

# Moment-based *CVaR* Estimation: Quasi-Closed Formulas

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## Abstract

The evaluation of quantiles (or *VaR*, *Value at Risk*) and that of *CVaR*, *Conditional Value at Risk* in risk management (or mean excess in actuarial sciences) is a very crucial point in the decision processes. Indeed, there are often great difficulties (or impossibilities) in finding the kind of distribution followed by the key random variable (r.v.). Several approaches suggest suitable transformations of the standard normal variable which may represent a good proxy of the key r.v. The transformations considered here are based on the first three (Normal-Power) or four (Johnson's systems) moments. In this paper, for any kind of r.v., closed or quasi-closed formulas for *CVaR* are given (starting from their first four moments) using the Normal-Power and Johnson's systems. Here there is also the analysis of the efficiency of the Johnson's method with reference to financial applications.

**Keywords:** *Risk Measures, Conditional Value at Risk, Skewness, Kurtosis*

**JEL Classification:** G11, C63

**AMS Classification:** 91B30, 91G60

## 1 Introduction

In many actuarial and financial applications, the evaluation of quantiles and that of mean excess (or *TailVaR* or *Worst Conditional Expectation* see [4], or *Expected Shortfall* see [9], or again *Conditional Value at Risk* especially in risk and portfolio management, see also [1] and [18]) is a very crucial point in the decision processes. Indeed, there are often great difficulties (or impossibilities) in finding the kind of distribution followed by the key random variable. Several approaches, based on the historical method of translation by Edgeworth [7] suggest suitable transformations of the standard normal variable which represent a good proxy of the key variable. The transformations are based on the first three (Normal-Power, see for example [6]) or four moments (Johnson's systems, [12] and [13]). Normal-Power (N-P) is well known in actuarial application, but it is

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meaningful only for r.v. with particular characteristics (see [6]), while the Johnson's systems may be applied to a wide set of r.v. Some interesting practical applications of Johnson's systems to  $VaR$  evaluation of derivatives or portfolios containing derivatives may be found in Riskmetrics [17] and Rouvinez [19].

The relevance of quantiles (or, substantially,  $VaR$  in risk management) in evaluating risk is well known in insurance and finance, but they in general do not represent a good measure of risk because they are not coherent risk measures in the sense of Artzner et al. ([4]). Indeed, the only coherent measure of risk is  $CVaR$  (or mean excess) considering regular distributions. In general, there are no closed-formulas for  $CVaR$  evaluation. Starting from the so-called N-P approximation and the more efficient Johnson's systems, it is possible to find closed or quasi-closed (based on numerical integration) formulas for  $CVaR$  estimation. In particular, in this paper there is the analysis of the application of the new  $CVaR$  formulas obtained under the Johnson's systems. Moreover, it is worthwhile to underline that N-P and Johnson's approaches require only the estimation of moments (three or four) in order to find  $CVaR$ , both for discrete and continuous distributions, and this represents a very appealing feature from a practical point of view. The new formulas have been applied to skewed distributions with excess kurtosis index far from zero, starting from financial applications found in literature. In more detail, in this paper funds of hedge funds (see [10]), stock market indices and exchange rates (see [16]), market indices compounded by different kinds of assets (see [20]) and last, equity futures contracts (see [5]) have been considered and the results seem to be satisfactory.

The paper is organized as follows: in section 1 the basic notations have been introduced and the moments methods have been described. In section 2 there are the definitions of  $VaR$  and  $CVaR$  and the classical formulas using normality assumption and then using the Normal-Power framework. In the same section the Johnson's systems have been applied obtaining closed-formulas for  $VaR$ . The Johnson's framework includes the log-normal distribution so here there is the closed formula for  $CVaR$  found in actuarial literature with reference to mean-excess loss estimation assuming log-normality. The very interesting original results regard the closed formula obtained under the other two kinds of transformation of the Johnson's systems, called unbounded and bounded. In particular, there is again a closed formula in the unbounded case and a quasi-closed formula in the bounded situation. Section 4 contains the numerical applications: starting from data concerning financial assets,  $VaR$  and  $CVaR$  have been evaluated under N-P and Johnson's systems and the results have been compared with those obtained by the authors of the above cited papers. In particular, Dowd and Cotter [5] suggest non-parametric estimation methods to estimate  $VaR$ ,  $CVaR$  and spectral risk measures (not considered here) for five of the most

liquid futures (S&P500, FTSE100, DAX, Hang Seng and Nikkei225 indices) over January 1991 and December 2003; the Johnson moment method has been applied giving interesting results. In Section 5 there are the conclusions and comments.

## 2 Moment based quantiles for r.v. with unknown distributions

### 2.1 Moments

In many practical situations, especially dealing with portfolios containing derivatives, it is impossible to identify exactly the kind of distribution of the stochastic change of value or return on the portfolio. This means that it is, at the same time, impossible to find the return cumulative distribution function and the inverse *CDF* or quantiles or *VaR*. If it is possible to estimate the main moments, there are methodologies that provide a proxy distribution based on transformations of normal variables distribution. This kind of approach is called moments method, originally due to the pioneering work by Edgeworth [7]. If  $\tilde{X}$  is the random variable to be transformed, the characterizing moments and the most important (moment based) indices to be estimated are:

$$\mu = E(\tilde{X}) \text{ mean or expected mean; } \sigma = \sigma(\tilde{X}) \text{ standard deviation;}$$

$$s_1 = s_1(\tilde{X}) = \frac{E(\tilde{X} - \mu)^3}{\sigma^3} \text{ skewness index; } \beta_1 = s_1^2 \text{ squared skewness;}$$

$$\beta_2 = \beta_2(\tilde{X}) = \frac{E(\tilde{X} - \mu)^4}{\sigma^4} \text{ kurtosis index; } \beta_2 - 3 = \text{excess kurtosis index}$$

### 2.2 Normal-Power Method

The Normal-Power method is based on the idea that the parameters mean, standard deviation and skewness may characterize sufficiently a distribution and that the r.v.  $\tilde{X}$  may be well approximated by a quadratic function of a standard normal variable  $\tilde{N}$ . The kind of transformation found in this case (see [6]) is the following:

$$\tilde{N} = -\frac{3}{s_1} + \sqrt{\frac{9}{s_1^2} + \frac{6}{s_1} \frac{\tilde{X} - \mu}{\sigma} + 1} = h(\tilde{X}) \quad \tilde{X} > \mu \quad (1)$$

Since  $h(\cdot)$  is a monotonically increasing function of  $\tilde{X}$ , to determine the  $c - th$  quantile, that is the  $x^*$  satisfying the following condition:

$$Prob(\tilde{X} \leq x^*) = 1 - c$$

it is possible to exploit the transformation rewriting the condition in terms of  $h(\cdot)$ :

$$Prob[h(\tilde{X}) \leq h(x^*)] = P[N \leq h(x^*)] \implies h(x^*) \cong N^{-1}(1 - c) = z$$

where  $N(x)$  is the value at  $x$  of the normal standard *CDF* and  $z = N^{-1}(1 - c)$  its inverse function evaluated at  $1 - c$ . Thus, it is:

$$\frac{3}{s_1} - \sqrt{\frac{9}{s_1^2} + \frac{6}{s_1} \frac{x^* - \mu}{\sigma} + 1} = z \quad (2)$$

Keeping outside the quantile  $x^*$ , that is inverting the function  $h$  and evaluating the inverse at  $-z$ , gives the final formula:

$$x^* \cong \sigma \left[ (z^2 - 1) \frac{s_1}{6} + z \right] + \mu \quad (3)$$

This approach is often used in actuarial application, but it is valid only for the right-hand tail (by definition 1, it must be  $\tilde{X} > \mu$ ) and it provides unreliable results for values of the skewness index lower than 1 (see [6]).

### 2.3 Johnson Systems

A refined method is based on Johnson systems (Johnson [12], [13]). It is based again on the idea of matching a set of moments of a distribution with a particular transformation of a standard normal distribution (the seminal work on the method of translation is due to Edgeworth, [7]). The systems created by Johnson together with the normal distribution itself are able to describe a wide variety of distributions with a great degree of precision ([12]). In the version of the Johnson systems adopted also in practice (see RiskMetrics [17] and [19]), the starting point is the estimation of mean, standard deviation, skewness and kurtosis indices.

The general transformation suggested by Johnson depends on four parameters  $A$ ,  $B$ ,  $C$  and  $D$ :

$$\tilde{N} = C + Dh \left( \frac{\tilde{X} - A}{B} \right), \quad \tilde{N} \approx N(0, 1) \quad (4)$$

where  $h$  is a non-decreasing monotonic function independent from the variable's moments, with positive  $B$  and  $D$ . The choice of the function  $h$  depends on the values of the skewness and kurtosis indices. Johnson considers the  $(\beta_1, \beta_2)$  plane and determines three zones. First of all, there is an impossible area, described by the condition

$$\beta_2 - \beta_1 - 1 < 0 \Leftrightarrow \rho(R_P^2, R_P) > 1$$

Coming now to the simplest form of Johnson's transformations, called log-normal and denoted by  $S_L$ , it is:

$$S_L: \quad h\left(\frac{\tilde{X} - A}{B}\right) = \ln\left(\frac{\tilde{X} - A}{B}\right) = \ln(\tilde{Y}) \quad \tilde{Y} = \frac{\tilde{X} - A}{B} > 0 \quad (5)$$

Indeed, this means  $\tilde{X}$  log-normal. It is possible to show that all pairs  $(\beta_1, \beta_2)$  characterizing the log-normal transformation satisfy the following parametric equations:

$$\begin{cases} \beta_1 = (\omega - 1)(\omega + 2)^2 \\ \beta_2 = \omega^4 + 2\omega^3 + 3\omega^2 - 3 \end{cases} ; \quad \omega = \exp(D^{-2})$$

The log-normal curve in the  $(\beta_1, \beta_2)$  plane divide the feasible area in two zones: the upper corresponds to the  $S_U$  unbounded transformation and the lower to the  $S_B$  bounded transformation (see fig. 1):

$$S_B: \quad h(\tilde{Y}) = \ln\left(\frac{\tilde{Y}}{1 - \tilde{Y}}\right) \quad 0 < \tilde{Y} < 1 \quad (6)$$

$$S_U: \quad h(\tilde{Y}) = \ln\left(\tilde{Y} + \sqrt{1 + \tilde{Y}^2}\right) \quad -\infty < \tilde{Y} < +\infty \quad (7)$$

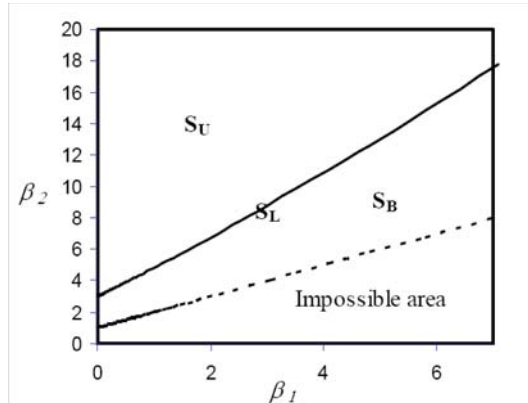


Figure 1. Regions in  $(\beta_1, \beta_2)$  plane

Once identified the kind of transformation on the basis of the values of the pair  $(\beta_1, \beta_2)$ , an estimation of the four parameters  $A$ ,  $B$ ,  $C$  and  $D$  is needed. It is not simple and not always possible to find a solution matching the input data (the moments). A numerical procedure is given by Hill [11] and Hill and Holder [14].

At the end, the estimated parameters may be used in order to find the desired percentile in terms of the corresponding standard normal percentile (as like as done in section 2.2):

$$P(\tilde{X} \leq x^*) = P\left[\tilde{N} \leq C + Dh\left(\frac{x^* - A}{B}\right)\right] = 1 - c$$

$$\implies C + Dh\left(\frac{x^* - A}{B}\right) = N^{-1}(1 - c) = z$$

where  $N(x)$  is the value at  $x$  of the standard normal *CDF*. Eventually, it is:

$$x^* = A + Bh^{-1}\left(\frac{z - C}{D}\right) \quad (8)$$

### 3 *VaR* and *CVaR* under moments method

In general, if the variable  $\tilde{L}$  is the loss of a position (or portfolio of positions) over a time horizon  $[t, t + \tau]$ , for a fixed probability  $1 - c$ , the Value at Risk (*VaR*) is defined by the condition:

$$Prob(\tilde{L} > VaR) = 1 - c \quad (9)$$

The previous condition is equivalent to:

$$Prob(\tilde{L} \leq VaR) = c = F_{\tilde{L}}(VaR) \quad (10)$$

In other words, *VaR* is the  $c$ -quantile of the distribution of the loss. In general, often the *CDF* of the loss cannot be inverted, so a generalized inverse of  $F$  (see [8] and [9]) may be defined:

$$F_L^{\leftarrow}(t) = \inf\{x \in \Re | F_L^{\leftarrow}(x) \geq t\} \quad 0 < t < 1$$

The generalized inverse of a *CDF* is also called the quantile function of  $F_L$ . *VaR* is defined as  $F_L^{\leftarrow}(c)$  (the value in  $c$  of the loss generalized inverse *CDF*) or the  $c$ -quantile of  $F_L$ .

In terms of the return  $\tilde{X}$  on a position, formulas may be easily modified starting from the relationship:

$$\tilde{X} = -\frac{\tilde{L}}{w}$$

where  $w$  is the initial value of the position. This gives  $VaR$  in terms of the generalized inverse of the return, that is  $-wF_{\tilde{X}}^{\leftarrow}(1-c)$ .

In many applications, the initial value is neglected and the percentage  $VaR$  is evaluated (in what follows the symbol  $VaR$  stays for percentage  $VaR$ ):

$$VaR = -F_{\tilde{X}}^{\leftarrow}(1-c) \quad (11)$$

However, both in the Normal-Power (see 3) and in the Johnson frameworks (see 8), the percentage  $VaR$  is given by a closed formula and it is equal to the opposite of the quantile  $x^*$  (obviously in the second case the kind of transformation and the parameters  $A$ ,  $B$ ,  $C$  and  $D$  have to be previously determined using a numerical procedure in order to apply equation 8).

Coming now to *Conditional VaR (CVaR)* or *Expected Shortfall (ES)*, here we refer to [9] (see also [1] and [18]). In the assumptions of limited absolute expected loss (formally  $E(|\tilde{X}|) < \infty$ ), the expected shortfall for a fixed probability  $c$  is:

$$ES_c = \frac{1}{1-c} \int_c^1 F_{\tilde{L}}^{\leftarrow}(t) dt$$

If the loss is integrable and its distribution function is continuous, for any  $c \in (0, 1)$  it is:

$$ES_c = E\left(\tilde{L} / \tilde{L} > F_{\tilde{L}}^{\leftarrow}(c)\right) = E\left(\tilde{L} / \tilde{L} > VaR\right) = CVaR \quad (12)$$

The last expression means that for continuous distributions *Conditional VaR* is the expected loss conditioned to worst loss values (exceeding  $VaR$ ).

The most important critique to  $VaR$  as suitable risk measure is linked to the idea that risk measures must satisfy a set of coherence properties: see the seminal paper by Artzner et al. [4].  $VaR$ , in general, suffers for the lack of subadditivity. On the other hand,  $CVaR$  is a coherent risk measure.

Coming back now to the above definition 12 of  $CVaR$  in terms of the loss  $\tilde{L}$  in the continuous case, it is well known that there exists a closed formula (see, for example [2] and [3]), for  $CVaR$  if  $\tilde{L}$  is normally distributed (described in the next section) and under other kind of distributions (see again [3] and [8]). In the next subsections, a review of the most important known closed formulas is presented together with the new formulas obtained for the bounded and unbounded transformation in the Johnson's system.

### 3.1 VaR and CVaR under return normality

Assuming normality of the variable return  $\tilde{X}$ , formally:

$$\tilde{X} \approx Normal(\mu, \sigma)$$

and looking simply the percentage VaR given in general by 11, it is easy to show that it is:

$$VaR\% = - \left[ \mu + \sigma N^{-1}(1 - c) \right] = -(\mu + \sigma z) = NormalVaR \quad (13)$$

**Theorem on closed CVaR under normality.** *If the return on a position  $\tilde{X}$  is normally distributed with parameters  $\mu$  and  $\sigma$ , the conditional (percentage)VaR is given by:*

$$CVaR = -\mu + \sigma \frac{n(z)}{1 - c} = NormalCVaR \quad (14)$$

where  $n(z)$  is the value of the standard normal density function evaluated in  $z$ .

### 3.2 VaR and CVaR in the Normal-Power framework

If  $\tilde{X}$  is the return of a position over a fixed time horizon, from paragraph 3 and from 3 the percentage VaR is given by:

$$VaR = -\sigma \left[ (z^2 - 1) \frac{s_1}{6} + z \right] - \mu \quad (15)$$

The last formula may be rewritten in the following way:

$$VaR = -\mu - \sigma z - \sigma \left[ (z^2 - 1) \frac{s_1}{6} \right] = NormalVar - \sigma \left[ (z^2 - 1) \frac{s_1}{6} \right]$$

that is VaR now is given by the VaR obtained under normality minus a term depending on skewness (see [6]).

With regard to CVaR, the following theorem holds:

**Theorem on closed CVaR under the Normal-Power method.** *The closed formula for CVaR of  $\tilde{X}$  under Normal-Power transformation is:*

$$CVaR_{NP} = -\mu + \sigma \frac{n(z)}{1 - c} \left\{ 1 + \frac{s_1}{6} z \right\} \quad (16)$$

*Remark 1. As like as for VaR, it is possible to express CVaR<sub>NP</sub> in terms of NormalCVaR, in fact 16 may be rewritten as follows:*

$$CVaR_{NP} = NormalCVaR + \sigma \frac{n(z)}{1-c} \frac{s_1}{6} z$$

*that is CVaR<sub>NP</sub> may be decomposed in NormalCVaR and a term depending on skewness.*

### 3.3 VaR and CVaR in the Johnson framework

Exploiting the idea of Johnson, it is possible to transform any variable  $\tilde{X}$  in function of a standard normal variable with a very good degree of precision, as recalled at the beginning of paragraph 2.3. This make it possible to find closed formula for CVaR for a wide set of random variables (with unknown or untractable distributions) once found the kind of transformation and the parameters in the Johnson framework. In more detail, in the next subsections there are the extensions of the theorem on closed CVaR under normality (see par. 4.1): the theorems below give a strictly closed-formula for both  $S_L$  and  $S_U$  transformation, while a quasi-closed formula may be obtained in the  $S_B$  case. The closed-formula for CVaR under the  $S_L$  transformation may be applied in case of loss log-normality and the closed CVaR formula is well known in literature (see the expected shortfall formula in [3]) while the CVaR formulas (theorem 1 and 2) for the unbounded and bounded transformations are new. However, it is worthwhile to say that in all the financial numerical applications the transformation obtained applying Johnson are  $S_U$  and  $S_B$ , never  $S_L$ .

#### 3.3.1 Log-normal transformation

Putting together 4 and 5 in this case it is:

$$\tilde{N} = C + D \ln \left( \frac{\tilde{X} - A}{B} \right) \Rightarrow \ln \left( \frac{\tilde{X} - A}{B} \right) = \frac{\tilde{N} - C}{D}$$

Indeed, in general lognormality means that all the characteristics of the distribution may be resumed by only two parameters. Usually, to indicate that  $\tilde{X}$  follows a lognormal distribution of parameters  $m$  and  $s$  it is sufficient to say that  $\ln(\tilde{X})$  is endowed by a normal distribution with parameters  $m$  and  $s$  and, again, this may be expressed in the following way:

$$\ln(\tilde{X}) = m + s \cdot \tilde{N}$$

In this special case it is possible to find all superior moments of  $\tilde{X}$  starting from  $m$  and  $s$ , in fact it is:

$$\begin{cases} \mu = E(\tilde{X}) = \exp\left(m + \frac{s^2}{2}\right) \\ \sigma^2 = \sigma^2(\tilde{X}) = \mu^2 (\exp(s^2) - 1) \end{cases} \quad (17)$$

$$s_1(\tilde{X}) = \frac{e^{3m} \left( 2 \exp\left(3\frac{s^2}{2}\right) - 3 \exp\left(5\frac{s^2}{2}\right) + \exp\left(9\frac{s^2}{2}\right) \right)}{\sigma^3} \quad (18)$$

$$\beta_2(\tilde{X}) = \frac{\exp(4m + 2s^2) (3 + 6e^{s^2} - 4e^{3s^2} + e^{6s^2})}{\sigma^4} \quad (19)$$

Thus, starting from the estimation of mean, standard deviation, skewness and kurtosis indeed there are only two unknowns ( $m$  and  $s$ ) which may be easily found solving the system 17. The solutions are:

$$s^2 = \ln \left[ 1 + \left( \frac{\sigma}{\mu} \right)^2 \right]; \quad m = \ln \mu - \frac{s^2}{2}$$

Comparing the values of "theoretical" skewness and kurtosis given by 18 and 19 with the estimated value it is possible to obtain a more precise matching than those given by the Hill's routines (see [11], [14] and [21]) applying the Johnson systems. Moreover, in the assumption of log-normality of the return  $\tilde{X}$  in the sense previously described, that is:

$$\ln(\tilde{X}) = m + s \cdot \tilde{N}$$

so that, under log-normality, the parameters of the Johnson systems are given by:

$$A = 0; \quad B = 1; \quad C = -\frac{m}{s}; \quad D = \frac{1}{s}$$

Coming back to the general form given by Johnson, the inverse function of the log-normal  $S_L$  transformation (see 5), is:

$$\tilde{X} = A + B \exp\left(\frac{\tilde{N} - C}{D}\right) \quad (20)$$

Obviously, it is:

$$\tilde{X} = \exp(m + s \cdot \tilde{N})$$

and percentage  $VaR$  in terms of the return  $\tilde{X}$  is given by:

$$VaR = -\exp(m + s \cdot z)$$

This is a well-known result for staticians and actuaries and there is a closed formula also for  $CVaR$  as recalled in the next theorem.

**Theorem on closed  $CVaR$  under Log-Normality** (see [3]). *The closed formula for  $CVaR$  of  $\tilde{X}$  under  $S_L$  transformation is:*

$$CVaR_{S_L} = -\frac{\exp(m + s^2/2)}{1 - c} N(z - s) = -\frac{\mu}{1 - c} N(z - s) \quad (21)$$

### 3.3.2 Bounded transformation

Coming now to the bounded  $S_B$  transformation (see definition 6), the inverse is given by:

$$\tilde{X} = A + B \left[ 1 + \exp\left(-\frac{\tilde{N} - C}{D}\right) \right]^{-1} \quad (22)$$

so that percentage  $VaR$  in terms of the return  $\tilde{X}$  on a position is given by:

$$VaR = -A - B \left[ 1 + \exp\left(-\frac{z - C}{D}\right) \right]^{-1} \quad (23)$$

**Theorem 1.** *The quasi-closed formula for  $CVaR$  of  $\tilde{X}$  under  $S_B$  transformation is:*

$$CVaR_{S_B} = -A - B \frac{1}{1 - c} \left\{ \frac{1}{\sqrt{2\pi}} \left[ \int_{-\infty}^z \frac{e^{-\frac{u^2}{2}}}{1 + \exp\left(-\frac{u - C}{D}\right)} du \right] \right\} \quad (24)$$

### 3.3.3 Unbounded transformation

Considering  $S_U$  transformation, it is (see definition 7):

$$\tilde{N} = C + D \ln \left( \frac{\tilde{X} - A}{B} + \sqrt{1 + \left(\frac{\tilde{X} - A}{B}\right)^2} \right) \quad -\infty < \frac{\tilde{X} - A}{B} < +\infty \quad (25)$$

The inverse of the transformation is given by:

$$\tilde{X} = A + \frac{B}{2} \left[ \exp\left(\frac{\tilde{N} - C}{D}\right) - \exp\left(-\frac{\tilde{N} - C}{D}\right) \right] = A + B \sinh\left(\frac{\tilde{N} - C}{D}\right) \quad (26)$$

So, if the variable  $\tilde{X}$  is the loss on a position,  $VaR$  is given by the following condition:

$$VaR = -A - B \sinh\left(\frac{z - C}{D}\right)$$

**Theorem 2.** *The closed-formula for CVaR of  $\tilde{X}$  under  $S_U$  transformation is:*

$$CVaR_{S_U} = -A - \frac{B}{2} \frac{1}{1-c} \left[ \exp\left(\frac{1-2CD}{2D^2}\right) N\left(z - \frac{1}{D}\right) - \exp\left(\frac{1+2CD}{2D^2}\right) N\left(z + \frac{1}{D}\right) \right] \quad (27)$$

## 4 Financial Applications

The problem of dealing with non normal distribution is often discussed in financial literature and many authors focus their attention on indices (such as skewness and kurtosis) and statistical tests to verify non-normality. A lot of papers concern the importance of fat tails of returns distributions. Further, the problem of risk measuring with non normal distribution has been widely investigated. Some of the papers concerning the topics recalled above, report the basic statistics about the financial objects they analyze. In particular, in the examined papers, authors consider returns on funds of hedge funds (belonging to various areas and with different risk profiles, see [10]), stock market indices (such as the Standard & Poor's 500 Composite, Dow-Jones Industrial, Nikkei and so on) and exchange rates (see [16]), market indices compounded by different kinds of assets (aggregate bonds, U.S. equity, international equity, emerging markets equity, real estate investment trusts, hedge funds of funds, private equity and commodities, see [20]). The greatest values for (absolute) estimated skewness and kurtosis may be found in Peirò [16] and they are, respectively, 4.1 and 105.172, which are both referred to the daily returns on Dow-Jones Industrial (DJ) over the period 1980-1993. In the more recent paper by Sheikh and Qiao [20] the tests are based on 15 years of monthly returns until 2007 and the maximum absolute skewness is less than 1 while the maximum kurtosis is 12.32. In Table 1-3 there are the results obtained assuming return normality, using the Normal-Power approach and, last, using the Johnson's systems, starting from the described data and using different levels of probability  $1-c$ . Here, the routines by Hill [11] and Hill and Holder [14] as been adapted in Visual Basic and the numerical integrations (occurring in the bounded transformation) has been made using Mathematica<sup>®</sup> [15].

		N VaR	N-P VaR	JSN VaR	N CVaR	N-P CVaR	JSN CVaR
Gregoriou, Sedarz, Zhu (EJOR 2005)	<i>Efficient hedge funds (1997-2001)</i>						
	Funds of hedge funds	8,29%	4,75%	7,43%	9,72%	4,75%	9,73%
	Event driven	11,35%	8,76%	11,08%	13,19%	9,54%	14,12%
	Market neutral	10,56%	2,83%	6,97%	12,41%	1,56%	8,71%
	Global macro	12,97%	8,22%	10,79%	15,11%	8,44%	13,19%
	Global international	17,00%	14,24%	15,86%	19,60%	15,72%	18,89%
	Global emerging	30,93%	33,80%	36,91%	35,83%	39,87%	48,26%
	Global established	16,02%	9,61%	13,68%	18,66%	9,65%	17,16%
	Sector	15,94%	14,17%	16,93%	18,58%	16,09%	21,95%
	<i>Non-efficient hedge funds (1997-2001)</i>						
	Funds of hedge funds	5,72%	3,33%	4,73%	6,69%	3,32%	5,90%
	Event driven	9,02%	6,98%	9,18%	10,47%	7,60%	12,02%
	Market neutral	6,09%	1,73%	2,46%	7,14%	1,00%	2,58%
	Global macro	10,50%	6,80%	7,33%	12,18%	6,97%	8,04%
	Global international	13,11%	10,96%	12,85%	15,13%	12,11%	15,88%
	Global emerging	18,94%	20,63%	22,88%	21,82%	24,19%	30,56%
	Global established	12,46%	7,59%	8,55%	14,46%	7,62%	9,37%
	Sector	15,71%	14,03%	16,54%	18,22%	15,85%	21,16%
	<i>Efficient hedge funds (1999-2001)</i>						
	Funds of hedge funds	6,17%	3,42%	4,12%	7,29%	3,41%	4,73%
	Event driven	11,88%	9,20%	11,60%	13,78%	10,01%	14,74%
	Market neutral	7,82%	1,76%	2,64%	9,27%	0,76%	2,78%
	Global macro	13,60%	8,73%	11,50%	15,80%	8,95%	14,09%
	Global international	16,60%	13,79%	15,19%	19,24%	15,29%	18,04%
	Global emerging	23,79%	26,19%	28,22%	27,89%	31,27%	36,58%
	Global established	16,27%	9,87%	11,36%	18,90%	9,91%	12,73%
	Sector	15,47%	13,72%	15,38%	18,07%	15,61%	18,98%
	<i>Non-efficient hedge funds (1999-2001)</i>						
	Funds of hedge funds	5,04%	2,86%	2,83%	5,92%	2,86%	3,04%
	Event driven	7,64%	5,88%	6,56%	8,89%	6,42%	7,71%
	Market neutral	4,90%	1,26%	0,38%	5,78%	0,66%	0,38%
	Global macro	0,19%	-0,15%	-0,20%	0,34%	-0,13%	-0,17%
	Global international	12,04%	10,03%	10,24%	13,93%	11,11%	11,54%
	Global emerging	15,23%	16,69%	17,44%	17,72%	19,77%	21,67%
	Global established	12,72%	7,85%	7,45%	14,72%	7,88%	7,85%
	Sector	15,72%	14,06%	15,35%	18,17%	15,85%	18,44%
	<i>Financial indices (1980-1993)</i>						
	SP	-2,17%	0,33%	-1,08%	-1,83%	1,69%	0,89%
	DJ	2,52%	5,83%	3,78%	2,89%	7,55%	6,07%
	NI	2,76%	3,07%	3,46%	3,17%	3,60%	5,07%
	FT	2,28%	3,48%	3,19%	2,62%	4,31%	4,72%
	CB	2,53%	3,41%	3,40%	2,90%	4,14%	4,89%
CA	2,27%	2,66%	2,85%	2,61%	3,16%	3,86%	
CI	1,84%	2,44%	2,44%	2,11%	2,96%	3,59%	
BC	3,20%	4,11%	4,22%	3,68%	4,95%	5,80%	
IG	2,26%	2,55%	2,81%	2,60%	3,01%	3,82%	
<i>Exchange rates (1980-1993)</i>							
YD	1,61%	1,79%	1,91%	1,84%	2,10%	2,44%	
PD	1,61%	1,62%	1,81%	1,85%	1,87%	2,32%	
MD	1,63%	1,70%	1,84%	1,86%	1,96%	2,31%	
<i>Asset classes (1997-2007)-with volatility before "Unsmoothing"</i>							
International equity	35,69%	45,39%	45,11%	40,88%	54,52%	54,56%	
Emerging markets equity	58,72%	73,38%	74,13%	67,27%	87,87%	92,88%	
Hedge funds of funds	15,28%	17,65%	18,83%	17,51%	20,84%	24,74%	
Private equity	53,67%	57,40%	60,91%	61,49%	66,73%	75,50%	
Commodities	54,25%	60,42%	63,25%	62,15%	70,83%	78,59%	
<i>Asset classes (1997-2007)-with volatility after "Unsmoothing"</i>							
International equity	43,25%	55,00%	54,67%	49,55%	66,07%	66,12%	
Emerging markets equity	73,95%	92,42%	93,37%	84,73%	110,68%	116,98%	
Hedge funds of funds	23,01%	26,57%	28,34%	26,36%	31,37%	37,24%	
Private equity	69,88%	74,74%	79,31%	80,06%	86,89%	98,31%	
Commodities	67,12%	74,75%	78,25%	76,89%	87,62%	97,22%	
Peirò (JBF 1999)	<i>Financial indices (1980-1993)</i>						
	SP	-2,17%	0,33%	-1,08%	-1,83%	1,69%	0,89%
	DJ	2,52%	5,83%	3,78%	2,89%	7,55%	6,07%
	NI	2,76%	3,07%	3,46%	3,17%	3,60%	5,07%
	FT	2,28%	3,48%	3,19%	2,62%	4,31%	4,72%
	CB	2,53%	3,41%	3,40%	2,90%	4,14%	4,89%
	CA	2,27%	2,66%	2,85%	2,61%	3,16%	3,86%
	CI	1,84%	2,44%	2,44%	2,11%	2,96%	3,59%
	BC	3,20%	4,11%	4,22%	3,68%	4,95%	5,80%
	IG	2,26%	2,55%	2,81%	2,60%	3,01%	3,82%
	<i>Exchange rates (1980-1993)</i>						
	YD	1,61%	1,79%	1,91%	1,84%	2,10%	2,44%
PD	1,61%	1,62%	1,81%	1,85%	1,87%	2,32%	
MD	1,63%	1,70%	1,84%	1,86%	1,96%	2,31%	
Sheikh, Qiao (JAI 2010)	<i>Asset classes (1997-2007)-with volatility before "Unsmoothing"</i>						
	International equity	35,69%	45,39%	45,11%	40,88%	54,52%	54,56%
	Emerging markets equity	58,72%	73,38%	74,13%	67,27%	87,87%	92,88%
	Hedge funds of funds	15,28%	17,65%	18,83%	17,51%	20,84%	24,74%
	Private equity	53,67%	57,40%	60,91%	61,49%	66,73%	75,50%
	Commodities	54,25%	60,42%	63,25%	62,15%	70,83%	78,59%
	<i>Asset classes (1997-2007)-with volatility after "Unsmoothing"</i>						
	International equity	43,25%	55,00%	54,67%	49,55%	66,07%	66,12%
	Emerging markets equity	73,95%	92,42%	93,37%	84,73%	110,68%	116,98%
	Hedge funds of funds	23,01%	26,57%	28,34%	26,36%	31,37%	37,24%
	Private equity	69,88%	74,74%	79,31%	80,06%	86,89%	98,31%
	Commodities	67,12%	74,75%	78,25%	76,89%	87,62%	97,22%

Table 1:  $1 - c = 1\%$

		<i>N VaR</i>	<i>N-P VaR</i>	<i>JSN VaR</i>	<i>N CVaR</i>	<i>N-P CVaR</i>	<i>JSN CVaR</i>
<i>Gregoriou, Sadzro, Zhu (EJOR 2005)</i>	<i>Efficient hedge funds (1997-2001)</i>						
	<b>Funds of hedge funds</b>	5,41%	4,04%	4,26%	7,17%	4,45%	6,27%
	<b>Event driven</b>	7,66%	6,65%	6,69%	9,92%	7,93%	9,47%
	<b>Market neutral</b>	6,83%	3,84%	4,39%	9,11%	3,17%	6,02%
	<b>Global macro</b>	8,65%	6,82%	6,99%	11,30%	7,65%	9,37%
	<b>Global international</b>	11,78%	10,71%	10,72%	14,98%	12,86%	13,92%
	<b>Global emerging</b>	21,06%	22,17%	20,96%	27,11%	29,32%	31,07%
	<b>Global established</b>	10,72%	8,24%	8,56%	13,97%	9,04%	11,80%
	<b>Sector</b>	10,63%	9,95%	9,77%	13,89%	12,52%	14,30%
	<i>Non-efficient hedge funds (1997-2001)</i>						
	<b>Funds of hedge funds</b>	3,77%	2,85%	2,96%	4,97%	3,13%	4,08%
	<b>Event driven</b>	6,11%	5,32%	5,33%	7,90%	6,33%	7,79%
	<b>Market neutral</b>	3,98%	2,30%	2,08%	5,27%	1,92%	2,31%
	<b>Global macro</b>	7,14%	5,70%	5,62%	9,20%	6,35%	6,66%
	<b>Global international</b>	9,04%	8,21%	8,20%	11,53%	9,88%	11,12%
	<b>Global emerging</b>	13,15%	13,81%	12,82%	16,70%	18,00%	19,25%
	<b>Global established</b>	8,43%	6,55%	6,57%	10,90%	7,16%	7,78%
	<b>Sector</b>	10,67%	10,02%	9,87%	13,76%	12,47%	14,09%
	<i>Efficient hedge funds (1999-2001)</i>						
	<b>Funds of hedge funds</b>	3,93%	2,87%	2,86%	5,31%	3,19%	3,64%
	<b>Event driven</b>	8,06%	7,02%	7,06%	10,40%	8,34%	9,93%
	<b>Market neutral</b>	4,90%	2,56%	2,17%	6,69%	2,03%	2,46%
	<b>Global macro</b>	9,17%	7,29%	7,48%	11,89%	8,14%	10,01%
	<b>Global international</b>	11,28%	10,19%	10,19%	14,54%	12,38%	13,30%
	<b>Global emerging</b>	15,54%	16,47%	15,75%	20,60%	22,45%	23,60%
	<b>Global established</b>	10,97%	8,50%	8,46%	14,22%	9,30%	10,24%
	<b>Sector</b>	10,25%	9,57%	9,52%	13,45%	12,11%	13,18%
	<i>Non-efficient hedge funds (1999-2001)</i>						
	<b>Funds of hedge funds</b>	3,27%	2,42%	2,24%	4,35%	2,68%	2,60%
	<b>Event driven</b>	5,13%	4,45%	4,46%	6,67%	5,32%	5,76%
	<b>Market neutral</b>	3,14%	1,74%	0,38%	4,22%	1,42%	0,38%
	<b>Global macro</b>	-0,12%	-0,25%	-0,29%	0,07%	-0,19%	-0,24%
<b>Global international</b>	8,24%	7,46%	7,42%	10,57%	9,02%	9,14%	
<b>Global emerging</b>	10,22%	10,78%	10,54%	13,29%	14,41%	14,84%	
<b>Global established</b>	8,69%	6,81%	6,25%	11,16%	7,42%	6,98%	
<b>Sector</b>	10,78%	10,14%	10,10%	13,80%	12,53%	13,37%	
<i>Petro (JBF 1999)</i>	<i>Financial indices (1980-1993)</i>						
	<b>SP</b>	-2,86%	-1,89%	-3,00%	-2,44%	-0,51%	-1,71%
	<b>DJ</b>	1,77%	3,05%	1,59%	2,23%	4,78%	3,07%
	<b>NI</b>	1,94%	2,06%	1,73%	2,44%	2,68%	2,87%
	<b>FT</b>	1,60%	2,06%	1,52%	2,02%	2,94%	2,62%
	<b>CB</b>	1,78%	2,12%	1,71%	2,24%	2,92%	2,81%
	<b>CA</b>	1,59%	1,74%	1,57%	2,01%	2,31%	2,40%
	<b>CI</b>	1,29%	1,53%	1,20%	1,63%	2,09%	2,02%
	<b>BC</b>	2,25%	2,60%	2,26%	2,84%	3,53%	3,52%
	<b>IG</b>	1,58%	1,69%	1,53%	2,00%	2,22%	2,35%
	<i>Exchange rates (1980-1993)</i>						
	<b>YD</b>	1,13%	1,20%	1,14%	1,42%	1,56%	1,63%
	<b>PD</b>	1,14%	1,14%	1,10%	1,43%	1,44%	1,55%
	<b>MD</b>	1,15%	1,18%	1,14%	1,44%	1,50%	1,58%
<i>Sheikh, Qiao (JAI 2010)</i>	<i>Asset classes (1997-2007)-with volatility before "Unsmoothing"</i>						
	<b>International equity</b>	25,23%	28,98%	28,38%	31,64%	39,10%	38,70%
	<b>Emerging markets equity</b>	41,52%	47,18%	45,00%	52,06%	63,34%	63,23%
	<b>Hedge funds of funds</b>	10,81%	11,72%	10,87%	13,55%	15,37%	15,94%
	<b>Private equity</b>	37,95%	39,39%	38,35%	47,59%	50,46%	52,49%
	<b>Commodities</b>	38,36%	40,74%	39,48%	48,10%	52,85%	54,37%
	<i>Asset classes (1997-2007)-with volatility after "Unsmoothing"</i>						
	<b>International equity</b>	30,58%	35,12%	34,39%	38,35%	47,39%	46,90%
	<b>Emerging markets equity</b>	52,29%	59,43%	56,68%	65,57%	79,78%	79,64%
	<b>Hedge funds of funds</b>	16,27%	17,65%	16,36%	20,40%	23,14%	23,99%
	<b>Private equity</b>	49,41%	51,29%	49,94%	61,96%	65,70%	68,35%
	<b>Commodities</b>	47,45%	50,41%	48,84%	59,51%	65,38%	67,26%

Table 2:  $1 - c = 5\%$

		<i>N VaR</i>	<i>N-P VaR</i>	<i>JSN VaR</i>	<i>N CVaR</i>	<i>N-P CVaR</i>	<i>JSN CVaR</i>
<i>Gregoriou, Sedzro, Zhu (EJOR 2005)</i>	<i>Efficient hedge funds (1997-2001)</i>						
	<b>Funds of hedge funds</b>	3,88%	3,36%	2,93%	5,88%	4,07%	4,90%
	<b>Event driven</b>	5,69%	5,31%	4,81%	8,25%	6,93%	7,56%
	<b>Market neutral</b>	4,84%	3,72%	3,21%	7,43%	3,49%	4,88%
	<b>Global macro</b>	6,35%	5,66%	5,20%	9,35%	6,93%	7,69%
	<b>Global international</b>	9,00%	8,59%	8,22%	12,62%	11,22%	11,63%
	<b>Global emerging</b>	15,80%	16,22%	14,38%	22,65%	24,12%	24,18%
	<b>Global established</b>	7,89%	6,96%	6,28%	11,57%	8,31%	9,55%
	<b>Sector</b>	7,80%	7,54%	6,75%	11,49%	10,59%	11,20%
	<i>Non-efficient hedge funds (1997-2001)</i>						
	<b>Funds of hedge funds</b>	2,74%	2,39%	2,15%	4,09%	2,87%	3,30%
	<b>Event driven</b>	4,56%	4,27%	3,77%	6,58%	5,54%	6,12%
	<b>Market neutral</b>	2,86%	2,22%	1,77%	4,32%	2,10%	2,12%
	<b>Global macro</b>	5,34%	4,80%	4,57%	7,68%	5,79%	5,86%
	<b>Global international</b>	6,87%	6,56%	6,11%	9,70%	8,60%	9,09%
	<b>Global emerging</b>	10,07%	10,32%	8,88%	14,09%	14,95%	14,93%
	<b>Global established</b>	6,28%	5,58%	5,33%	9,08%	6,60%	6,84%
	<b>Sector</b>	7,98%	7,74%	7,02%	11,49%	10,63%	11,19%
	<i>Efficient hedge funds (1999-2001)</i>						
	<b>Funds of hedge funds</b>	2,74%	2,33%	2,14%	4,29%	2,89%	3,06%
	<b>Event driven</b>	6,02%	5,63%	5,11%	8,68%	7,31%	7,96%
	<b>Market neutral</b>	3,34%	2,46%	1,78%	5,37%	2,28%	2,21%
	<b>Global macro</b>	6,81%	6,10%	5,61%	9,89%	7,40%	8,23%
	<b>Global international</b>	8,45%	8,04%	7,70%	12,14%	10,71%	11,06%
	<b>Global emerging</b>	11,15%	11,50%	10,36%	16,88%	18,10%	18,18%
	<b>Global established</b>	8,15%	7,22%	6,78%	11,83%	8,56%	8,90%
	<b>Sector</b>	7,47%	7,21%	6,78%	11,09%	10,20%	10,60%
	<i>Non-efficient hedge funds (1999-2001)</i>						
	<b>Funds of hedge funds</b>	2,32%	2,00%	1,81%	3,55%	2,44%	2,30%
	<b>Event driven</b>	3,80%	3,54%	3,38%	5,54%	4,64%	4,82%
	<b>Market neutral</b>	2,21%	1,68%	0,38%	3,43%	1,57%	0,38%
	<b>Global macro</b>	-0,28%	-0,33%	-0,36%	-0,07%	-0,24%	-0,28%
	<b>Global international</b>	6,21%	5,92%	5,82%	8,85%	7,83%	7,85%
<b>Global emerging</b>	7,55%	7,76%	7,36%	11,03%	11,77%	11,81%	
<b>Global established</b>	6,54%	5,84%	5,36%	9,34%	6,86%	6,38%	
<b>Sector</b>	8,14%	7,90%	7,57%	11,57%	10,73%	11,04%	
<i>Peiró (JBF 1999)</i>	<i>Financial indices (1980-1993)</i>						
	<b>SP</b>	-3,22%	-2,85%	-3,59%	-2,75%	-1,47%	-2,52%
	<b>DJ</b>	1,37%	1,85%	0,93%	1,89%	3,58%	2,14%
	<b>NI</b>	1,51%	1,55%	1,15%	2,07%	2,23%	2,14%
	<b>FT</b>	1,24%	1,41%	0,97%	1,71%	2,32%	1,91%
	<b>CB</b>	1,38%	1,51%	1,12%	1,90%	2,35%	2,09%
	<b>CA</b>	1,23%	1,28%	1,08%	1,70%	1,90%	1,85%
	<b>CI</b>	1,00%	1,09%	0,78%	1,38%	1,69%	1,49%
	<b>BC</b>	1,74%	1,87%	1,53%	2,40%	2,87%	2,68%
	<b>IG</b>	1,22%	1,26%	1,05%	1,69%	1,84%	1,81%
	<i>Exchange rates (1980-1993)</i>						
	<b>YD</b>	0,88%	0,90%	0,82%	1,21%	1,30%	1,30%
	<b>PD</b>	0,88%	0,88%	0,81%	1,21%	1,22%	1,25%
	<b>MD</b>	0,90%	0,91%	0,84%	1,23%	1,26%	1,28%
	<i>Sheikh, Qiao (JAI 2010)</i>	<i>Asset classes (1997-2007)-with volatility before "Unsmoothing"</i>					
<b>International equity</b>		19,66%	21,07%	20,47%	26,92%	31,87%	31,36%
<b>Emerging markets equity</b>		32,35%	34,48%	32,21%	44,30%	51,77%	50,57%
<b>Hedge funds of funds</b>		8,42%	8,76%	7,70%	11,53%	12,74%	12,52%
<b>Private equity</b>		29,57%	30,11%	28,34%	40,49%	42,39%	42,65%
<b>Commodities</b>		29,89%	30,78%	28,96%	40,93%	44,07%	44,01%
<i>Asset classes (1997-2007)-with volatility after "Unsmoothing"</i>							
<b>International equity</b>		23,82%	25,54%	24,80%	32,63%	38,62%	38,01%
<b>Emerging markets equity</b>		40,74%	43,43%	40,56%	55,79%	65,20%	63,69%
<b>Hedge funds of funds</b>		12,67%	13,19%	11,59%	17,36%	19,17%	18,84%
<b>Private equity</b>		38,50%	39,21%	36,90%	52,72%	55,20%	55,54%
<b>Commodities</b>		36,97%	38,08%	35,82%	50,63%	54,52%	54,45%

Table 3:  $1 - c = 10\%$

Again, in Dowd and Cotter [5] data consist of daily returns for five of the most liquid futures (S&P500, FTSE100, DAX, Hang Seng and Nikkei225 indices) over January 1991 and December 2003. They have found a maximum for absolute skewness equal to 0.3307 and 13.8301 for kurtosis. In their work Dowd and Cotter suggest non-parametric estimation methods to estimate  $VaR$ ,  $CVaR$  and spectral risk measures (not considered here), so it is possible to compare their results with those obtained using the Johnson moment method (see Table 4).

1% Long						
	VaR JSN	VaR Dowd	Abs diff.	CVaR JSN	CVaR Dowd	Abs diff.
S&P	0,0293	0,0319	0,0026	0,0388	0,0408	0,0020
FTSE	0,0299	0,0304	0,0005	0,0380	0,0387	0,0007
DAX	0,0415	0,0404	0,0011	0,0551	0,0527	0,0024
HANGSENG	0,0505	0,0512	0,0007	0,0700	0,0699	0,0001
NIKKEI	0,0376	0,0371	0,0005	0,0468	0,0472	0,0004
1% Short						
	VaR JSN	VaR Dowd	Abs diff.	CVaR JSN	CVaR Dowd	Abs diff.
S&P	0,0291	0,0291	0,0000	0,0380	0,0385	0,0005
FTSE	0,0297	0,0329	0,0032	0,0375	0,0398	0,0023
DAX	0,0387	0,0425	0,0038	0,0503	0,0544	0,0041
HANGSENG	0,0555	0,0520	0,0035	0,0775	0,0699	0,0076
NIKKEI	0,0381	0,0368	0,0013	0,0477	0,0476	0,0001

5% Long						
	VaR JSN	VaR Dowd	Abs diff.	CVaR JSN	CVaR Dowd	Abs diff.
S&P	0,0168	0,0175	0,0007	0,0248	0,0256	0,0008
FTSE	0,0183	0,0185	0,0002	0,0256	0,0261	0,0005
DAX	0,0238	0,0236	0,0002	0,0351	0,0340	0,0011
HANGSENG	0,0276	0,0294	0,0018	0,0425	0,0445	0,0020
NIKKEI	0,0238	0,0239	0,0001	0,0325	0,0336	0,0011
5% Short						
	VaR JSN	VaR Dowd	Abs diff.	CVaR JSN	CVaR Dowd	Abs diff.
S&P	0,0173	0,0181	0,0008	0,0248	0,0254	0,0006
FTSE	0,0185	0,0175	0,0010	0,0256	0,0260	0,0004
DAX	0,0232	0,0243	0,0011	0,0331	0,0358	0,0027
HANGSENG	0,0297	0,0294	0,0003	0,0465	0,0440	0,0025
NIKKEI	0,0237	0,0247	0,0010	0,0328	0,0329	0,0001

10% Long						
	VaR JSN	VaR Dowd	Abs diff.	CVaR JSN	CVaR Dowd	Abs diff.
S&P	0,0119	0,0119	0,0000	0,0194	0,0200	0,0006
FTSE	0,0134	0,0135	0,0001	0,0206	0,0209	0,0003
DAX	0,0168	0,0169	0,0001	0,0275	0,0269	0,0006
HANGSENG	0,0193	0,0208	0,0015	0,0327	0,0345	0,0018
NIKKEI	0,0178	0,0173	0,0005	0,0265	0,0270	0,0005
10% Short						
	VaR JSN	VaR Dowd	Abs diff.	CVaR JSN	CVaR Dowd	Abs diff.
S&P	0,0126	0,0117	0,0009	0,0197	0,0200	0,0003
FTSE	0,0137	0,0131	0,0006	0,0207	0,0205	0,0002
DAX	0,0170	0,0160	0,0010	0,0264	0,0278	0,0014
HANGSENG	0,0205	0,0203	0,0002	0,0355	0,0341	0,0014
NIKKEI	0,0175	0,0177	0,0002	0,0265	0,0270	0,0005

Table 4

## Concluding remarks

Quantile based risk evaluation is relevant in many actuarial and financial applications. However, identifying the right distribution for positions random returns or for the key random variable loss may be difficult or impossible while it is easier to find at least the most important moments of the distributions involved. In this paper two well known moment methods to evaluate quantiles are reviewed. The first, called Normal-Power method, is based on mean, standard deviation and skewness while the second, known as Johnson's systems, considers the kurtosis index as well. In this paper there is a review of the methods and of the quantile ( $VaR$ ) formulas provided under them. Moreover, new closed formulas are shown for  $CVaR$  for the unbounded and bounded transformation provided in the Johnson's systems. This makes it possible to evaluate  $CVaR$  whatever the distribution of the key random variable. With the aim of testing the goodness of the Johnson approach the new closed formulas have been applied to skewed distributions with high kurtosis obtaining satisfactory results. In particular, in comparison with the results obtained by Dowd and Cotter [5], the absolute value of the differences in  $VaR$  belong to the interval [0.2 b.p.; 0.5 b.p.] while differences in  $CVaR$  goes from 38 b.p. to 76.b.p.

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