of extreme value theory in finance is Diebold et al.(1998). Here, we introduce a model which is able to mix the existence of extreme events, volatility and long memory behavior. In the second step, we consider modelings from the methods based on the Markovitz "optimization" approach. The alternative models we develop are stochastic processes called LM-BL-GARCH (Long Memory-Bilinear-Generalized Autoregressive Conditional Heteroskedasticity), Storti and Vitale (2003), Diongue et al.(2010). Their interest lies in the fact that they model the existence of persistence, the existence of jumps as well as the existence of volatility, putting these dynamics on the price or on the volatility. This modeling will be associated with residuals characterized by distribution laws which belong to the class of the generalized hyperbolic distributions class or the extreme value distributions category.

Note that this approach does not go against the stress tests that were conducted by 91 European banks in May 2010 (results in July 2010). The objective of the extended stress test exercise is to assess the overall resilience of the EU banking sector and the banks' ability to absorb further possible shocks on credit and market risks, including sovereign risks, and to assess the current dependence on public support measures. This approach can be regarded as a complementary methodology to the various current initiatives and also a way to ensure the health of banking system in the instable period which is harmful to the context of a latent economic recovery.

Some parts of the empirical studies are not available by this first draft. We will add the results by the final version in August. Thanks for your concern and understanding.

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Appendix .A Figure

Figure 1: Daily Return Sample of 5 Index 1990-2005

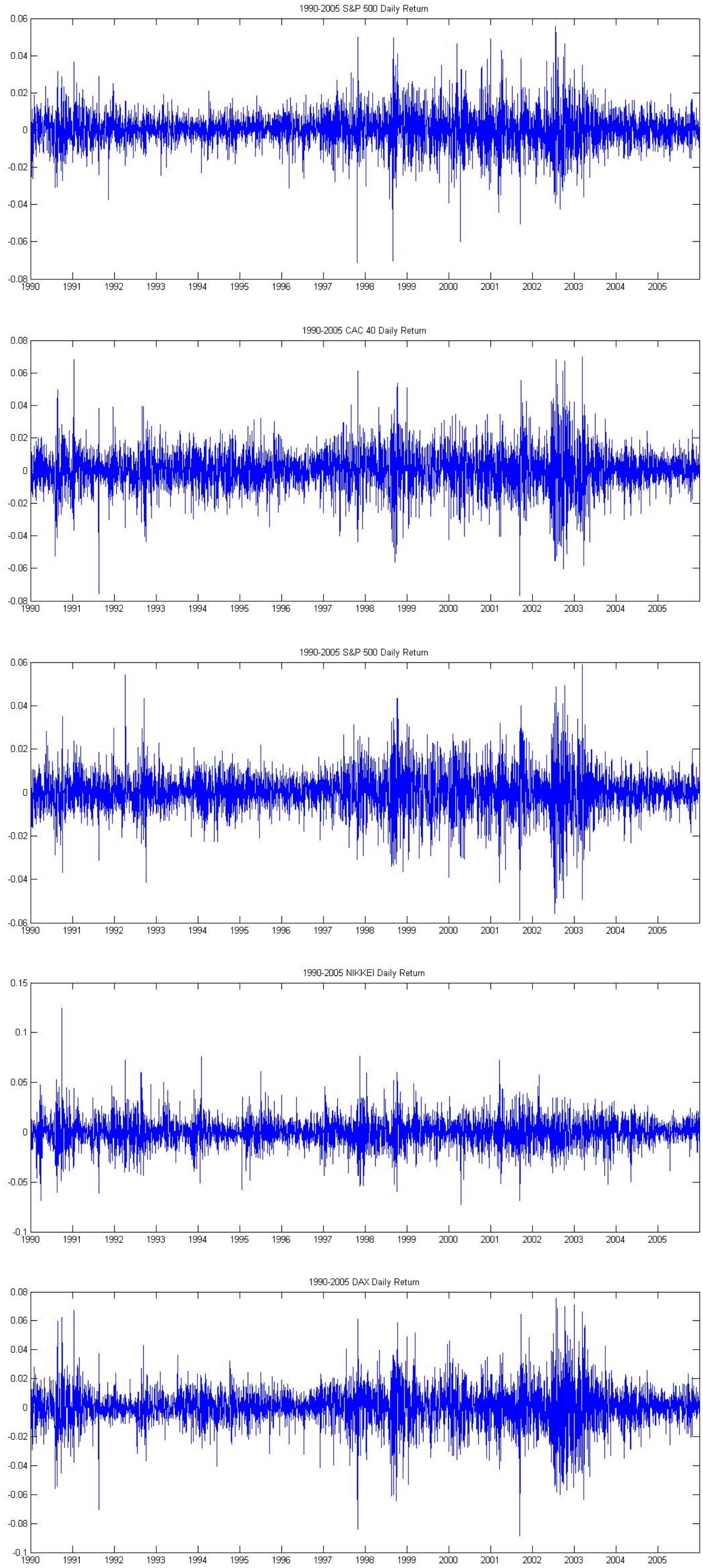


Figure 2: In-sample Value at Risk of AR2 Model

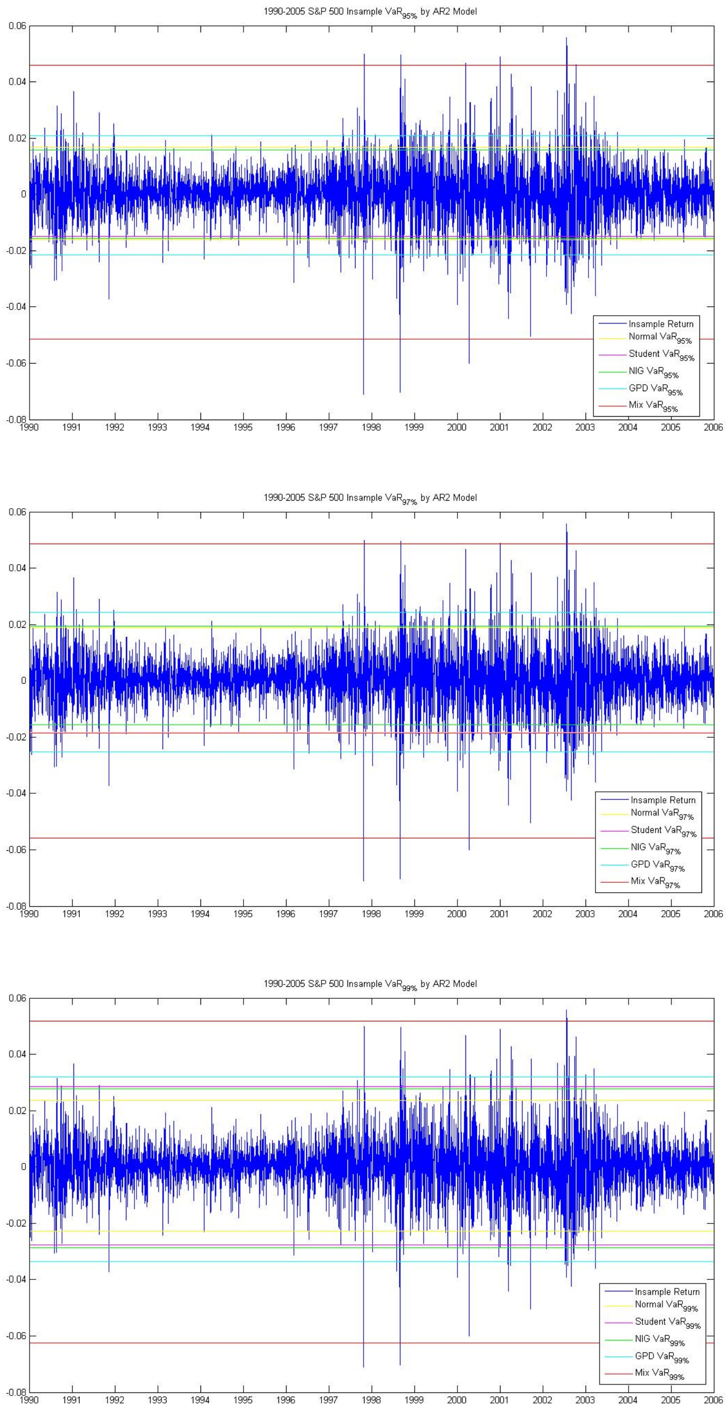


Figure 3: In-sample Value at Risk of GARCH Model

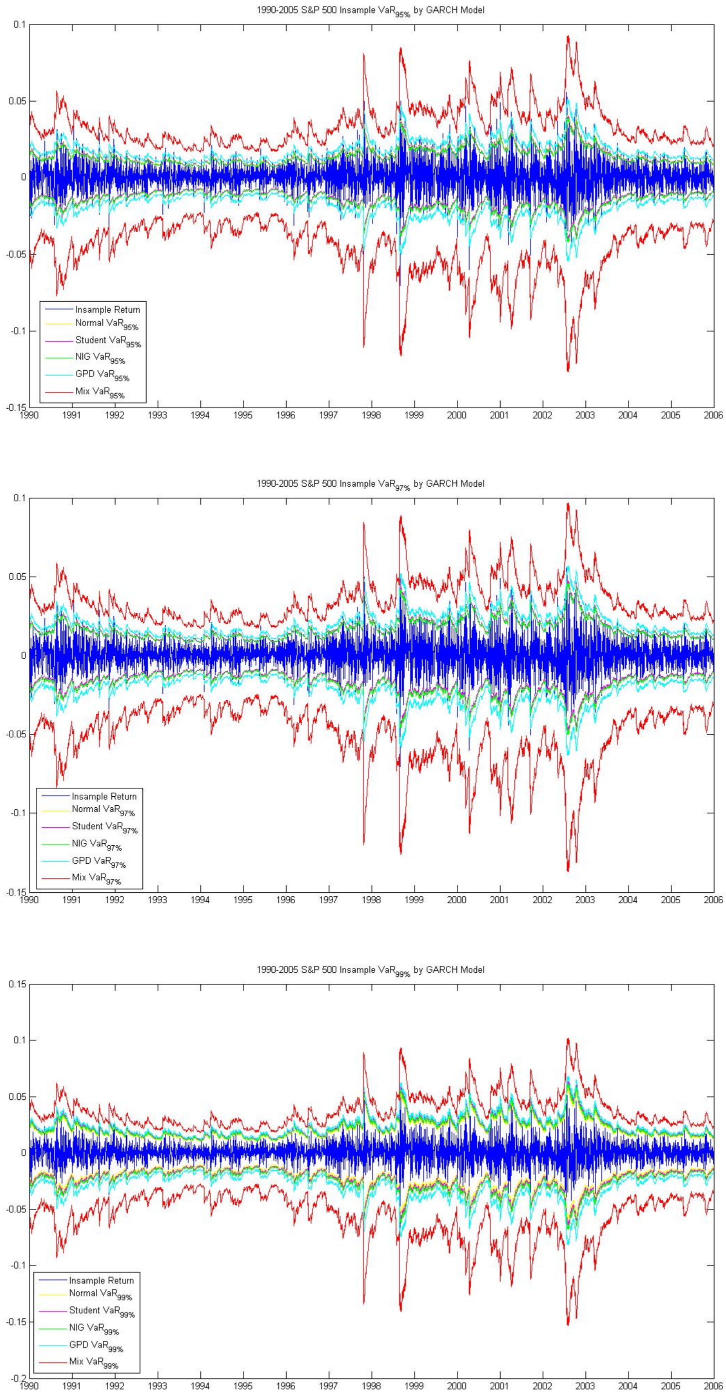


Figure 4: In-sample Value at Risk of IGARCH Model

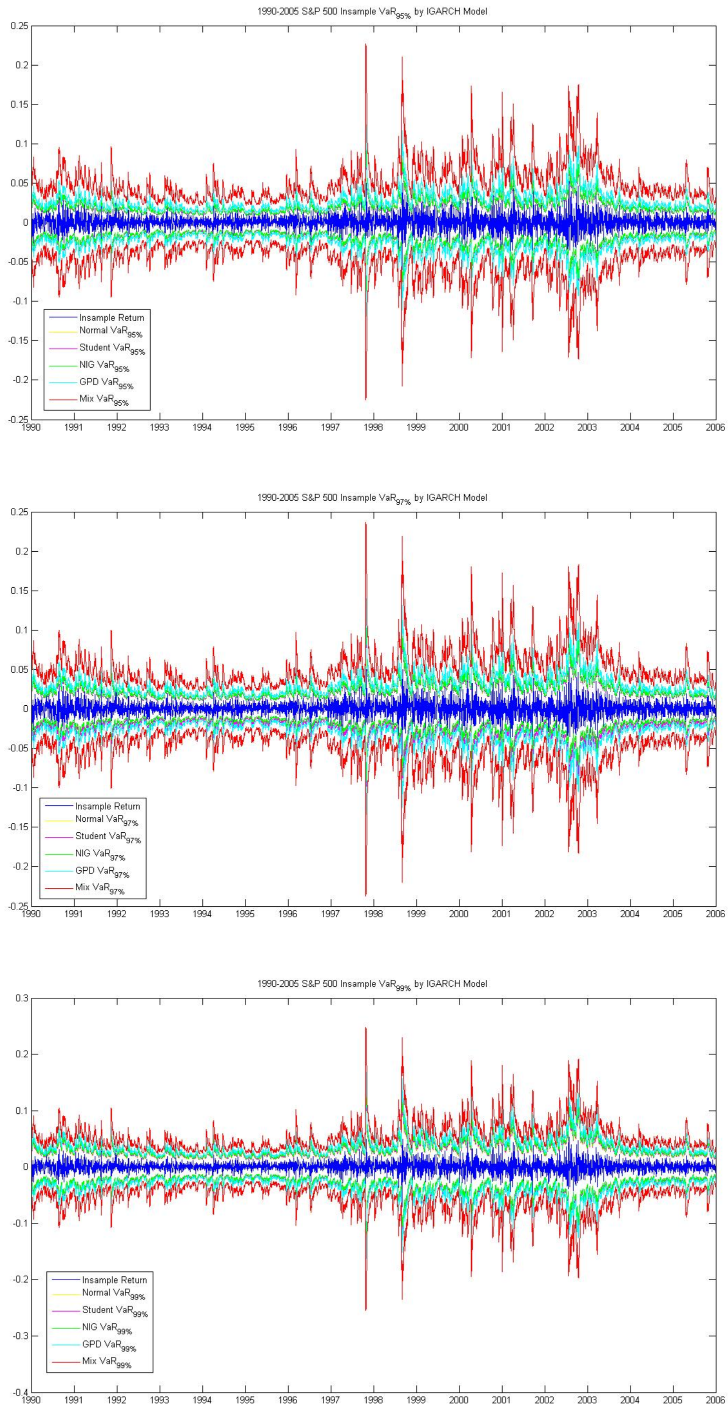
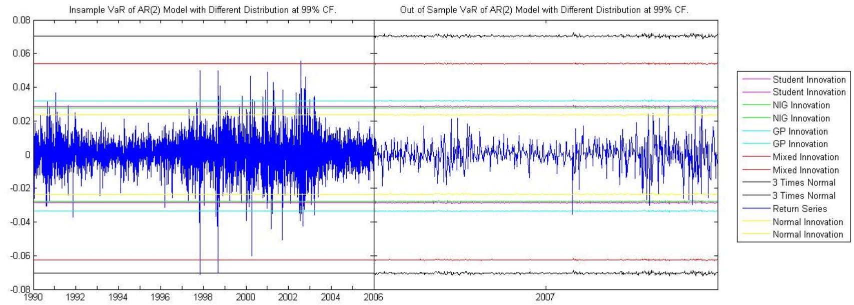


Figure 5: 1 Day VaR with AR(2) Model and Five Distributions of Innovation



Appendix .B Table

Table 1: Parameter Estimation of AR Model

Model	Parameters	Loss Function	AIC*	FPE**
AR(1)	r_1=-0.004605	9.93598e-005	-9.2163	0.9941e-004
AR(2)	r_1=-0.004706, r_2=-0.01792	9.92771e-005	-9.2166	0.9937e-004
AR(3)	r_1=-0.005272, r_2=-0.01808, r_3=-0.0348	9.91792e-005	-9.2171	0.9932e-004
AR(4)	r_1=-0.005662, r_2=-0.01813, r_3=-0.03491, r_4=-0.01088	9.91744e-005	-9.2167	0.9936e-004

*Akaike Information Criterion for estimated model, **Akaike Final Prediction Error for estimated model

Table 2: Statistic Summary Of Daily Return of Sample

	Mean	Max	Min	Median	Standard Deviation	Skewness	Kurtosis
S&P 500	3.0226e-04	0.0557	-0.0711	1.0634e-04	0.0100	-0.1005	7.0124
CAC 40	2.0530e-04	0.0700	-0.0768	0	0.0131	-0.0987	6.0209
FTSE 100	2.0149e-04	0.0590	-0.0589	1.8312e-05	0.0101	-0.1057	6.3483
NIKKEI 225	-2.1123e-04	0.1243	-0.0723	0	0.0145	0.1794	6.5232
DAX	2.6479e-04	0.0755	-0.0887	2.8748e-04	0.0141	-0.2121	6.7968

Table 3: Estimation of Long Memory Parameter by Whittle Estimation

	S&P 500 INDEX	CAC 40 INDEX	NIKKEI 225	FTSE 100 INDEX	DAX INDEX
$\hat{d_L}$	-0.0240 (2.6034e-016)	1.7187e-004 (2.3864e-018)	1.7279e-004 (3.2542e-019)	0.1954 (3.3323e-015)	1.9580e-004 (3.2814e-018)
$\hat{d_V}$	0.1582 (2.4993e-015)	0.0025 (3.0807e-017)	-0.0242 (3.9919e-016)	-0.0121 (2.6902e-016)	-0.0163 (1.0414e-017)

Table 4: Estimation of Parameters for AR(2) Risk Measures

	S&P 500	CAC 40	NIKKEI 225	FTSE 100	DAX
ϕ_1	0.004689 (9.92296e-005)				
ϕ_2	0.01792 (9.93246e-005)				
μ_N	0.0003 (0.0000,0.0006)				
σ_N	0.0100 (0.0098,0.0102)				
γ	3.3974 (2.9802,3.8146)				
μ_t	0.0004 (0.0002,0.0007)				
σ_t	0.0068 (0.0065,0.0071)				
α_{NIG}	79.4298 (1.6351e-012)				
β_{NIG}	-3.2480 (9.7749e-015)				
μ_{NIG}	0.0006 (7.3763e-018)				
δ_{NIG}	0.0080 (9.3722e-017)				
α_{GDP}	-0.0007 (-0.0402, 0.0387)				
β_{GDP}	0.0069 (0.0066 0.0074)				
ξ_{M1}	-0.8355 (-0.8258,-0.8452)				
ξ_{M2}	0.0461 (0.0358,0.0564)				
σ_{M1}	-0.3933 (-0.4613,-0.3253)				
σ_{M2}	0.0295 (0.0256,0.0340)				
θ_M	0.0070 (0.0068,0.0072)				

Table 5: Estimation of Parameters for GARCH Risk Measures

	S&P 500	CAC 40	NIKKEI 225	FTSE 100	DAX
α_0	0.004689 (9.92296e-005)				
α_1	0.01792 (9.93246e-005)				
β_1					
μ_N	0.0003 (0.0000,0.0006)				
σ_N	0.0100 (0.0098,0.0102)				
γ	3.3974 (2.9802,3.8146)				
μ_t	0.0004 (0.0002,0.0007)				
σ_t	0.0068 (0.0065,0.0071)				
α_{NIG}	79.4298				
β_{NIG}	-3.2480				
μ_{NIG}	0.0006				
δ_{NIG}	0.0080				
α_{GDP}					
β_{GDP}					
ξ_{M1}	-0.8355 (-0.8258,-0.8452)				
ξ_{M2}	0.0461 (0.0358,0.0564)				
σ_{M1}	-0.3933 (-0.4613,-0.3253)				
σ_{M2}	0.0295 (0.0256,0.0340)				
θ_M	0.0070 (0.0068,0.0072)				

Table 6: Estimation of Parameters for IGARCH Risk Measures

	S&P 500	CAC 40	NIKKEI 225	FTSE 100	DAX
α_0	0.004689 (9.92296e-005)				
α_1	0.01792 (9.93246e-005)				
β_1	0.01792 (9.93246e-005)				
μ_N	0.0003 (0.0000,0.0006)				
σ_N	0.0100 (0.0098,0.0102)				
γ	3.3974 (2.9802,3.8146)				
μ_t	0.0004 (0.0002,0.0007)				
σ_t	0.0068 (0.0065,0.0071)				
α_{NIG}	79.4298				
β_{NIG}	-3.2480				
μ_{NIG}	0.0006				
δ_{NIG}	0.0080				
α_{GDP}					
β_{GDP}					
ξ_{M1}	-0.8355 (-0.8258,-0.8452)				
ξ_{M2}	0.0461 (0.0358,0.0564)				
σ_{M1}	-0.3933 (-0.4613,-0.3253)				
σ_{M2}	0.0295 (0.0256,0.0340)				
θ_M	0.0070 (0.0068,0.0072)				

Table 7: Estimation of Parameters for FIGARCH Risk Measures

	S&P 500	CAC 40	NIKKEI 225	FTSE 100	DAX
α_0	0.004689 (9.92296e-005)				
α_1	0.01792 (9.93246e-005)				
β_1					
μ_N	0.0003 (0.0000,0.0006)				
σ_N	0.0100 (0.0098,0.0102)				
γ	3.3974 (2.9802,3.8146)				
μ_t	0.0004 (0.0002,0.0007)				
σ_t	0.0068 (0.0065,0.0071)				
α_{NIG}	79.4298				
β_{NIG}	-3.2480				
μ_{NIG}	0.0006				
δ_{NIG}	0.0080				
α_{GDP}					
β_{GDP}					
ξ_{M1}	-0.8355 (-0.8258,-0.8452)				
ξ_{M2}	0.0461 (0.0358,0.0564)				
σ_{M1}	-0.3933 (-0.4613,-0.3253)				
σ_{M2}	0.0295 (0.0256,0.0340)				
θ_M	0.0070 (0.0068,0.0072)				

		S&P 500		DJIA		CAC40		FTSE 100		N225	
AR(2)	Normal	0.0240	(14/252)	0.0287	(12/252)	0.0307	(5/256)	0.0289	(9/253)	0.0398	(7/243)
	T-Student	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	NIG	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	GEV	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	Mixed	0.0877	(0/252)	0.0855	(0/252)	0.0917	(0/256)	0.0800	(0/253)	0.1117	(0/243)
GARCH (1,1)	Normal	0.0671	(1/252)	0.0545	(2/252)	0.0557	(2/256)	0.0443	(3/253)	0.0486	(4/243)
	T-Student	0.0735	(0/252)	0.0599	(2/252)	0.0575	(2/256)	0.0467	(2/253)	0.0526	(2/243)
	NIG	0.0735	(0/252)	0.0599	(2/252)	0.0575	(2/256)	0.0467	(2/253)	0.0526	(2/243)
	GEV	0.1187	(0/252)	0.1148	(2/252)	0.1203	(0/256)	0.1081	(0/253)	0.1163	(0/243)
	Mixed	0.1187	(0/252)	0.1148	(2/252)	0.1203	(0/256)	0.1081	(0/253)	0.1163	(0/243)
IGARCH (1,1,1)	Normal	0.0600	(2/252)	0.0499	(2/252)	0.0557	(2/256)	0.0500	(2/253)	0.0461	(1/243)
	T-Student	0.0579	(2/252)	0.0518	(2/252)	0.0575	(2/256)	0.0515	(1/253)	0.0480	(1/243)
	NIG	0.0579	(2/252)	0.0518	(2/252)	0.0575	(2/256)	0.0515	(1/253)	0.0480	(1/243)
	GEV	0.1140	(0/252)	0.0935	(0/252)	0.1203	(0/256)	0.0991	(0/253)	0.1414	(0/243)
	Mixed	0.1140	(0/252)	0.0935	(0/252)	0.1203	(0/256)	0.0991	(0/253)	0.1414	(0/243)
FIGARCH (1,d*,1)	Normal	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	T-Student	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	NIG	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	GEV	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	Mixed	0.1174	(2/252)	0.1143	(0/252)	0.1051	(0/256)	0.1083	(0/253)	0.1160	(0/243)
BL-FIGARCH (1,d*,1)	Normal	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	T-Student	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	NIG	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	GEV	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	Mixed	0.1174	(2/252)	0.1143	(0/252)	0.1051	(0/256)	0.1083	(0/253)	0.1160	(0/243)

Table 8: 1-day VaR in short position of five Stock Markets based one AR Model, GARCH Model, IGARCH Model and FIGARCH Model with three different distributions

		S&P 500		DJIA		CAC40		FTSE 100		N225	
AR(2)	Normal	0.0240	(14/252)	0.0287	(12/252)	0.0307	(5/256)	0.0289	(9/253)	0.0398	(7/243)
	T-Student	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	NIG	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	GEV	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	Mixed	0.0877	(0/252)	0.0855	(0/252)	0.0917	(0/256)	0.0800	(0/253)	0.1117	(0/243)
GARCH (1,1)	Normal	0.0671	(1/252)	0.0545	(2/252)	0.0557	(2/256)	0.0443	(3/253)	0.0486	(4/243)
	T-Student	0.0735	(0/252)	0.0599	(2/252)	0.0575	(2/256)	0.0467	(2/253)	0.0526	(2/243)
	NIG	0.0735	(0/252)	0.0599	(2/252)	0.0575	(2/256)	0.0467	(2/253)	0.0526	(2/243)
	GEV	0.1187	(0/252)	0.1148	(2/252)	0.1203	(0/256)	0.1081	(0/253)	0.1163	(0/243)
	Mixed	0.1187	(0/252)	0.1148	(2/252)	0.1203	(0/256)	0.1081	(0/253)	0.1163	(0/243)
IGARCH (1,1,1)	Normal	0.0600	(2/252)	0.0499	(2/252)	0.0557	(2/256)	0.0500	(2/253)	0.0461	(1/243)
	T-Student	0.0579	(2/252)	0.0518	(2/252)	0.0575	(2/256)	0.0515	(1/253)	0.0480	(1/243)
	NIG	0.0579	(2/252)	0.0518	(2/252)	0.0575	(2/256)	0.0515	(1/253)	0.0480	(1/243)
	GEV	0.1140	(0/252)	0.0935	(0/252)	0.1203	(0/256)	0.0991	(0/253)	0.1414	(0/243)
	Mixed	0.1140	(0/252)	0.0935	(0/252)	0.1203	(0/256)	0.0991	(0/253)	0.1414	(0/243)
FIGARCH (1,d*,1)	Normal	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	T-Student	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	NIG	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	GEV	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	Mixed	0.1174	(2/252)	0.1143	(0/252)	0.1051	(0/256)	0.1083	(0/253)	0.1160	(0/243)

Table 9: 1-day VaR in long position of five Stock Markets based one AR Model, GARCH Model, IGARCH Model and FIGARCH Model with three different distributions

		S&P 500		DJIA		CAC40		FTSE 100		N225	
AR(2)	Normal	0.0240	(14/252)	0.0287	(12/252)	0.0307	(5/256)	0.0289	(9/253)	0.0398	(7/243)
	T-Student	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	NIG	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	GEV	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	Mixed	0.0877	(0/252)	0.0855	(0/252)	0.0917	(0/256)	0.0800	(0/253)	0.1117	(0/243)
GARCH (1,1)	Normal	0.0671	(1/252)	0.0545	(2/252)	0.0557	(2/256)	0.0443	(3/253)	0.0486	(4/243)
	T-Student	0.0735	(0/252)	0.0599	(2/252)	0.0575	(2/256)	0.0467	(2/253)	0.0526	(2/243)
	NIG	0.0735	(0/252)	0.0599	(2/252)	0.0575	(2/256)	0.0467	(2/253)	0.0526	(2/243)
	GEV	0.1187	(0/252)	0.1148	(2/252)	0.1203	(0/256)	0.1081	(0/253)	0.1163	(0/243)
	Mixed	0.1187	(0/252)	0.1148	(2/252)	0.1203	(0/256)	0.1081	(0/253)	0.1163	(0/243)
IGARCH (1,1,1)	Normal	0.0600	(2/252)	0.0499	(2/252)	0.0557	(2/256)	0.0500	(2/253)	0.0461	(1/243)
	T-Student	0.0579	(2/252)	0.0518	(2/252)	0.0575	(2/256)	0.0515	(1/253)	0.0480	(1/243)
	NIG	0.0579	(2/252)	0.0518	(2/252)	0.0575	(2/256)	0.0515	(1/253)	0.0480	(1/243)
	GEV	0.1140	(0/252)	0.0935	(0/252)	0.1203	(0/256)	0.0991	(0/253)	0.1414	(0/243)
	Mixed	0.1140	(0/252)	0.0935	(0/252)	0.1203	(0/256)	0.0991	(0/253)	0.1414	(0/243)
FIGARCH (1,d*,1)	Normal	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	T-Student	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	NIG	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	GEV	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	Mixed	0.1174	(2/252)	0.1143	(0/252)	0.1051	(0/256)	0.1083	(0/253)	0.1160	(0/243)

Table 10: 1-day ES in short position of five Stock Markets based one AR Model, GARCH Model, IGARCH Model and FIGARCH Model with three different distributions

		S&P 500		DJIA		CAC40		FTSE 100		N225	
AR(2)	Normal	0.0240	(14/252)	0.0287	(12/252)	0.0307	(5/256)	0.0289	(9/253)	0.0398	(7/243)
	T-Student	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	NIG	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	GEV	0.0301	(8/252)	0.0234	(5/252)	0.0362	(4/256)	0.0243	(6/253)	0.0342	(5/243)
	Mixed	0.0877	(0/252)	0.0855	(0/252)	0.0917	(0/256)	0.0800	(0/253)	0.1117	(0/243)
GARCH (1,1)	Normal	0.0671	(1/252)	0.0545	(2/252)	0.0557	(2/256)	0.0443	(3/253)	0.0486	(4/243)
	T-Student	0.0735	(0/252)	0.0599	(2/252)	0.0575	(2/256)	0.0467	(2/253)	0.0526	(2/243)
	NIG	0.0735	(0/252)	0.0599	(2/252)	0.0575	(2/256)	0.0467	(2/253)	0.0526	(2/243)
	GEV	0.1187	(0/252)	0.1148	(2/252)	0.1203	(0/256)	0.1081	(0/253)	0.1163	(0/243)
	Mixed	0.1187	(0/252)	0.1148	(2/252)	0.1203	(0/256)	0.1081	(0/253)	0.1163	(0/243)
IGARCH (1,1,1)	Normal	0.0600	(2/252)	0.0499	(2/252)	0.0557	(2/256)	0.0500	(2/253)	0.0461	(1/243)
	T-Student	0.0579	(2/252)	0.0518	(2/252)	0.0575	(2/256)	0.0515	(1/253)	0.0480	(1/243)
	NIG	0.0579	(2/252)	0.0518	(2/252)	0.0575	(2/256)	0.0515	(1/253)	0.0480	(1/243)
	GEV	0.1140	(0/252)	0.0935	(0/252)	0.1203	(0/256)	0.0991	(0/253)	0.1414	(0/243)
	Mixed	0.1140	(0/252)	0.0935	(0/252)	0.1203	(0/256)	0.0991	(0/253)	0.1414	(0/243)
FIGARCH (1,d*,1)	Normal	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	T-Student	0.0642	(0/252)	0.0541	(0/252)	0.0538	(2/256)	0.0443	(3/253)	0.0481	(4/243)
	NIG	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	GEV	0.0707	(2/252)	0.0594	(0/252)	0.0570	(2/256)	0.0467	(2/253)	0.0520	(2/243)
	Mixed	0.1174	(2/252)	0.1143	(0/252)	0.1051	(0/256)	0.1083	(0/253)	0.1160	(0/243)

Table 11: 1-day ES in long position of five Stock Markets based one AR Model, GARCH Model, IGARCH Model and FIGARCH Model with three different distributions

Appendix .C Estimation Methods

.C.1 Likelihood Function

$$Normal\ Distribution: \quad L(r_1,\ldots,r_T|u_N,\sigma_N) = (\frac{1}{\sqrt{2\pi}\sigma_N})^T e^{-\frac{1}{2}\sum_{i=1}^{T}(\frac{r_i-u_N}{\sigma_N})^2} \tag{10}$$

$$Student\ Distribution: \quad L(r_2,\ldots,r_t|\gamma,u_t,\sigma_t) = [\frac{\Gamma(\frac{\gamma+1}{2})}{\Gamma(\frac{\gamma}{2})}]^T(\frac{\sigma_t}{\pi\gamma})^{\frac{T}{2}}\Pi_{i=1}^{T}[1+\frac{\sigma_t(r_i-u_t)^2}{\gamma}]^{-\frac{\gamma+1}{2}} \tag{11}$$

$$NIGDistribution: \tag{12}$$

$$\tag{13}$$

$$\tag{14}$$