Analytical CoVaR*

KiHoon Jimmy Hong†

Faculty of Economics, University of Cambridge

Job Market Paper

Abstract

This paper proposes an analytical form of CoVaR, which is compatible with the existing VaR and stress-test based risk management framework, adding value by capturing systemic risk of a portfolio. Analytical CoVaR is a computationally inexpensive risk tool used to screen systemic risk. It is reasonably accurate and also has desired statistical properties hence is suitable for use by investment companies, brokerage firms, mutual funds and any business that evaluates risk. Empirical backtest results show that historical analytical CoVaR exceedances pass both Failure Frequency Test and Conditional Test. The paper also presents a theoretical frameworks within which to investigate how analytical CoVaR captures systemic risk when asset and market returns are serially independent and when they are serially correlated. Analytical CoES is presented as an extension to capture loss beyond CoVaR.

Keywords: Finance, Systemic Risk, VaR, CoVaR, ES, Delta-Normal Method, Backtest, Bivariate Ornstein – Uhlenbeck Process, MLE, GLS, IRLS

*I am indebted to Dr. Stephen Satchell, my PhD supervisor, for his valuable guidance and continuous support. I would also like to thank Christophe Becker, Hashem Pesaran, Michael Dempster, Miguel Ferreira, Pedro Santa-Clara, Ron Giammarino and Usha Mittoo for useful discussions and helpful comments.

†Corresponding address at: Faculty of Economics, University of Cambridge, Sidgwick Avenue, Cambridge, CB3 9DD. Email: kjh47@cam.ac.uk.

1 Introduction

The co-movement of an individual equity return and the market return is primarily driven by the fundamentals of a company during normal periods. But it tends to increase during the times of crisis. There are two typical reasons for this. First, equity and market are strongly affected by common exposure to operational and financial risk sources during the times of crisis. Second, there exists propagation of distress associated with a decline in the market value, more specifically balance sheet contraction, of individual institutions. Measuring systemic risk has received attention recently. Acharya, Pedersen, Philippon and Richardson (2010) present a simple model of systemic risk that can be used to measure the contributions of individual financial institutions to overall systemic risk. Brownlees and Engle (2010) focuses on constructing measures of systemic risk, Marginal Expected Shortfall, based on public market data. Adrian and Brunnermeier (2010) (A&B here after) introduced Conditional Value at Risk (CoVaR), which is the overall Value at Risk of a financial institution conditional on another financial institution being under distress.

The main contribution of this paper can be summarized in one line: Analytical CoVaR is a computationally inexpensive risk tool to screen systemic risk that is reasonably accurate and that is compatible with existing VaR and stress test based risk management framework. In order to employ CoVaR in daily risk management procedure of financial institutions and to solve a portfolio optimization problem, the development of analytical methods is required. Running a complicated risk engine of full valuation method that is used by most of risk management departments is computationally burdensome and would not be suited for the use in dynamic intra – day trading decision making. An analytical approximation would be used like a simple back of the envelop calculation and enables traders / portfolio managers to swiftly access potential systemic risk exposure of their positions. In this respect, analytical CoVaR is necessary to make CoVaR a practical tool to existing risk management framework. Analytical form of a risk measure is an approximation hence is not accurate. It would be important to access the accuracy (or inaccuracy) of the analytical measure and show that it does what it is intended to do. Chapter 4 of the paper presents empirical evidences that Analytical CoVaR captures systemic risk properly hence it is an appropriate measure. Given it is a reasonably accurate but precise measure of systemic risk, analytical CoVaR would raise a red flag to initiate a further detailed full scale investigation of portfolio's systemic risk. Therefore, there would be two practical uses of Analytical CoVaR. First, as a quick and computationally less expensive way of gauging systemic risk of a portfolio by traders and portfolio managers and second, as a quick signal that would raise a red flag to initiate a further investigation of a trading book or a portfolio by risk managers.

The objective of this paper is to (i) propose an analytical form of CoVaR that shares the advantages of analytical VaR and is readily applicable to financial institutions' existing risk management system (ii) provide empirical and theoretical frameworks within which to investigate how analytical CoVaR captures systemic risk. The analytical CoVaR studied in this paper is suitable for use by investment companies, brokerage firms, mutual funds and any business that evaluates financial risk. There are two aspects of A&B's CoVaR that I intend to address and amend in this paper. (1) A&B concentrates their efforts to employ CoVaR as a regulatory policy tool hence their model only applies to macroeconomic variables. It is hard to capture day to day systemic risk of individual financial institutions or their trading books and portfolios with macroeconomic variables and this paper will use daily return data of financial institutions' stock prices. (2) A&B defines CoVaR to capture the risk of a specific financial institution's systemic risk with respect to another. This definition is very useful to capture the contribution of individual financial institution to overall systemic risk in the market. However, this is different from how other literatures define systemic risk. A&B's "exposure CoVaR", which intends to measure systemic risk of a portfolio with respect to the entire financial market will be used in this paper to be consistent with other literatures. For the simplicity, here after I will refer this measure as "CoVaR".

The remainder of this paper is organized as follows. Section 2 defines systemic risk and explains why VaR failed during the recent financial crisis. Section 3 presents the benefits of analytical form and proposes analytical CoVaR under bivariate normality assumption then derives the confidence interval of it. Section 4 provides empirical backtest using Kupiec's failure frequency test and Christoffersen's conditional test to find that analytical CoVaR exceedances pass both tests while analytical VaR exceedances does not pass the conditional test. Section 5 investigates how analytical CoVaR captures the systemic risk by assuming 1) price process follow geometric Brownian motion and 2) detrended log price process follow the bivariate Ornstein-Uhlenbeck (OU) model of Hong and Satchell (2011). Empirical evidence is then presented. Section 6 presents analytical Conditional Expected Shortfall (CoES) as an extension to capture loss beyond CoVaR and discusses various aspects and potential applications of CoVaR. And section 7 concludes the paper.

2 Systemic Risk and Failure of VaR

Since systemic risk is often confused with systematic risk, it would be meaningful to look at few of its definitions. (i) Kaufman defines systemic risk as the risk of collapse of an entire financial system or entire market, as opposed to the risk associated with any one individual entity, group or component of a system. According to Kaufman, systemic risk can be seen as "financial system instability, potentially catastrophic, caused or exacerbated by idiosyncratic events or conditions in financial intermediaries". (ii) The Bank for International Settlements (BIS) defines systemic risk as "the risk that the failure of a participant to meet its contractual obligations may in turn cause other participants to default with a chain reaction leading to broader financial difficulties" (BIS 1994, 177). These definitions emphasize correlation with causation, and they require close and direct connections among institutions or markets. (iii) According to Kaufman and Scott (2003), systemic risk refers to the risk or probability of breakdowns in an entire system, as opposed to breakdowns in individual parts or components, and is evidenced by co-movements among most or all the parts. (iv) Similarly, Schwarcz (2008) defines systemic risk as the risks imposed by interlinkages and interdependencies in a system or market, where the failure of a single entity or cluster of entities can cause a cascading failure, which could potentially bankrupt or bring down the entire system or market.

(Insert Table 1)

Table 1 summarizes American and British financial institutions' log return correlation against the S&P500. The sample period runs from January 1, 2004 to July 22, 2009 and was collected from Yahoo Finance. The pre-crisis period is from January 1, 2004 to February 26, 2007 and the crisis period is from February 27, 2007 to July 22, 2009. On February 27, 2007 the Federal Home Loan Mortgage Corporation (Freddie Mac) announced that it will no longer buy the most risky subprime mortgages and mortgage-related securities. This is taken as the earliest day of the 2008 financial crisis. Conversely, July 23, when 2009, Citigroup announced that it completed a previously announced exchange offer with private investors of convertible preferred securities and a previously announced matching exchange offer with the U.S. Government is taken as the first sign of recovery from the crisis. The sample start date of January 1, 2004 was chosen to secure enough number of Pre-Crisis period sample observations to demonstrate long term correlation. It is clear that the log return correlation jumps from the Pre-Crisis to Crisis period. As shown in table I, in the presence of extreme events such as equity market crash there exists additional correlation between portfolio return and market return.

The most common measure of risk value at risk (VaR) measures the amount of loss a portfolio may incur if the portfolio stays unchanged over a given time period under normal market conditions at a given level of confidence. VaR does not provide information on how bad the loss of the portfolio may be if a sharp adverse movement were to occur under these normal market conditions. VaR is only valid under normal market conditions and a series of theoretical assumptions. During normal times, the co-movement of financial institutions' assets and liabilities is driven by fundamentals; in these circumstances VaR provides a valid risk measure. However in the time of market turmoil such as recent financial crisis when the co-movement between the market and financial institutions' asset increased significantly, VaR is was not able to reflect such systemic risk properly, this is because VaR focuses on the risk of an individual institution in isolation. Such increases of co-movement give rise to systemic risk, the risk that institutional distress spreads widely and distorts the supply of credit and capital to the real economy. A single institution's risk does not necessarily reflect systemic risk. However CoVaR is designed to solve such problems as it represents the overall Value at Risk of a portfolio or a financial institution conditional on financial market being distressed.

3 Analytical CoVaR

3.1 Analytical Approach

Two methods of computing VaR, can be used; (i) the local valuation and (ii) the full valuation methods. The local valuation methods assess risk by approximating possible changes of the value of a portfolio with local derivatives, "Greeks". The full valuation methods fully reprice a portfolio over various scenarios. This classification reflects a tradeoff between speed and accuracy. The tradeoff between speed and computational cost becomes very important as the size of portfolio and the complexity of the risk exposure increase. Analytical VaR is an application of traditional mean variance portfolio analysis. It is sometimes called delta-normal method or covariance matrix approach. This approach is a local valuation method using delta, the first order derivative of a portfolio value. This approach is analytical because VaR is derived from closed-form solutions. To assess the risk of a portfolio and generate VaR under this local valuation approach, only risk exposures and the variance – covariance matrix are needed. Although the normality condition imposed on return distribution is the downside, such assumption also allows aggregation and disaggregation of risk. Jorion (2001) clearly presents the objective / justification of such analytical approach. He mentions that "This method is important not only for its own sake but also because it illustrates the 'mapping' principle in risk management. Even the more sophisticated simulation methods cannot possibly model the enormous number of risk factors in financial markets. They have to rely on simplifications. Hence the first step in understanding risk is to decompose financial instruments into their fundamental building blocks." The most important advantages of analytical CoVaR is that it is a straight forward, simple and computationally inexpensive method of screening risk to determine the overall risk of a portfolio at the expense of some accuracy. This approach is useful in day to day risk management. This approach can save significant time and effort by employing an analytical risk measure as a daily screening process. Once notable risk is identified with this method, more thorough analysis can be executed.

3.2 Point Measure

In this section, CoVaR conditioned that financial market return is at q% VaR level is expressed in terms of the conditional distributions of a portfolio and the market return distribution. The information about these conditional distributions is equivalent to the information about the portfolio's cumulative distribution functions and their mutual correlations, which is a prerequisite to solving any portfolio optimization problem.

CoVaR of a portfolio is defined as

$$\Pr\left(R^P \leq CoVaR_q^{P|M} \mid R_q^M = VaR_q^M\right) = q \tag{1}$$

where
R^P: return of a portfolio, R^M: market return
$CoVaR_q^{P|M}$: CoVaR of the portfolio given market return is at lowest q% level.
VaR_q^M: q% Value at Risk of market return

I again emphasize the distinction between this approach and that taken by A&B. A&B capturing the probability of entire portfolio of a financial institution being under CoVaR conditioning that another institution is under distress. Hence in equation(1), where i and j stands for different individual financial institutions therefore equation(1) is equivalent to the "exposure CoVaR" in A&B.

Proposition 1 *Under normality assumption, q% Analytical form of VaR and CoVaR can be represented as*

$$VaR_q^P = -\mu_P + \lambda_q \sigma_P \tag{2}$$

$$CoVaR_q^{P|M} = -\mu_P + \rho_{P,M}\frac{\sigma_P}{\sigma_M}\left(VaR_q^M + \mu_M\right) + \lambda_P\sqrt{\left(1-\rho_{P,M}^2\right)}\sigma_P \tag{3}$$

where
$VaR_q^M = -\mu_M + \lambda_q \sigma_M$
q: the threshold quantile (confidence level)
λ_P: the value of normal probability distribution at q %

μ_M: the mean return of market and portfolio

σ_M: the standard deviation of market and portfolio return

$\rho_{P,M}$: the correlation between the market return and the portfolio return

The derivation of Proposition 1 can be found in Appendix A. In this proposition VaR and CoVaR are expressed in terms of the loss occurred. If these values are positive, that means a loss has occurred. And if the sensitivities of these values are positive, that means loss increases with respect to the variable of interest. Analytical CoVaR of equation (3) should share the same advantages and characteristics of analytical VaR of equation (2), which is frequently used in risk management and portfolio construction.

Corollary 1 *The equation (3) can be rewritten in a form of Sharpe's one factor model*

$$R_q^{P|M} + r_f = -\beta_{P,M}\left(R_q^M - r_f\right) + \epsilon \tag{4}$$

where $\epsilon = \lambda_P \sqrt{\left(1 - \rho_{P,M}^2\right)}\sigma_P$

The proof of Corollary 1 is found in the Appendix B. Similar to Capital Asset Pricing Model (CAPM), this form of the equation expresses excess return of the equity with respect to the risk premium of the market return. Due to the fact that equation (4) is expressed in terms of loss, the equation takes the opposite signs when compared to Sharpe's one factor model. It can be rephrased as if it is expressed in terms of returns. This result highlights that univariate CoVaR is an explanatory model of conditional portfolio return with respect to the market return.

3.3 Analytical CoVaR Confidence Interval

Previous section focused on CoVaR as a point estimate of portfolio risk. For the purpose of practical usage, in addition to a point CoVaR estimate, a precision measure of CoVaR estimate is also important. Such a precision measure of CoVaR would capture the uncertainty of point CoVaR estimate and would therefore indicate the point estimate's reliability. The existing literature suggests capturing the uncertainty in VaR estimates in the form of VaR confidence intervals. Moraux (2009) suggest using analytical VaR confidence band based on normally distributed returns whereas Jorion (1996) suggests normal and Student t distributed returns. Dowd (2000) suggests investigating VaR distributions under normality using Monte Carlo Simulation and Hong et al. (2010) showed four different methods to capture VaR and its confidence interval for any distribution. Such approaches can also be applied to CoVaR. In this section, I derive the asymptotic distribution for analytical CoVaR under an iid assumption, which explicitly allows including conditional skewness or conditional kurtosis in CoVaR computation and is easily amenable to implementation. This approach should be seen as one that is complimentary to the many existing large-sample or simulation-based techniques.

Proposition 2 *q% CoVaR confidence interval has the width of*

$$\frac{2cv\sqrt{\omega^2}}{\sqrt{N}} \tag{5}$$

where

$\omega^2 = \sigma_{P|M}^2 \left(1 + G\sigma_{P|M} k_3^{P|M} + G^2 \sigma_{P|M}^2 \left(k_4^{P|M} - 1\right)\right)$
$G = \rho \frac{R_q^M - \mu_M}{2\sigma_P \sigma_M} + \frac{\lambda_q \sqrt{1-\rho^2}}{2\sigma_P}$
and cv is the relevant critical value.

The proof of the Proposition 2 is found in the Appendix C. Proposition 2 suggests that an analytical range estimate of CoVaR can be easily constructed as long as valid sample four conditional moments (mean, variance, skewness and kurtosis) of asset returns exist. The asymptotic normality assumed provides that the result will always be valid in large samples if central limit theorems apply. The equation (5) shows that the proposed analytical confidence CoVaR interval permits the inclusion of skewness or kurtosis and is easily amenable to implementation. Such asymptotic approach, based on higher moments but not knowing the true distribution should lead to improvements in the accuracy of a risk measure relative to the usual normal distribution-based methodology, as claimed in Hong et al. (2010).

4 Empirical Backtesting

In order to validate the proposed analytical CoVaR, empirical evaluation of the risk measure is essential. To backtest analytical CoVaR, I select 20 random stocks from S&P500 and generate 1/n portfolio with uniform weight of 5%. The sample period, from July 24, 2007 to September 23, 2010 with 800 daily return objectives, was selected. The validity of the null hypothesis for VaR and CoVaR is tested via Kupiec's Failure Frequency Test and Christoffersen's Conditional Testing Backtest. Note that this is not in-sample result where risk measure is computed based on sample data then the same sample period is tested. At time t, it is assumed that only the information until time t-1 is available, hence analytical VaR and CoVaR are computed from the historical data up to t-1, then it tests whether those measures are breached by portfolio return at t.

4.1 Kupiec's Failure Frequency Test

The failure frequency (FF) test developed by Kupiec (1995) being the most well known and the simplest backtest method, was used. This test is concerned with whether or not the reported risk measure (VaR and CoVaR) is violated more (or less) than p of the sample. If it is violated more than p, then the accuracy of the underlying risk model is called into question. The hypothesis to be tested would be

H_0 :Number of Historical Breaches is not significantly different from 5%×N
H_1 :Otherwise

Using a sample of N observations, FF test phrases basic frequency in likelihood ratio (LR) form,

$$LR_{FF} = 2\ln\left[\left(\frac{1-x/N}{1-p}\right)^{N-x}\left(\frac{x/N}{p}\right)\right] \tag{6}$$

where x is the number of failures (breaches) in the sample N is the sample size and p is the probability of a failure under the null hypothesis. Under the null hypothesis, test statistic has a (Chi-Square distribution with one degree of freedom). If the proportion of violations, , is equal to then the takes the value zero, indicating no evidence of any inadequacy in the underlying risk measure. As the frequency of violations, x, differs from p, the grows indicating mounting evidence that the proposed risk measure either systematically understates or overstates the portfolio's underlying level of risk. In short, it boils down to a test of whether the empirical frequency is sufficiently close to the predicted frequency p. If this likelihood ratio is larger than the chi-square critical value, then it indicates that the null hypothesis is acceptable and that provide risk measure is acceptable / valid, vice and versa.

Analytical VaR and CoVaR are computed with 100 day lookback period to count historical exceedances which x is derived. Then are calculated to test the null hypothesis. Table 1 presents the first 50 days of computation.

(Insert Table 2)

Note that VaR and CoVaR figures are converted to return measures (negative in general) to help understanding while previous analysis expressed VaR and CoVaR as loss (positive in general). As we can see in Table 2, for 1/n portfolio, both analytical VaR and analytical CoVaR pass the FF test in historical backtesting in the sample period therefore, for the sample period CoVaR appears to be a valid risk measure.

(Insert Table 3)

Basic frequency test, such as the one that is presented in this section, is based on a simple intuition, is easy to apply and does not require a great deal of information. However, there are well known disadvantages. As Dowd (2005) points out, it focuses exclusively on the frequency of exceedances over the sample period hence throw away information about the temporal pattern of exceedances, which could be important because many risk models predict that exceedances should be independently identically distributed. Kupiec (1995) also reports increasing sample period from one year to two years increases the chance of detecting the systematic under reporting of VaR from 65% to roughly 90% but at the cost of reducing the frequency with which the adequacy of VaR measure can be assessed. In this respect, the sample period used in this section, which is little more than 3 years, may be considered long enough although it could be too long. A formal stress test on the sample period would be an interesting topic for further research.

4.2 Christoffersen's Conditional Testing Backtest

To overcome the problems of FF test, Christoffersen (1998) suggests conditional testing Backtest (CT) which first separates out the particular predictions being tested and then test each prediction. The first step is same as the FF test, which generates the 'correct' frequency of exceedances. This is the prediction of correct unconditional coverage. The second step is to test to determine whether sample exceedances are independent of each other. This is important insofar as it suggests that exceedances should not be clustered over time. Evidence of exceedances clustering would suggest that the model is misspecified, even if the model passes the prediction of correct unconditional coverage. Therefore the hypothesis to be tested for unconditional coverage test is identical to that of Kupiec's FF test while the hypothesis for the independence would be

H_0 :The historical breaches are independent to each other
H_1 :Otherwise

Since Kupiec's FF test is the unconditional coverage test, LR_{FF} is rephrased as LR_{UC} in this section. Turning to the independence prediction, let n_{ij} be the number of days that state j occurred after state i occurred the previous day, where the state 0 refers to non-exceedances and 1 refers to exceedances, and let π_{ij} be the probability of state j in any given day, given that the previous day's state was i. Under the hypothesis of independence, the test statistic is

$$LR_{ind} = 2\ln\left[\frac{(1-\widehat{\pi}_{01})^{n_{00}}\,\widehat{\pi}_{01}^{n_{01}}\,(1-\widehat{\pi}_{11})\,\widehat{\pi}_{01}^{n_{11}}}{(1-\widehat{\pi}_{2})^{n_{00}+n_{11}}\,\widehat{\pi}_{2}^{n_{01}+n_{11}}}\right] \tag{7}$$

where the estimates of the probability are defined as

$\widehat{\pi}_{01} = \frac{n_{01}}{n_{00}+n_{01}}$, $\widehat{\pi}_{11} = \frac{n_{11}}{n_{10}+n_{11}}$ and $\widehat{\pi}_{2} = \frac{n_{01}+n_{11}}{n_{00}+n_{01}+n_{10}+n_{11}}$

LR_{ind} is also distributed as a $\chi^2(1)$. Table 3 presents number of days, n_{ij} and the estimates of the probability, $\widehat{\pi}_{ij}$

(Insert Table 4)

It follows that under the combined hypothesis of correct coverage and independence, the hypothesis of correct conditional coverage, the test statistic is:

$$LR_{CC} = LR_{UC} + LR_{ind} \tag{8}$$

which is distributed as $\chi^2(2)$. As Dowd (2005) points out, the Christoffersen approach enables us to test both coverage and independence hypotheses at the same time. Table 4 presents likelihood ratios and the test result.

(Insert Table 5)

With the current sample, LR_{ind} cannot be computed with CoVaR exceedances because state 11, where two exceedances occurred consecutively, never happened. Hence $n_{11} = 0$, making $\widehat{\pi}_{11} = 0$ and therefore as result the $\mathrm{LR}_{ind} = 2\ln(0)$, which is not defined. This provides strong evidence that for the current sample the CoVaR exceedances are independent to each other. With the current sample, analytical VaR performs very poorly in independent test although it passes the unconditional test. This might be because the sample period includes the period of recent financial crisis. This can be interpreted as, during the sample period, (1) the unconditional expected VaR exceedances was not statistically higher than what we intended but (2) VaR exceedances were serially correlated. This result is because VaR does not pick up the increased correlation between the market and portfolio return. Hence when the portfolio loss exceeds VaR today, breach is likely to happen again tomorrow and this might be due to autocorrelation in portfolio return series.

5 Models of Price Processes

In the previous section, analytical CoVaR is empirically validated via backtest. This section intends to provide a theoretical frameworks within which to investigate and validate analytical CoVaR. The objective of this section is to investigate the sensitivities of VaR and CoVaR with respect to the underlying correlation between market and portfolio returns. In order to do this, I first present a model that assumes portfolio and market values follow bivariate log Wiener process. From this, I expect to see a straight forward result of why VaR failed and how CoVaR captures systemic risk in the simplest possible construction. Then I present the second case where the log price of a portfolio and the market follow OU Process. In the previous we saw VaR breaches are serially correlated and this is because there is autocorrelation in portfolio returns. OU process introduces autocorrelation in returns hence the key message that I wish to deliver by employing OU process is the impact of autocorrelation in the sensitivity of those measures with respect to the underlying correlation between portfolio and market returns. I do not believe that either process is the true process governing all returns. However, such what-if model helps to gain greater insights in understanding how analytical CoVaR captures the systemic risk.

5.1 If Prices follow Geometric Brownian Motion

This section investigates analytical VaR and CoVaR when returns are iid. Assume that the prices follows a bivariate geometric Brownian motion, a form that is very popular and well known to model stock price behavior used in many of previous works such as Black and Scholes (1973) and Hull (2007).

$$dP(t) = \mu P(t)dt + \sigma P(t)W(t) \tag{9}$$

where $P(t) = \begin{pmatrix} P_P(t) \\ P_M(t) \end{pmatrix}$, $\mu(t) = \begin{pmatrix} \mu_P(t) \\ \mu_M(t) \end{pmatrix}$, $\sigma(t) = \begin{pmatrix} \sigma_P(t) \\ \sigma_M(t) \end{pmatrix}$,
$W(t) = \begin{pmatrix} W_P(t) \\ W_M(t) \end{pmatrix}$, and $E\left[dW_M(t)dW_P(t)\right] = \kappa dt$

P stands for price, subscript P indicates that the parameter is for a portfolio, subscript M indicates that the parameter is for the market. μ_M and σ_M indicate the mean and the variance of underlying process q_M. W_t denotes a Wiener process. $W_M(t)$ and $W_P(t)$ are correlated Wiener processes with correlation coefficient κ, i.e. . Therefore κ can be seen as the underlying systemic risk of a portfolio with respect to the market. For most financial assets, we expect κ to be positive. The system of equation (9) can be easily solved using Ito's Lemma. Assuming the price process starts at s,

$$\begin{array}{c} P_P(t) = P_P(s) \exp\left(\left(\mu_P - \frac{1}{2}\sigma_P^2\right)(t-s) + \sigma\left(W_P(t) - W_P(s)\right)\right) \\ P_M(t) = P_M(s) \exp\left(\left(\mu_M - \frac{1}{2}\sigma_M^2\right)(t-s) + \sigma\left(W_M(t) - W_M(s)\right)\right) \end{array} \tag{10}$$

Remark 1 *The solution to the system of equations (9)*

$$R(t+1,t) = \mu - \frac{1}{2}\sigma^2 + \overline{H}\left(W(t+1) - W(t)\right) \tag{11}$$

where

$$r_A(t+1,t) := \log P_P(t+1) - \log P_P(t)$$

$$H = \begin{pmatrix} \sigma_P\sqrt{1-\kappa^2} & \kappa\sigma_P \\ 0 & \sigma_M \end{pmatrix} \overline{H} = \begin{pmatrix} \sigma_P & 0 \\ 0 & \sigma_M \end{pmatrix} HH' = \begin{pmatrix} \sigma_P^2 & \kappa\sigma_P\sigma_M \\ \kappa\sigma_P\sigma_M & \sigma_M^2 \end{pmatrix}$$

The proof of Remark 1 is found in Appendix D. Note that $E\left[W_M(t)\,W_P(t)\right] = E\left[W_M(t+1)\,W_P(t+1)\right] = \kappa$ and $E\left[W_i(t+1)\,W_j(t)\right] = 0$, $i = M, P$. i = M, P and j = M, P. The mean, variance and covariance of $r_M(t+1,t)$ and $r_P(t+1,t)$ is summarized as below. Again note that, as previously mentioned, these are conditional on time 0 information.

$$\begin{array}{ll} E\left(r_M(t+1,t)\right) = \mu_{rM} = \mu_M - \frac{1}{2}\sigma_M^2, & Var\left(r_M(t+1,t)\right) = \sigma_{rM}^2 = \sigma_M^2 \\ E\left(r_P(t+1,t)\right) = \mu_{rP} = \mu_P - \frac{1}{2}\sigma_P^2, & Var\left(r_M(t+1,t)\right) = \sigma_{rP}^2 = \sigma_P^2 \\ Cov\left[r_P(t+1,t), r_M(t+1,t)\right] = \sigma_{r\,\mathrm{Pr}\,M} = \sigma_P\sigma_M\kappa & \end{array}$$

As expected, log returns $R(t+1,t) := \log P(t+1) - \log P(t)$ are normally distributed. Therefore the correlation between market return and the portfolio return can be written as

$$Correl\left[r_P(t+1,t), r_M(t+1,t)\right] \neq \rho_{r\,\mathrm{Pr}\,M} = \frac{\sigma_{r\,\mathrm{Pr}\,M}}{\sigma_{rP}\sigma_{rM}} = \kappa$$

And the beta can be expressed as

$$\beta_{P,M} = \frac{\sigma_P}{\sigma_M}\kappa \tag{12}$$

Remark 2 *Under the current framework, portfolio analytical VaR and CoVaR can be expressed as*

$$VaR_q^P = -\mu_P + \lambda_q \sigma_P \tag{13}$$

$$CoVaR_q^{P|M} = -\mu_P + \lambda_q \sigma_P \kappa + \lambda_q \sqrt{1-\kappa^2}\sigma_P \tag{14}$$

Proof of Remark 2 is omitted. Remark 2 confirms that under no return autocorrelation, portfolio value at risk is independent to the cross correlation between market and asset return. This is the precise reason why CoVaR is introduced.

Proposition 3 *The sensitivity of portfolio VaR and CoVaR with respect to* κ*, under independent return series and assuming equations (9) to (14), can be expressed as*

$$\frac{\partial VaR_q^P(t)}{\partial \kappa} = 0 \tag{15}$$

$$\frac{\partial CoVaR_q^P(t)}{\partial \kappa} = \lambda_q \sigma_P \left(1 - \frac{\kappa}{\sqrt{1-\kappa^2}}\right) \tag{16}$$

The proof of Proposition 3 is omitted. Under no autocorrelation in return, VaR is invariant to κ. The sensitivity of CoVaR with respect to the underlying correlation κ is determined by the interaction between κ and $\sqrt{1-\kappa^2}$ which are coming from $\sqrt{1-\rho_{P,M}^2}$term in equation (3). Although the assumption that the prices follow a bivariate geometric Brownian motion is unrealistic, the advantage of such assumption is the simplicity and it provides straight forward implications. Equation (15) and (16) show that that the sensitivity of $\sqrt{1-\rho_{P,M}^2}$ term with respect to the underlying correlation determines the effectiveness of CoVaR. This result will be reinforced in next section.

5.2 Prices follow Log OU Process

I present a variation of Hong and Satchell (2011), which assumes asset and market returns follow bivariate log Ornstein-Uhlenbeck (OU) process. The model allows asset and market price processes to be correlated. This correlation may correspond to common factors due to the stock's dependence on factors such as size or value and the markets co-variation with these factors. Assume that the logarithm of the asset prices $\log P_A(t)$ and $\log P_M(t)$ have linear trends $\mu_A t$ and $\mu_M t$ respectively. We consider the process

$$q_M(t) := \log P_M(t) - \mu_M t \qquad q_A(t) := \log P_A(t) - \mu_A t$$

which we call the de-trended log price process to emphasize that the vector $Q(t)$ contains no deterministic trend component. Assume that $Q(t)$ satisfies the following pair of stochastic differential equations,

$$dq_P(t) = (-\theta_{P1} q_P(t) + \theta_{P2} q_M(t))\, dt + \sigma_P dW_P(t) \tag{17}$$

$$dq_M(t) = -\theta_M q_M(t) dt + \sigma_M dW_M(t) \tag{18}$$

where the same notation for parameters are used as the previous section. θ and σ are the parameters of the OU process with the obvious use of subscripts. There is no reason to believe that either a portfolio or the market return converges faster than the other. Hence there should be no restrictions on the value of θ_{P1}, θ_{P2} and θ_M. Equation (17) allows portfolio value to error-correct around the market price; this is one particular parameterization: alternative ones could be considered as well. Equations (17) and (18) do not presume stationarity or trending in either a portfolio or the market. Equation (18) allows for a continuous-time error correction mechanism about 0, if theta is positive. This model has been successfully used by Lo and Wang (1994, 1995) to analyze option prices when the data are auto correlated.

Remark 3 *The solution to the system of equations (17) and (18) is given by*

$$
\begin{aligned}
R(t+h,t) &= \begin{pmatrix} \mu_A \\ \mu_M \end{pmatrix} h \\
&+ \begin{pmatrix} e^{-\theta_M h}-1 & \frac{\theta_{P2}}{\theta_{P1}-\theta_M}\left(e^{-\theta_M h}-e^{-\theta_{P1} h}\right) \\ 0 & e^{-\theta_M h}-1 \end{pmatrix} Q(t) \\
&+ \int_t^{t+h} \begin{pmatrix} e^{-\theta_{P1}(t-u)} & \frac{\theta_{P2}}{\theta_{P1}-\theta_M}\left(e^{-\theta_M (t-u)}-e^{-\theta_{P1}(t-u)}\right) \\ 0 & e^{-\theta_M (t-u)} \end{pmatrix} \overline{H} dW(u)
\end{aligned}
\tag{19}
$$

A proof of the Remark 3 is found in Appendix E. Note that this is not the only possible representation of this process. The same process and therefore the same solution can be found by taking the difference between 0 and t+1 of the underlying Q (rather than return) An exact discrete representation of $Q(t)$ and hence $R(t+1,t)$ has also been computed. Alternative representations of the log normal OU process are analysed in the Appendix F.

The properties of the market and portfolio return series that $Q(t)$ generates can be derived from the equation (19). The mean, variance and covariance of $r_M(t+1,t)$ and $r_P(t+1,t)$, based on Remark and the appendices are summarized below. These are all calculated conditional on time 0 information.

$$
E\left(r_M\left(t+h,t\right)\right) = \mu_{rM} + q_M(0)e^{-\theta_M t}\left(e^{-\theta_M}-1\right)
$$

$$
Var\left(r_M(t+1,t)\right) = \sigma^2_{rM} = \left(\frac{e^{2\theta_M}-1}{2\theta_M}\right)\sigma^2_M
$$

$$
\begin{aligned}
E\left(r_P\left(t+h,t\right)\right) &= \mu_{rP} + q_P(0)e^{-\theta_{P1} t}\left(e^{-\theta_{P1}}-1\right) \\
&+\frac{\theta_{P2}}{\theta_{P1}-\theta_M} q_M(0)\left(e^{-\theta_M t}e^{-\theta_M(t+1)} - e^{-\theta_{P1} t}e^{-\theta_{P1}(t+1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
Var\left(r_P(t+1,t)\right) = \sigma^2_{rP} &= \left(\frac{\sigma_M\theta_{P2}}{\theta_{P1}-\theta_M}\right)^2\left(\frac{e^{2\theta_M}-1}{2\theta_M}+\frac{e^{2\theta_{P1}}-1}{2\theta_{P1}}-\frac{e^{(\theta_{P1}+\theta_M)}-1}{\theta_M+\theta_{P1}}\right) \\
&+\frac{e^{2\theta_{P1}}-1}{2\theta_{P1}}\sigma^2_M + \left(\frac{2\theta_{p2}}{\theta_{P1}-\theta_M}\right)\left(\frac{e^{(\theta_{P1}+\theta_M)}-1}{\theta_{P1}+\theta_M}-\frac{e^{2\theta_{P1}}-1}{2\theta_{P1}}\right)\sigma_M\sigma_P\kappa
\end{aligned}
$$

In this model, both market and portfolio return processes are subject to autocorrelation, i.e. $\rho_{rMMt} \neq 0$ and $\rho_{rPPt} \neq 0$. The proof of this statement is omitted (see page 18, Lo and Wang 1994). This is desirable because returns measured over shorter periods may suffer from autocorrelation. The covariance of portfolio return can be written as :

$$Cov\left[r_P\left(t+1,t\right),r_M\left(t+1,t\right)\right]=\sigma^2_{r\,\Pr M}$$
$$=\left(\frac{e^{2\theta_M}-1}{2\theta_M}-\frac{e^{(\theta_{P1}+\theta_M)}-1}{\theta_M+\theta_{P1}}\right)\frac{\theta_{P2}}{\theta_{P1}-\theta_M}\sigma^2_M+\frac{e^{(\theta_{P1}+\theta_M)}-1}{\theta_M+\theta_{P1}}\sigma_M\sigma_P\kappa \tag{20}$$

Therefore the correlation between market and a portfolio return can be written as:

$$\rho_{r\,\Pr M}=\frac{\sigma_{r\,\Pr M}}{\sigma_{rP}\sigma_{rM}} \tag{21}$$

Equation (21) also can be written in terms of the underlying parameters, $\sigma_P,\sigma_M,\theta_{P1},\theta_{P2},\theta_M,\kappa$ etc but it would not be necessary to include the entire term for our purpose, let alone it is too long and complicated. For all values of θ_{P1} and θ_M such that $\theta_{P1}\neq\theta_M$, assuming the one period population beta under non-zero autocorrelation in return series assuming equations (17) to (21) and can be expressed as:

$$\beta_{P,M}=\left(1-\frac{e^{(\theta_{P1}+\theta_M)}-1}{\theta_M+\theta_{P1}}\cdot\frac{2\theta_{P2}}{\theta_{P1}+\theta_M}\right)\frac{\theta_{P2}}{\theta_{P1}-\theta_M}+\frac{e^{(\theta_{P1}+\theta_M)}-1}{\theta_M+\theta_{P1}}\cdot\frac{2\theta_M}{\theta_{P1}+\theta_M}\beta_G \tag{22}$$

where

$$\beta_G=\frac{\sigma_P}{\sigma_M}\kappa \tag{23}$$

The proof of the Equation (22) is found in Appendix G. β_G is the portfolio beta when prices follow geometric Brownian motion. The sensitivity of a portfolio beta with respect to can be expressed as:

$$\frac{\partial\beta_{P,M}}{\partial\kappa}=\frac{1-e^{(\theta_{P1}+\theta_M)}}{1-e^{2\theta_M}}\cdot\frac{2\theta_M}{\theta_{P1}+\theta_M} \tag{24}$$

It is computationally more efficient to express analytical CoVaR with beta than correlation in this model, since we can avoid dealing with square roots of parameters in the denominator.

Remark 4 *Under the current framework and substituting the correlation with beta, portfolio analytical VaR and CoVaR can be expressed as*

$$VaR_q^P = -\mu_{rP} + \lambda_q \sigma_{rP} \tag{25}$$

$$CoVaR_q^{P|M} = -\mu_{rP} + \beta_{P,M}\left(VaR_q^P + \mu_{rM}\right) + \lambda_q \sigma_{rP}\sqrt{1 - \beta_{P,M}^2 \frac{\sigma_{rP}^2}{\sigma_{rP}^2}} \tag{26}$$

The proof of Remark 4 is omitted. It is computationally less burden to compute analytical CoVaR with portfolio beta than with correlation between portfolio and market returns. Note that μ_{rP}, σ_{rP} and $\beta_{P,M}$ are the functions of κ and therefore VaR is sensitive to the underlying log price correlation via portfolio volatility.

Proposition 4 *The sensitivity of portfolio VaR and CoVaR with respect to κ under non-zero autocorrelation in return series and assuming equations (17) to (26), can be expressed as*

$$\frac{\partial VaR_q^P(t)}{\partial \kappa} = \frac{\lambda_q \sigma_M \sigma_P}{2\sigma_{rP}(\kappa)}\left(\frac{2\theta_{P2}}{\theta_{P1} - \theta_M}\right)\left(\frac{e^{(\theta_M + \theta_{P1})} - 1}{\theta_M + \theta_{P1}} - \frac{e^{(2\theta_{P1})} - 1}{2\theta_{P1}}\right) \tag{27}$$

$$\begin{aligned} \frac{\partial CoVaR_q^{P|M}(t)}{\partial \kappa} &= \frac{1 - e^{(\theta_M + \theta_{P1})}}{1 - e^{2\theta_{P1}}} \cdot \frac{2\theta_M}{\theta_M + \theta_{P1}} \cdot \sqrt{\frac{e^{2\theta_M} - 1}{2\theta_M}} \lambda_q \sigma_M \\ &+ \frac{\lambda_q}{\sqrt{B}} \times \left(\frac{\partial \sigma_{rP}}{\partial \kappa} - \beta_{P,M} \frac{\partial \beta_{P,M}}{\partial \kappa} \frac{\sigma_{rM}^2}{\sigma_{rP}^2}\right) \end{aligned} \tag{28}$$

where $B = 1 - \beta_{P,M}^2 \frac{\sigma_{rM}^2}{\sigma_{rP}^2}$

The proof of Proposition 4 is found in Appendix H. Equation (27) explains why VaR fails when the financial market is under distress. The only source of κ sensitivity in analytical VaR is the portfolio variance, which is inversely related with the sensitivity. Such inverse relationship makes portfolio VaR less sensitive to the underlying correlation when the correlation suddenly jumps to a very high level. Note that $\frac{\lambda_q}{\sqrt{B}} > 0$ and $\frac{\partial \beta_{P,M}}{\partial \kappa} > 0$ hence the sign of equation (28) depends on the magnitude of θ_M, θ_{P1}, θ_{P2}, and κ. As shown in equation (27), $\frac{\partial \sigma_{rP}}{\partial \kappa}$ is inversely related to κ, $\beta_{P,M}$ is positively related to κ and $\frac{1}{\sigma^2_{rP}}$ is inversely related to κ. Therefore increases in κ has an opposite effect to $\beta_{P,M}$ and $\frac{1}{\sigma^2_{rP}}$. Although it would not be meaningful to analyze the precise magnitude of the tradeoff due to the complexity of equation (28), there is a clear indication that the effectiveness of analytical CoVaR in capturing systemic risk comes from the tradeoff between the sensitivity of portfolio and market return correlation and the sensitivity of the portfolio return volatility with respect to the systemic risk. As algebraically shown in Proposition 3 and Proposition 4, VaR of a portfolio can always be reduced by taking less risk, however it does not necessarily reduce CoVaR. In order to reduce CoVaR, managers need to either decrease assets that are highly correlated with the market or increase the assets that are not correlated with the market. Therefore theoretically, replacing assets that are highly correlated to the market with less correlated ones would reduce CoVaR while maintaining the same VaR.

5.3 Parameterization of Log OU Process

As previously mentioned, Proposition 4 is complicated that it would be helpful to show equation (27) and (28) visually. There are two reasons for parameterization, (i) to perform a sensitivity analysis of analytical CoVaR with respect to κ, (ii) to locate sensible value of parameters. Historical correlation data cannot be altered hence we are not able to carry out what-if type sensitivity analysis with respect to κ. The sample period is from July 23, 2009 to April 6, 2011 and I use Goldman Sachs data then the parameters of bivariate log OU process are estimated. The only reason for parameterization is to locate a sensible set of OU parameter figures hence could be any data and GS and the sample period are chosen for no particular reason. Then the parameters are chosen based on the estimated figures to show equation (27) and (28) numerically. The OU model in this section has certain peculiarities; whilst the de-trended log price equation is simple, the returns equation is complex. It can be shown that the matrix return equation is VARMA(1,1) in returns (See Appendix F). Substituting out the Q(t) term yields a moving average representation in terms of the original error term. This leads to complexity in estimation. Regressing on Q(t), we get similar structure to our de-trended log price equations., so we shall estimate the model in de-trended log prices.

θ_M, θ_{P1}, θ_{P2}, σ_M, σ_P and κ are estimated using maximum likelihood, which is equivalent to generalized least squares (GLS) with one iteration (see appendix I for the equivalence of MLE and EGLS). Iteratively reweighted least squares (IRLS) is used when heteroskedasticity or correlations, or both are present among the error terms of the model, but where little is known about the covariance structure of the errors independently of the data. In the first iteration, OLS or GLS with a provisional covariance structure is carried out and the residuals are obtained from the fit. Based on the residuals, an improved estimate of the covariance structure of the errors can usually be obtained. A subsequent GLS iteration is then performed using this estimate of the error structure to define the weights. The process can be iterated to convergence, but in many cases, as Carroll (1982) and Cohen et al. (1993) have shown that iterating once is sufficient to achieve an efficient estimate (see appendix J for the derivation of the estimation method).

(Insert Figure 1)

Figure 1 confirms the argument made in the previous section that analytical VaR is insensitive to underlying correlation while analytical CoVaR is sensitive to it.

6 Discussion, Extension and Further Research

6.1 Use of Publicly Available Return Data

Presented methodology is based on publicly available return data of a financial institution. This is supported by many others including Acharya et al. (2010) and Brownlees and Engle (2010). It is advantageous in that it allows CoVaR to be applied to a firm specific risk model rather than a more general macroeconomic systemic risk model. However this assumes that the investors value the company's risk properly, which does not always the case. Hence the methodology is useful as a simple screening method of firm's systemic risk management to raise a red flag. Analytical CoVaR can be applied to different books and portfolios within a firm to provoke a further detailed investigation. This point corresponds to the main objective of introducing analytical CoVaR as previously stated. Analytical CoVaR is useful in that it serves as a computationally inexpensive systemic risk screening measure that would provoke detailed investigations of risk exposures when an unusual systemic risk is identified.

6.2 Accuracy of Analytical Form

It is clearly indicated that the most important advantage of analytical CoVaR is its simplicity. Analytical CoVaR is a computationally inexpensive method of screening to grab overall systemic risk of a portfolio at the expense of some accuracy. Given that the analytical method is precise but not accurate, one may question how inaccurate the analytical measure. The best place to start the further investigation would be previous literatures assessing analytical VaR accuracy. There are plenty of academic works showing that analytical VaR is reasonably accurate. Garbade (1986), J.P. Morgan (1994), and Hsieh (1993) argue that changes in the portfolio function can be well-approximated by the delta of the portfolio. Chapter 9.5 of Jorion (2001) empirically show the accuracy of delta normal VaR and conclude that it gives a coverage that is very close to the ideal number although it seems to underestimate VaR slightly. Table $9-3$ of Jorion (2001) summarizes the tradeoff between accuracy and the computation speed to illustrate the usefulness of delta normal VaR.

Any investigation of VaR depends on the effectiveness of linear (delta method) approximation of the unknown independent return distribution while assessing the accuracy of analytical CoVaR involves unknown conditional return distribution hence a different approach would be needed in exploring the accuracy of analytical CoVaR. Some empirical evidences are already presented in Chapter 4 of this paper where the empirical backtest showed that analytical CoVaR accurately capture the probability of historical breaches. However the problems of the two methods are well known and the further theoretical investigation would be needed. I ponder the test on the accuracy of the analytical form would become test of the validity of the normality assumption made when deriving the analytical form. I leave this to a later paper where I plan to present a simple model to investigate the accuracy of analytical CoVaR. The paper would start with the assumption that a particular stock and S&P500 history are correlated bivariate Weibull distributed just to locate some parameter values which we can take to be the true ones. We know the exact values of moments and the

VaR of the Weibull distribution, so simulating random Weibull distributions based on the original financial data would be the starting point. Weibull distribution is selected as an example due to its popularity and the non-normality. The correlated bivariate Weibull distribution would allow creating conditional distribution of return where we can assess the accuracy the risk measure.

6.3 Extension: Loss Beyond CoVaR, Analytical CoES

Loss beyond VaR can be captured by ES, which is an average expected loss given that the loss exceeded VaR. CoES to CoVaR is the analogous of ES to VaR. The advantage of CoES relative to CoVaR is that it provides less incentive to load on to tail risk below the percentile that defines CoVaR. CoES is the expected portfolio return given that is lower than CoVaR hence is defined as

$$CoES_q^{P|M} = E\left(R^P \mid R^P \leq CoVaR_q^{P|M}\right) \tag{29}$$

Note that this definition of CoES is for individual portfolio, not for an aggregated system hence is different from that of A&B (2010). There are two conditions applied to CoES, (1) given that a portfolio return is lower than CoVaR, (2) that the market return is at q% VaR (the market is under distress). The first condition comes from ES and the second comes from CoVaR. Therefore CoES can be interpreted as the expected return (or loss) of a portfolio when portfolio return is already lower than CoVaR given market is under distress. Although in different expressions, the CoES is equivalent to Systemic Expected Shortfall (SES) of Acharya et al. (2010), which measures the contribution of the individual bank's contribution to systemic risk. It is a straight forward to extend analytical CoVaR to analytical CoES using the conditional moments of portfolio under normality.

Proposition 5 *The analytical form of CoES can be expressed as*

$$CoES_q^{P|M} = -\mu_P - \rho\frac{\sigma_P}{\sigma_M}\left(VaR_q^M + \mu_M\right) + \frac{\phi\left(\lambda_q\right)}{1-q}\sqrt{1-\rho_{P,M}^2}\sigma_P \tag{30}$$

where $\phi(\cdot)$ is the value of the standard normal density function and the other notations are the same as before. The derivation of Proposition 5 is in Appendix K. Analytical CoES introduces a simple and straight forward way to aggregate and disaggregate individual systemic risk contribution of each bank.

7 Conclusion

In this paper, analytical form of a point and a range approximation of CoVaR are proposed. Analytical CoVaR is advantageous in that it saves much computing time and effort. It can act as a screening risk measure that raises red flags when it needs more in-depth investigation on the risk of a portfolio. Then empirical backtests using Kupiec's failure frequency test and Christoffersen's conditional backtest are performed to find that the historical in-sample analytical CoVaR exceedances pass both tests while VaR exceedances pass only Kupiec's frequency failure test. This result can be interpreted as, during the sample period, (1) the unconditional expected VaR exceedances was not statistically higher than what we intended but (2) VaR exceedances were serially correlated. I have advocated the use of the bivariate cross correlated geometric Brownian motion and log OU model to theoretically illustrate how analytical CoVaR captures the systemic risk. From the comparison of the Brownian motion model and the bivariate log OU model, it is shown that under no autocorrelation in returns, analytical VaR is not sensitive to the underlying correlation while when autocorrelation is present, it becomes less sensitive as the level of the underlying price correlation between the market and a portfolio becomes high. In both cases, analytical VaR fails to capture the increase in systematic risk if the correlation between the market and a portfolio return suddenly jumps to a very high level, which occurs when the financial market is under distress. It is identified that the interaction between the correlation and the volatility of the portfolio return, which both are sensitive to the underlying systemic risk, governs the effectiveness of CoVaR in capturing the systemic risk. The result conforms to existing literatures and also is intuitive. A discussion regarding different aspects of analytical CoVaR and its potential application is presented and a theoretical model of assessing the accuracy of analytical CoVaR is suggested as a further research topic. Analytical CoES, which could capture individual institution's contribution to systemic risk, is presented as a straight forward extension that could overcome a criticism on CoVaR that it overlooks the tail return distribution beyond CoVaR.

References

[1] Acharya,V., Pedersen, L., Philippon, T. and Richardson, M. P. (2010): “Measuring Systemic Risk”, AFA 2011 Denver Meetings Paper

[2] Adrian, T. and Brunnermeier, M. K. (2009): “CoVaR”, FRB of New York Staff Report No. 348

[3] Bank for International Settlements (1994): 64th Annual Report. Basel, Switzerland: BIS

[4] Brownlees, C. T. and Engle, R. F. (2011): “Volatility, Correlation and Tails for Systemic Risk Measurement”

[5] Carroll, Raymond J. (1982). "Adapting for Heteroscedasticity in Linear Models". The Annals of Statistics, vol. 10(4), 1224–1233

[6] Christoffersen, P. (1998): "Evaluating Interval Forecasts," International Economic Review, vol. 39(4), 841-62

[7] Cohen, Michael; Dalal, Siddhartha R.; Tukey,John W. (1993). "Robust, Smoothly Heterogeneous Variance Regression". Journal of the Royal Statistical Society. Series C (Applied Statistics), vol. 42 (2), 339–353

[8] Dowd K. (2000): “Assessing VaR accuracy”, Derivatives Quarterly, vol. 6(3), 61-63

[9] Dowd, K. (2005): “Measuring market risk”, 2nd ed., Chichester and New York, John Wiley & Sons

[10] Eremin, E.V. (1999): “A Central Limit Theorem from Linear Estimators of Measurement Results”, Measurement Techniques, vol. 42(II), 1009 - 1016

[11] Garbade, Ken, ,1986, “Assessing Risk and Capital Adequacy for Treasury Securities, ” Topics in Money and Securities Markets, Bankers Trust

[12] Hsieh, D., 1993, “Implications of Nonlinear Dynamics for Financial Risk Management”, Journal of Financial and Quantitative Analysis, vol. 28(1), 41-64

[13] Hong, J.K., Knight, J., Satchell, S. and Scherer, B. (2010): “Using Approximate Results for Validating VaR”, Journal of Risk Model Validation, vol. 4(3), 69 – 81

[14] Hong, J.K. and Satchell, S (2011): "The Sensitivity of Systematic Risk to the Return Measurement Interval when Log Prices follow an Ornstein-Uhlenbeck Process", Working Paper

[15] Hull, J. (2006): "Options, futures, and other derivatives", Upper Saddle River, N.J: Pearson/Prentice Hall.

[16] Jorion, P. (1996): "Risk2: Measuring the Risk in Value at Risk". Financial Analyst Journal, Dec., 47-56

[17] Jorion P. (2001): "Value at Risk: the New Benchmark for Managing Financial Risk", 2nd ed. Mc Graw Hill

[18] J.P. Morgan, 1994, "RiskMetrics", Second Edition, J.P. Morgan

[19] Kaufman G. and Scott K. E. (2003): "What Is Systemic Risk, and Do Bank Regulators Retard or Contribute to It?", The Independent Review, vol. VII(3), 371– 391

[20] Kupiec, P. (1995): "Techniques for Verifying the Accuracy of Risk Measurement Models", Journal of Derivatives, vol. 3(2), 73 − 84

[21] Lo A. W. and Wang J. (1994): "Implementing option pricing models when asset returns are predictable", National Bureau of Economic Research, Working Paper No. 4720

[22] Lo A. W. and Wang J. (1995): "Implementing Option Pricing Models When Asset Returns Are Predictable", Journal of Finance, American Finance Association, vol. 50(1), 87-129

[23] Moraux F. (2009): "Large Sample Confidence Intervals for Normal VaR", Journal of Risk Management in Financial Institutions, vol. 4(2), 189-200

[24] Ryu, K.(1993): "An extension of Marshall and Olkin multivariate exponential distribution", Journal of American Statistical Association, vol. 88, 1458–1465

[25] Schwarcz S. L. (2008): "Systemic Risk", Duke Law School Legal Studies Research Paper Series, Research Paper No. 163

8. Appendix

A. Proof of Proposition 1

Define $f(R^P)$: pdf of R^P, $F(R^P)$: cdf of R^P, $F_P^{-1}(\hat{q})$: the inverse of $F(R^P)$ at $\hat{q}$
q: the threshold quantile (confidence level),
$R^P(q)$: the q order quantile function of cdf

Then we know that $\Pr(R^P \le R^P(q)) = q$.

$VaR_{\hat{q}}^P$, VaR with $1-\hat{q}$ confidence level, is defined as $\Pr(R^P \le VaR_{\hat{q}}^P) = \hat{q}$ and interpreted as the loss below some reference point, L, over a given period of time, where there is a probability of $\hat{q}$ of incurring a loss equal to or larger than L. (i.e. $VaR_{\hat{q}}^P = L - R^P(\hat{q})$). When the reference loss is set equal to zero, $VaR_{\hat{q}}^P$ becomes $-R^P(\hat{q}) = -F_P^{-1}(\hat{q})$.

Therefore, using the definition of VaR, analytical VaR of a portfolio can be represented as

$$VaR_{\hat{q}}^P = VaR(\hat{q}, R^P) = -F_P^{-1}(\hat{q}) = z^P(\hat{q}) = -\mu_P + \lambda_{\hat{q}}\sigma_P$$

Since the mean of market return is $E[R^M]$, analytical VaR of the market can be expressed as

$$VaR_{\hat{q}}^M = -E[R^M] + \lambda_{\hat{q}}\sigma_M$$

Assume $R^P \sim N(\mu_P, \sigma_P)$, $R^M \sim N(\mu_M, \sigma_M)$ and $R^{P|M} \sim N(\mu_{P|M}, \sigma_{P|M})$
Let $\rho_{P,M}$ be correlation between portfolio return and market return.

$\Pr\left(R^P \le CoVaR_{\hat{q}}^{P|M} \middle| R^M = VaR_{\hat{q}}^M\right) = \hat{q}$, CoVaR with $1-\hat{q}$ confidence level, is defined as the loss below some reference point, L given that the market return is equal to $VaR_{\hat{q}}^M$, where there is a probability of $\hat{q}$ of incurring a loss that is equal to or larger than L. Going through the same derivation as part 2 and employing conditional pdf, we get

$$CoVaR_q^{P|M} = CoVaR(q, R_q^{P|M}) = z^{P|M}(q) = -\mu_{P|M} + \lambda_q \sigma_{P|M}$$

where $z^{P|M}(\hat{q}) = -F_{P|M}^{-1}(\hat{q})$.

The conditional pdf of X given Y is:

$$f_{X|Y}(x|y) = \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}\,\sigma_X} e^{-\left(x - \rho\frac{\sigma_X}{\sigma_Y}y\right)^2 / 2\sigma_{X|Y}^2} \quad \text{where } \sigma_{X|Y}^2 = (1-\rho^2)\sigma_X^2$$

$$E[X|Y] = E[X] + \rho \frac{\sigma_X}{\sigma_Y}(Y - E[Y]) \text{ and } \sigma^2_{X|Y} = (1-\rho^2)\sigma^2_X : \; X|Y \sim N\left(E[X|Y], \sigma^2_{X|Y}\right)$$

Therefore, for bivariate normal variable, the conditional mean and the variance are

$$\mu_{P|M} = \mu_P + \rho_{P,M}\sigma_P / \sigma_M\left(R^M - \mu_M\right) \text{ and } \sigma_{P|M} = \sqrt{(1-\rho^2_{P,M})}\sigma_P$$

Therefore, substituting conditional mean and standard deviation, and using the Condition that $R^M = -VaR^M_{\hat{q}}$, analytical CoVaR of a portfolio with respect to the market can be expressed as

$$CoVaR^{P|M}_{\hat{q}} = -\mu_P + \rho_{P,M}\frac{\sigma_P}{\sigma_M}\left(VaR^M_{\hat{q}} + \mu_M\right) + \lambda_{\hat{q}}\sqrt{(1-\rho^2_{P,M})}\sigma_P \tag{A1}$$

B. Proof of Corollary 1

Sharpe's one factor model states that $\mu_P - r_f = \beta_{P,M}(\mu_M - r_f)$ where $\beta_{P,M} = \rho_{P,M}\sigma_P / \sigma_M$

Substituting $\mu_P = r_f + \beta_{P,M}(\mu_M - r_f)$ to Equation (A1), we get

$CoVaR^{P|M}_{\hat{q}} = -(1-\beta_{P,M})r_f - \beta_{P,M}VaR^M_{\hat{q}} + \lambda_{\hat{q}}\sqrt{(1-\rho^2_{P,M})}\sigma_P$

Note that $CoVaR^{P|M}_{\hat{q}}$ is the $\hat{q}$ percentile quantile portfolio return that is conditioned on market return is at a $\hat{q}$ percentile level. Hence it can be rewritten as $R^{P|M}_{\hat{q}}$. $VaR^M_{\hat{q}}$ is also the $\hat{q}$ percentile market return which can be rewritten as $R^M_{\hat{q}}$. Therefore the equation (A1) can be rewritten in a form that resembles the CAPM equation:

$$R^{P|M}_q + r_f = -\beta_{P,M}\left(R^M_q - r_f\right) + \varepsilon \tag{A2}$$

where $\varepsilon = \lambda_q\sqrt{(1-\rho^2_{P,M})}\sigma_P$

C. Proof of Proposition 2

Applying the delta method to the expression for CoVaR requires first derivatives of VaR with respect to the sample mean and variance. Replacing the variables by their probability limits, (population moments), we see that our first derivative vector, g, is shown to be

$$CoVaR^{P|M}_{\hat{q}} = -\mu_P - \rho_{P,M}\frac{\sigma_P}{\sigma_M}\left(VaR^M_{\hat{q}} - \mu_M\right) + q_\alpha\sqrt{(1-\rho^2_{P,M})}\sigma_P$$

$$\begin{pmatrix} \sqrt{N}\left(\overline{R} - \mu_{P|M}\right) \\ \sqrt{N}\left(s^2 - \sigma^2_{P|M}\right) \end{pmatrix} \xrightarrow{d} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{P|M} & \mu_3^{P|M} \\ \mu_3^{P|M} & \mu_4^{P|M} - \sigma^2_{P|M} \end{pmatrix}\right]$$

Although we use bivariate normal formula, since this section deals with the asymptotic properties, we do not necessarily assume normality as long as the central limit theorem holds for non-identical but independent case. Hong et al. (2010) present how central limit theorem is applied for iid case. The theorem in Eremin (1999) states that for non-identically distributed independent variables X_i, the Lyapunov condition of convergence of a sequence of distribution of normalized sums to a standard normal distribution law will be fulfilled if the ratio of non-random weight coefficients will be a bounded variable: $\left|a_i / a_j\right| \leq M, M < \infty$ Eremin (1999) proves the theorem too.

Now consider CoVaR to be the following function of parameters, $CoVaR = \mu_{P|M} + q_\alpha \sigma_{P|M}$ with q_α some known constant typically determined by some parameter – free distribution and corresponding to the inverse of a lower-tail probability, indeed in most contexts, we can view it as negative. $\mu_3^{P|M}$ and $\mu_4^{P|M}$ are the third and the fourth moments of conditional portfolio return distribution. Taking the derivative of CoVaR with respect to the mean and the standard deviation of the portfolio gives the first derivative vector, g,

$$g = \begin{pmatrix} 1 \\ \rho \dfrac{\left(R_\alpha^M - \mu_M\right)}{2\sigma_P \sigma_M} + \dfrac{q_\alpha \sqrt{1-\rho^2}}{2\sigma_P} \end{pmatrix}$$

Given that the delta method implies that the limiting distribution is multivariate normal with an asymptotic mean a vector of zeroes and also that the asymptotic covariance matrix is $g' \Sigma g$, the result follows.

$$\sqrt{N}\left(\overline{R} + \lambda s - \mu_{P|M} - \lambda \sigma_{P|M}\right) \xrightarrow{d} N\left(0, \varpi^2\right)$$

where the asymptotic variance of CoVaR distribution, ϖ^2, is

$$(1-\rho_{P,M}{}^2)\sigma_P^2 + \left[\rho \frac{\left(R_\alpha^M - \mu_M\right)}{2\sigma_P \sigma_M} + \frac{q_\alpha \sqrt{1-\rho^2}}{2\sigma_P}\right] \mu_3^{P|M} + \left[\rho \frac{\left(R_\alpha^M - \mu_M\right)}{2\sigma_P \sigma_M} + \frac{q_\alpha \sqrt{1-\rho^2}}{2\sigma_P}\right]^2 \left(\mu_4^{P|M} - \sigma_{P|M}^4\right)$$

Let $\rho \dfrac{\left(R_\alpha^M - \mu_M\right)}{2\sigma_P \sigma_M} + \dfrac{q_\alpha \sqrt{1-\rho^2}}{2\sigma_P} = G$

then, a ϖ^2 can be written in terms of conditional moments

$$\varpi^2 = \left(\sqrt{1-\rho_{P,M}^2}\,\sigma_P\right)^2 + G\left(\sqrt{1-\rho_{P,M}^2}\,\sigma_P\right)^3 k_3^{P|M} + G^2\left(\sqrt{1-\rho_{P,M}^2}\,\sigma_P\right)^4 \left(k_4^{P|M} - 1\right)$$

and this can be rewritten as

$$\varpi^2 = \sigma_{P|M}^2 \left(1 + G\sigma_{P|M} k_3^{P|M} + G^2 \sigma_{P|M}^2 \left(k_4^{P|M} - 1\right)\right) \quad \text{(A3)}$$

It is an immediate consequence that we can build a 95 or 99% confidence interval, the width of which is

$$\frac{2cv\sqrt{\varpi^2}}{\sqrt{N}}$$

Here cv is the relevant critical value.

D. Proof of Remark 1

$$\begin{aligned} P_P(t) &= P_P(0)\exp\left(\left(\mu_P - \frac{1}{2}\sigma_P^2\right)t + \sigma_P W_P(t)\right) \\ P_M(t) &= P_M(0)\exp\left(\left(\mu_M - \frac{1}{2}\sigma_M^2\right)t + \sigma_M W_M(t)\right) \end{aligned} \tag{A4}$$

Taking log of equation $P_P(t) = P_P(0)\exp\left(\left(\mu_P - \frac{1}{2}\sigma_P^2\right)t + \sigma_P W_P(t)\right)$, we get

$$\log(P_P(t)) = \log(P_P(0)) + \left(\mu_P - \frac{1}{2}\sigma_P^2\right)t + \sigma_P W_P(t)$$

and therefore

$$\log(P_P(t+1)) = \log(P_P(0)) + \left(\mu_P - \frac{1}{2}\sigma_P^2\right)(t+1) + \sigma_P W_P(t+1)$$

Hence

$$\log(P_P(t+1)) - \log(P_P(t)) = \left(\mu_P - \frac{1}{2}\sigma_P^2\right) + \sigma_P\left(W_P(t+1) - W_P(t)\right)$$

And the log return can be defined as

$$r_P(t+1,t) := \log(P_P(t+1)) - \log(P_P(t)) = \mu_P - \frac{1}{2}\sigma_P^2 + \sigma_P\left(W_P(t+1) - W_P(t)\right)$$

Applying the same process to market price process to get

$$r_M(t+1,t) := \log(P_M(t+1)) - \log(P_M(t)) = \mu_M - \frac{1}{2}\sigma_M^2 + \sigma_M\left(W_M(t+1) - W_M(t)\right)$$

Therefore

$$R(t+1,t) = \mu - \frac{1}{2}\sigma^2 + H\left(W(t+1) - W(t)\right) \tag{A5}$$

E. Proof of Remark 3

Combine the processes $q_A(t)$ and $q_M(t)$ into an 2 dimensional random process Q(t) and the independent Wiener processes $W_M(t)$ and $W_A(t)$ into a 2 dimensional Wiener process W(t). Denote Σ as a 2 x 2 diagonal matrix of volatilities, σ_A and σ_M. And let matrix A be a matrix of coefficients. i.e., we set

$$Q(t) := \begin{pmatrix} q_P(t) \\ q_M(t) \end{pmatrix}, \ \overline{H} = \begin{pmatrix} \sigma_P & 0 \\ 0 & \sigma_M \end{pmatrix}, \ \overline{\overline{H}} = \begin{pmatrix} \sqrt{1-\kappa^2} & \kappa \\ 0 & 1 \end{pmatrix},$$

$$H = \overline{H}\overline{\overline{H}} = \begin{pmatrix} \sigma_P \sqrt{1-\kappa^2} & \kappa\sigma_P \\ 0 & \sigma_M \end{pmatrix} \quad \Sigma := HH' = \begin{pmatrix} \sigma_P^2 & \kappa\sigma_P\sigma_M \\ \kappa\sigma_P\sigma_M & \sigma_M^2 \end{pmatrix},$$

$$dW(t) := \begin{pmatrix} dW_P(t) \\ dW_M(t) \end{pmatrix} = \overline{\overline{H}} dE(t) = \overline{\overline{H}} \begin{pmatrix} dE_P(t) \\ dE_M(t) \end{pmatrix} \text{ and } A := \begin{pmatrix} -\theta_{P1} & \theta_{P2} \\ 0 & -\theta_M \end{pmatrix},$$

Note that $dE(t) = \begin{pmatrix} dE_P(t) \\ dE_M(t) \end{pmatrix} \sim (0, Idt)$

Thus the system

$$\begin{pmatrix} dq_P(t) \\ dq_M(t) \end{pmatrix} = \begin{pmatrix} -\theta_{P1} & \theta_{P2} \\ 0 & -\theta_M \end{pmatrix} \begin{pmatrix} q_P(t) \\ q_M(t) \end{pmatrix} dt + H \begin{pmatrix} dE_P(t) \\ dE_M(t) \end{pmatrix}$$

becomes

$$dQ(t) = AQ(t)dt + HdE(t)$$

And therefore we have

$$\exp(-At)dQ(t) - \exp(-At)AQ(t)dt = \exp(-At)HdE(t)$$

Integrating both sides from t to s we get,

$$\exp(-As)dQ(s) - \exp(-At)AQ(t)dt = \int_t^s \exp(-At)HdE(t)$$

so

$$\begin{aligned} Q(s) &= \exp(-A(t-s))Q(t) + \exp(As)\int_t^s \exp(-Au)HdE(u) \\ &= \exp(-A(t-s))Q(t) + \exp(As)\int_t^s \exp(A(s-u))HdE(u) \end{aligned} \quad \text{(A6)}$$

The differential equations, (q_A, q_M) is Gaussian given its initial value $(q_A(0), q_M(0))$ at t = 0, as we can see from the following explicit solution:

Let $B := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $B^2 = \begin{pmatrix} a^2 & b(a+c) \\ 0 & c^2 \end{pmatrix}$, let $V_1 = 1$ and $V_2 = a + c$, then $B^n = \begin{pmatrix} a^n & bV_n \\ 0 & c^n \end{pmatrix}$

now $V_{n+1} = aV_n + c^n$ and therefore $V_{n+1} = \begin{cases} \dfrac{a^{n+1} - c^{n+1}}{a-c} \text{ if } a \neq c \\ (n+1)a^n \text{ if } a = c \end{cases}$

$$\text{Hence}\quad \begin{cases} \text{if } a \neq c \text{ then } e^{tA} = \begin{pmatrix} e^{at} & \dfrac{b}{a-c}\left(e^{at} - e^{ct}\right) \\ 0 & e^{ct} \end{pmatrix} \\ \text{if } a = c \text{ then } e^{tA} = \begin{pmatrix} e^{at} & b\left(e^{at} - 1\right) \\ 0 & e^{at} \end{pmatrix} \end{cases}$$

If a = c = 0, $\exp(At) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ When $A := \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} -\theta_{A1} & \theta_{A2} \\ 0 & -\theta_M \end{pmatrix}$, we obtain

$$e^{tA} = \begin{pmatrix} e^{-\theta_{A1} t} & \dfrac{\theta_{A2}}{\theta_{A1} - \theta_M}\left(e^{-\theta_M t} - e^{-\theta_{A1} t}\right) \\ 0 & e^{-\theta_M t} \end{pmatrix}$$

and therefore, setting s = t+1, the solution is given by

$$\begin{aligned} Q(t+1) &= \begin{pmatrix} e^{-\theta_{P1}(t+1)} & \dfrac{\theta_{P2}}{\theta_{P1} - \theta_M}\left(e^{-\theta_M (t+1)} - e^{-\theta_{P1}(t+1)}\right) \\ 0 & e^{-\theta_M (t+1)} \end{pmatrix} \cdot Q(0) \\ &+ \int_0^{t+1} \begin{pmatrix} e^{-\theta_{P1}(t+1-u)} & \dfrac{\theta_{P2}}{\theta_{P1} - \theta_M}\left(e^{-\theta_M (t+1-u)} - e^{-\theta_{P1}(t+1-u)}\right) \\ 0 & e^{-\theta_M (t+1-u)} \end{pmatrix} \cdot HdE(u) \end{aligned} \tag{A7}$$

Define $r_M(t) := \log P_M(t+1) - \log P_M(t)$, $r_P(t) := \log P_P(t+1) - \log P_P(t)$, and let $\varepsilon_M = e^{-\theta_M} - 1$, $\varepsilon_{2M} = e^{2\theta_M} - 1$, $\varepsilon_{P1} = e^{-\theta_{P1}} - 1$, $\varepsilon_{P2} = e^{2\theta_{P2}} - 1$ and $\varepsilon_{P1M} = e^{(\theta_M + \theta_{P1})} - 1$ then the return vector $R(t) = \begin{pmatrix} r_P(t) \\ r_M(t) \end{pmatrix} = Q(t+1) - Q(t) + \mu$ can be written, writing out (A7) twice at t+1 and at t and subtracting, as

$$\begin{aligned} R(t) &= \begin{pmatrix} \mu_P \\ \mu_M \end{pmatrix} + \begin{pmatrix} \varepsilon_M & \dfrac{\theta_{P2}}{\theta_{P1} - \theta_M}\left(\varepsilon_M - \varepsilon_{A1}\right) \\ 0 & \varepsilon_M(h) \end{pmatrix} \cdot Q(0) \\ &+ \int_t^{t+1} \begin{pmatrix} e^{-\theta_{P1}(t-u)} & \dfrac{\theta_{P2}}{\theta_{P1} - \theta_M}\left(e^{-\theta_M (t-u)} - e^{-\theta_{P1}(t-u)}\right) \\ 0 & e^{-\theta_{M1}(t-u)} \end{pmatrix} \cdot EdW(u) + V(t,0) \end{aligned} \tag{A8}$$

where $V(t,0)$ can be expressed in matrix notation

$$V(t,0) = \int_0^t \left(\exp(A(t+1-u)) - \exp(A(t-u))\right) HdE(u)$$

F. Alternative Representation of the log OU Process

From Appendix A, we know that the solution to the market system is given by, using A(1)

$$q_M(t) = e^{-\theta_M t} q_M(0) + \sigma_M \int_0^t e^{-\theta_M (t-u)} dW(u)$$
$$q_M(t+1) = e^{-\theta_M (t+1)} q_M(0) + \sigma_M \int_0^{t+1} e^{-\theta_M (t-u)} dW(u)$$

Therefore

$$q_M(t+1) - q_M(t) = \left(e^{-\theta_M} - 1\right) q_M(t) + e^{-\theta_M t} \sigma_M \int_t^{t+1} e^{\theta_M u} dW(u)$$

Using the properties of the Ito integral,the error term in the above expression has unconditional mean zero and time t conditional variance equal to

$$\frac{\sigma_M^2 (1 - \exp(2\vartheta_M))}{-2\vartheta_M}$$

Likewise,

$$Q(t+1) = \begin{pmatrix} e^{-\theta_{A1}} & \dfrac{\theta_{A2}}{\theta_{A1} - \theta_M}\left(e^{-\theta_M} - e^{-\theta_{A1}}\right) \\ 0 & e^{-\theta_M} \end{pmatrix} \cdot Q(t) + \int_t^{t+1} \begin{pmatrix} e^{-\theta_{A1}(t+1-u)} & \dfrac{\theta_{A2}}{\theta_{A1} - \theta_M}\left(e^{-\theta_M (t+1-u)} - e^{-\theta_{A1}(t+1-u)}\right) \\ 0 & e^{-\theta_M (t+1-u)} \end{pmatrix} \cdot HdE(u) \tag{A9}$$

By setting h to 1, we see that discretized market data has an exact AR(1) representation, whilst Q(t) has an exact VAR(1) representation. The error processes, being based on non-overlapping Brownian motions, are independent and identically distributed normal errors. Thus additive returns being the difference of Q, will be VARIMA (1, 1, 0) plus a drift term.

We can write this in matrix notation as

$$Q(t+1) = AQ(t) + V(t+1)$$

$$Q(t+1) - Q(t) = \left(A - I_2\right) Q(t) + V(t+1)$$

$$R(t+1, t) = \mu + \left(A - I_2\right) Q(t) + V(t+1)$$

We can see that the matrix return equation is VARMA(1, 1) in returns. If we wish to substitute out the Q(t) term, we would get a moving average representation in terms of the original error term in (A4) as in Perron et al(1988). This leads to complexity in estimation so we shall estimate the model in de-trended log prices.

If a=c=0, $\exp(At)=\begin{pmatrix}1 & \theta_{A2}\\ 0 & 1\end{pmatrix}$ we have the random walk case, we then can solve the system directly

And

$$
\begin{aligned}
&Q(t+1)=\begin{pmatrix}1 & \theta_{A2}\\ 0 & 1\end{pmatrix}\cdot Q(t)+\int_t^{t+1}\begin{pmatrix}1 & \theta_{A2}\\ 0 & 1\end{pmatrix}\cdot H dE(u)\\
&q_A(t+1)-q_A(t)=\theta_{A2}q_A(t)+\sigma_M\int_t^{t+1}dE_M(u)+\sigma_A\sqrt{1-\kappa^2}\int_t^{t+1}dE_A(u)\\
&q_M(t+1)-q_M(t)=\sigma_M\int_t^{t+1}dE_M(u)
\end{aligned}
\qquad \text{(A10)}
$$

G. Proof of Equation (22)

$$
Cov[r_P(t),r_M(t)]=\left(\frac{\varepsilon_{2M}}{2\theta_M}-\frac{\varepsilon_{P1M}}{\theta_{P1}+\theta_M}\right)\frac{\theta_{P2}}{\theta_{P1}-\theta_M}\sigma_M^2+\frac{\varepsilon_{P1M}}{\theta_{P1}+\theta_M}\sigma_P\sigma_M\kappa
$$

$$
Var(r_M(t))=\sigma_{rM}^2=\left(\frac{\varepsilon_{2M}}{2\theta_M}\right)\sigma_M^2
$$

$$
\begin{aligned}
\beta_{P,M}&=\frac{Cov[r_P(t),r_M(t)]}{Var[r_M(t)]}\\
&=\frac{\left(\frac{\varepsilon_{2M}}{2\theta_M}-\frac{\varepsilon_{PM}}{\theta_{P1}+\theta_M}\right)\frac{\theta_{P2}}{\theta_{P1}-\theta_M}\sigma_M^2+\frac{\varepsilon_{P1M}}{\theta_{P1}+\theta_M}\sigma_P\sigma_M\kappa}{\left(\frac{\varepsilon_{2M}}{2\theta_M}\right)\sigma_M^2}\\
&=\left(\frac{\varepsilon_{2M}}{2\theta_M}-\frac{\varepsilon_{P1M}}{\theta_{P1}+\theta_M}\right)\left(\frac{\varepsilon_{2M}}{2\theta_M}\right)^{-1}\frac{\theta_{P2}}{\theta_{P1}-\theta_M}+\frac{\varepsilon_{P1M}}{\theta_{P1}+\theta_M}\left(\frac{\varepsilon_{2M}}{2\theta_M}\right)^{-1}\frac{\sigma_P}{\sigma_M}\kappa\\
&=\left(1-\frac{e^{(\theta_{P1}+\theta_M)h}-1}{e^{2\theta_M h}-1}\cdot\frac{2\theta_M}{\theta_{P1}+\theta_M}\right)\frac{\theta_{P2}}{\theta_{P1}-\theta_M}+\frac{e^{(\theta_{P1}+\theta_M)h}-1}{e^{2\theta_M h}-1}\cdot\frac{2\theta_M}{\theta_{P1}+\theta_M}\frac{\sigma_P}{\sigma_M}\kappa
\end{aligned}
$$

Since the portfolio beta when prices follow geometric Brownian motion, $\beta_G=\frac{\sigma_P}{\sigma_M}\kappa$ then,

$$
\beta_{P,M}=\left(1-\frac{e^{(\theta_{P1}+\theta_M)h}-1}{e^{2\theta_M h}-1}\cdot\frac{2\theta_M}{\theta_{P1}+\theta_M}\right)\frac{\theta_{P2}}{\theta_{P1}-\theta_M}+\frac{e^{(\theta_{P1}+\theta_M)h}-1}{e^{2\theta_M h}-1}\cdot\frac{2\theta_M}{\theta_{P1}+\theta_M}\beta_G \qquad \text{(A11)}
$$

H. Proof of Proposition 4

1. VaR

$$\sigma_{rP}=\sqrt{\begin{aligned}&\frac{\sigma_P^2\left(e^{2\theta_{P1}}-1\right)}{2\theta_{P1}}+\left(\frac{\sigma_M\theta_{P2}}{\theta_{P1}-\theta_M}\right)^2\left(\frac{\left(e^{2\theta_M}-1\right)}{2\theta_M}+\frac{\left(e^{2\theta_{P1}}-1\right)}{2\theta_{P1}}-\frac{2\left(e^{(\theta_{P1}+\theta_M)}-1\right)}{\theta_M+\theta_{P1}}\right)\\&+\left(\frac{2\theta_{P2}}{\theta_{P1}-\theta_M}\right)\left(\frac{\left(e^{(\theta_{P1}+\theta_M)}-1\right)}{\theta_M+\theta_{P1}}-\frac{\left(e^{2\theta_{P1}}-1\right)}{2\theta_{P1}}\right)\sigma_M\sigma_P\kappa\end{aligned}}$$

Applying the chain rule, we get

$$\frac{\partial VaR_q^P(t)}{\partial\kappa}=\lambda_q\frac{\partial\sigma_{rP}}{\partial\kappa}$$

$$=\lambda_q\frac{\frac{1}{2}\left(\frac{\theta_{P2}}{\theta_{P1}-\theta_M}\right)\left(\frac{\left(e^{(\theta_{P1}+\theta_M)}-1\right)}{\theta_M+\theta_{P1}}-\frac{\left(e^{2\theta_{P1}}-1\right)}{2\theta_{P1}}\right)}{\sqrt{\begin{aligned}&\frac{\sigma_P^2\left(e^{2\theta_{P1}}-1\right)}{2\theta_{P1}}+\left(\frac{\sigma_M\theta_{P2}}{\theta_{P1}-\theta_M}\right)^2\left(\frac{\left(e^{\theta_M}-1\right)}{2\theta_M}+\frac{\left(e^{2\theta_{P1}}-1\right)}{2\theta_{P1}}-\frac{2\left(e^{(\theta_{P1}+\theta_M)}-1\right)}{\theta_M+\theta_{P1}}\right)\\&+\left(\frac{2\theta_{P2}}{\theta_{P1}-\theta_M}\right)\left(\frac{\left(e^{(\theta_{P1}+\theta_M)}-1\right)}{\theta_M+\theta_{P1}}-\frac{\left(e^{2\theta_{P1}}-1\right)}{2\theta_{P1}}\right)\sigma_M\sigma_P\kappa\end{aligned}}}$$

2. CoVaR

$$CoVaR_q^{P|M}=-\mu_{rP}+\beta_{P,M}(\kappa)\left(VaR_q^M+\mu_{rM}\right)+\lambda_q\sigma_{rP}(\kappa)\sqrt{1-\beta_{P,M}^2(\kappa)\frac{\sigma_{rM}^2}{\sigma_{rP}^2(\kappa)}}$$

Applying the chain rule, we get

$$\frac{\partial CoVaR_q^P(t)}{\partial\kappa}=\frac{\partial\beta_{P,M}}{\partial\kappa}\lambda_q\sigma_{rM}+\lambda_q\frac{\partial\sigma_{rP}}{\partial\kappa}\left(1-\beta_{P,M}^2\frac{\sigma_{rM}^2}{\sigma_{rP}^2}\right)^{\frac{1}{2}}$$
$$+\lambda_q\sigma_{rP}\frac{1}{2}\left(1-\beta_{P,M}^2\frac{\sigma_{rM}^2}{\sigma_{rP}^2}\right)^{-\frac{1}{2}}\left(-\sigma_{rM}^2\right)\left(\frac{\partial\beta_{P,M}^2}{\partial\kappa}\sigma_{rP}^{-2}+\frac{\partial\sigma_{rP}^{-2}}{\partial\kappa}\beta_{P,M}^2\right)$$

Let $C:=\frac{\partial\beta_{P,M}}{\partial\kappa}\left(VaR_q^M+\mu_{rM}\right)=\frac{1-e^{(\theta_{P1}+\theta_M)}}{1-e^{2\theta_M}}\cdot\frac{2\theta_M}{\theta_{P1}+\theta_M}\cdot\sqrt{\frac{e^{2\theta_M}-1}{2\theta_M}}\lambda_q\sigma_M$,
$B:=1-\beta_{P,M}^2\frac{\sigma_{rM}^2}{\sigma_{rP}^2}$ and $L=\lambda_q\beta_{P,M}\frac{\partial\beta_{P,M}}{\partial\kappa}\frac{\sigma_{rM}^2}{\sigma_{rP}}$

Applying Equation (21),

$$\frac{\partial CoVaR_q^P(t)}{\partial \kappa}$$

$$= C + \lambda_q \left(1 - \beta_{P,M}^2 \frac{\sigma_{rM}^2}{\sigma_{rP}^2}\right)^{-\frac{1}{2}} \times \left(\frac{\partial \sigma_{rP}}{\partial \kappa} \left(1 - \beta_{P,M}^2 \frac{\sigma_{rM}^2}{\sigma_{rP}^2}\right) - \frac{1}{2} \sigma_{rP} \sigma_{rM}^2 \left(\frac{\partial \beta_{P,M}^2}{\partial \kappa} \sigma_{rP}^{-2} + \frac{\partial \sigma_{rP}^{-2}}{\partial \kappa} \beta_{P,M}^2 \right)\right)$$

$$= C + \lambda_q \left(1 - \beta_{P,M}^2 \frac{\sigma_{rM}^2}{\sigma_{rP}^2}\right)^{-\frac{1}{2}} \times \left(\frac{\partial \sigma_{rP}}{\partial \kappa} - \frac{\partial \sigma_{rP}}{\partial \kappa} \beta_{P,M}^2 \frac{\sigma_{rM}^2}{\sigma_{rP}^2} - \frac{1}{2} \frac{\partial \beta_{P,M}^2}{\partial \kappa} \frac{\sigma_{rM}^2}{\sigma_{rP}} + \frac{\sigma_{rM}^2}{\sigma_{rP}^2} \frac{\partial \sigma_{rP}}{\partial \kappa} \beta_{P,M}^2 \right)$$

$$= C + \frac{\lambda_q}{\sqrt{B}} \times \left(\frac{\partial \sigma_{rP}}{\partial \kappa} - \beta_{P,M} \frac{\partial \beta_{P,M}}{\partial \kappa} \frac{\sigma_{rM}^2}{\sigma_{rP}} \right)$$

Therefore

$$\frac{\partial CoVaR_q^P(t)}{\partial \kappa} = \frac{1 - e^{(\theta_{P1} + \theta_M)}}{1 - e^{2\theta_M}} \cdot \frac{2\theta_M}{\theta_{P1} + \theta_M} \cdot \sqrt{\frac{e^{2\theta_M} - 1}{2\theta_M}} \lambda_q \sigma_M + \frac{\lambda_q}{\sqrt{B}} \times \left(\frac{\partial \sigma_{rP}}{\partial \kappa} - \beta_{P,M} \frac{\partial \beta_{P,M}}{\partial \kappa} \frac{\sigma_{rM}^2}{\sigma_{rP}^2} \right)$$

where $B := 1 - \beta_{P,M}^2 \frac{\sigma_{rM}^2}{\sigma_{rP}^2}$

I. Equivalence of MLE and EGLS

GLS

$$Q(t+1) = AQ(t) + u(t+1),\ u(t) \sim (0, \Omega)$$

$$Q(t+1) = \begin{pmatrix} q_A(t+1) \\ q_M(t+1) \end{pmatrix} \quad Q(t) = \begin{pmatrix} q_A(t) \\ q_M(t) \end{pmatrix} \quad u(t+1) = \begin{pmatrix} u_A(t+1) \\ u_M(t+1) \end{pmatrix}$$

$$A(h) = \begin{pmatrix} e^{-\theta_{A1}(t+h-u)} & \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(e^{-\theta_M (t+h-u)} - e^{-\theta_{A1}(t+h-u)} \right) \\ 0 & e^{-\theta_M (t+h-u)} \end{pmatrix}$$

$$\Omega = \begin{bmatrix} \left(\begin{array}{l} \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}} \sigma_A^2 + \frac{2\theta_{A2} \kappa \sigma_A \sigma_M}{\theta_{A1} - \theta_M} \left(\frac{1 - e^{-(\theta_{A1} + \theta_M)}}{\theta_{A1} + \theta_M} - \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}} \right) \\ + \left(\frac{\theta_{A2}}{\theta_{A1} - \theta_M} \right)^2 \left(\frac{1 - e^{-2\theta_M}}{2\theta_M} + \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}} - \frac{2\left(1 - e^{-(\theta_{A1} + \theta_M)}\right)}{\theta_{A1} + \theta_M} \right) \sigma_M^2 \end{array} \right) & \left(\begin{array}{l} \frac{1 - e^{-(\theta_{A1} + \theta_M)}}{\theta_{A1} + \theta_M} \kappa \sigma_A \sigma_M + \\ \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\frac{1 - e^{-2\theta_M}}{2\theta_M} - \frac{1 - e^{-(\theta_{A1} + \theta_M)}}{\theta_{A1} + \theta_M} \right) \sigma_M^2 \end{array} \right) \\ \left(\begin{array}{l} \frac{1 - e^{-(\theta_{A1} + \theta_M)}}{\theta_{A1} + \theta_M} \kappa \sigma_A \sigma_M + \\ \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\frac{1 - e^{-2\theta_M}}{2\theta_M} - \frac{1 - e^{-(\theta_{A1} + \theta_M)}}{\theta_{A1} + \theta_M} \right) \sigma_M^2 \end{array} \right) & \left(\frac{1 - e^{-2\theta_M}}{2\theta_M} \sigma_M^2 \right) \end{bmatrix}$$

1. Decompose $E(uu'|A)$: $E(uu'|A) = A\Omega A'$

2. Construct $\Omega^{\frac{1}{2}}$

3. Create a new regression by transforming $Q(t+1)$, $Q(t)$, A and u(t+1) as

$\widetilde{Q}(t+1)=\Omega^{-\frac{1}{2}}Q(t+1)$, $\widetilde{Q}(t)=\Omega^{-\frac{1}{2}}Q(t)$, $A_{\Omega}=\Omega^{-\frac{1}{2}}A\Omega^{\frac{1}{2}}$ and $\varepsilon(t+1)=\Omega^{-\frac{1}{2}}u(t+1)$

$$\widetilde{Q}(t+1)=A_{\Omega}\widetilde{Q}(t)+\varepsilon(t+1)$$

$$SSR=\sum_{t=1}^{T-1}\left[\varepsilon(t+1)\varepsilon'(t+1)\right]=\sum_{t=1}^{T-1}\left[\left(\widetilde{Q}(t+1)-A_{\Omega}\widetilde{Q}(t)\right)\left(\widetilde{Q}(t+1)-A_{\Omega}\widetilde{Q}(t)'\right)\right]$$
$$=\sum_{t=1}^{T-1}\left[\widetilde{Q}(t+1)\widetilde{Q}(t+1)'-2A_{\Omega}\widetilde{Q}(t)\widetilde{Q}(t+1)'+A_{\Omega}\widetilde{Q}(t)\widetilde{Q}(t)'A_{\Omega}'\right]$$

$$\frac{\partial SSR}{\partial A_{\Omega}}=-2\sum_{t=1}^{T-1}\left[\widetilde{Q}(t)\widetilde{Q}(t+1)'+2\widetilde{Q}(t)\widetilde{Q}(t)'A_{\Omega}'\right]=0$$

$$\hat{A}_{\Omega}=\sum_{t=1}^{T-1}\left[\left(\widetilde{Q}(t)\widetilde{Q}(t)'\right)^{-1}\widetilde{Q}(t)\widetilde{Q}(t+1)'\right]=\sum_{t=1}^{T-1}\left[\left(\Omega^{-\frac{1}{2}}Q(t)Q(t)'\Omega'^{-\frac{1}{2}}\right)^{-1}\Omega^{-\frac{1}{2}}Q(t)Q'(t+1)\Omega'^{-\frac{1}{2}}\right]$$

MLE

$$Q(t+1)=AQ(t)+u(t+1),\ u(t+1)\sim(0,\Omega)$$

Likelihood function:

$$\log L=\sum_{t=1}^{T-1}\left[-\frac{T}{2}\ln(2\pi)+\frac{1}{2}\ln\left|\Omega^{-1}\right|-\frac{1}{2}\Omega^{-1}u(t+1)'u(t+1)\right]$$

Note $Q(t)=\log P(t)-\mu t$ and the property of $\ln\left|\Omega^{-1}\right|=-\ln|\Omega|$ is used

Differentiate logL with respect to the elements in $\Omega=\begin{pmatrix}o_{11} & o_{12}\\ o_{21} & o_{22}\end{pmatrix}$ we get,

$$\frac{\partial\log L}{\partial o_j}=\frac{1}{2}\sum_{t=1}^{T-1}\left[r(\Omega)\frac{\partial\Omega^{-1}}{\partial o_j}\right]t-\sum_{t=1}^{T-1}\left[\Omega^{-1}u(t+1)'\frac{\partial u(t+1)}{\partial o_j}\right]-\frac{1}{2}\sum_{t=1}^{T-1}\left[u(t+1)'\frac{\partial\Omega^{-1}}{\partial o_j}u(t+1)\right]$$
$$=\sum_{t=1}^{T-1}\left[\Omega^{-1}u(t+1)'Q(t)\frac{\partial A}{\partial o_j}\right]+\frac{1}{2}\sum_{t=1}^{T-1}\left[\left(tr(\Omega)-u(t+1)u(t+1)'\right)\frac{\partial\Omega^{-1}}{\partial o_j}\right]=0$$

Since $\frac{\partial A}{\partial o_j}\neq\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$, we get

$$\sum_{t=1}^{T-1}\left[\Omega^{-1}u(t+1)'Q(t)\right]=0$$
$$\frac{1}{2}\sum_{t=1}^{T-1}\left[\left(tr(\Omega)-u(t+1)u(t+1)'\right)\frac{\partial\Omega^{-1}}{\partial o_j}\right]=0$$

From the first condition we get,

$$\hat{A}_{\Omega} = \Omega^{-\frac{1}{2}} \hat{A} \Omega^{\frac{1}{2}} = \sum_{t=1}^{T-1} \left[\left(\Omega^{-\frac{1}{2}} Q(t) Q(t)' \Omega'^{-\frac{1}{2}} \right)^{-1} \Omega^{-\frac{1}{2}} Q(t) Q'(t+1) \Omega'^{-\frac{1}{2}} \right]$$

J. Estimation Method

$$\begin{bmatrix} q_P(t+1) \\ q_M(t+1) \end{bmatrix} = \begin{pmatrix} e^{-\theta_{P1}} & \dfrac{\theta_{P2}}{\theta_{Pa} - \theta_M} \\ 0 & e^{-\theta_M} \end{pmatrix} \cdot \begin{bmatrix} q_P(t) \\ q_M(t) \end{bmatrix} + \begin{bmatrix} u_P(t+1) \\ u_M(t+1) \end{bmatrix} \qquad \text{(A12)}$$

where $u(t+1) = \begin{bmatrix} u_A(t+1) \\ u_M(t+1) \end{bmatrix} \sim (0, \Omega)$,

$HH' = \begin{pmatrix} \sigma_A^2 & \kappa \sigma_A \sigma_M \\ \kappa \sigma_A \sigma_M & \sigma_M^2 \end{pmatrix}$ as before $H = \begin{pmatrix} \sigma_A \sqrt{1-\kappa^2} & \kappa \sigma_A \\ 0 & \sigma_M \end{pmatrix}$

Let $A(h) = \begin{pmatrix} e^{-\theta_{A1}(t+h-u)} & \dfrac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(e^{-\theta_M (t+h-u)} - e^{-\theta_{A1}(t+h-u)} \right) \\ 0 & e^{-\theta_M (t+h-u)} \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$,

$$\Omega = AHH'A' = \int_t^{t+1} \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} \sigma_A^2 & \kappa \sigma_A \sigma_M \\ \kappa \sigma_A \sigma_M & \sigma_M^2 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} du$$

$$= \int_t^{t+1} \begin{pmatrix} a^2 \sigma_A^2 + 2ab\kappa \sigma_A \sigma_M + b^2 \sigma_A \sigma_M & ac\kappa \sigma_A \sigma_M + bc\sigma_M^2 \\ ac\kappa \sigma_A \sigma_M + bc\sigma_M^2 & c^2 \sigma_M^2 \end{pmatrix} du$$

We let h = 1,

1. $\int_t^{t+1} \left(ac\kappa \sigma_A \sigma_M + bc\sigma_M^2 \right) du$

$$\int_t^{t+1} acdh = \int_t^{t+1} e^{-(\theta_{A1}+\theta_M)(t+1-u)} dh = \left. \frac{e^{-(\theta_{A1}+\theta_M)(t+1-u)}}{\theta_{A1} + \theta_M} \right|_t^{t+1} = \frac{1 - e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1} + \theta_M}$$

$$\int_t^{t+1} bcdh = \int_t^{t+1} \frac{\theta_{A2}}{\theta_{A1} - \theta_M} e^{-\theta_M (t+1-u)} \left(e^{-\theta_M (t+1-u)} - e^{-\theta_{A1}(t+1-u)} \right) du$$

$$= \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \int_t^{t+1} \left(e^{-2\theta_M (t+1-u)} - e^{-(\theta_{A1}+\theta_M)(t+1-u)} \right) du$$

$$= \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\int_t^{t+1} e^{-2\theta_M (t+1-u)} du - \int_t^{t+1} e^{-(\theta_{A1}+\theta_M)(t+1-u)} du \right)$$

$$= \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\left(\left. \frac{e^{-2\theta_M (t+1-u)}}{2\theta_M} \right|_t^{t+1} \right) - \left(\left. \frac{e^{-(\theta_{A1}+\theta_M)(t+1-u)}}{\theta_{A1} + \theta_M} \right|_t^{t+1} \right) \right)$$

$$= \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\frac{1 - e^{-2\theta_M}}{2\theta_M} - \frac{1 - e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1} + \theta_M} \right)$$

$$\int_t^{t+1} \left(ac\kappa\sigma_A\sigma_M + bc\sigma_M^2 \right) du$$

$$\frac{1 - e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1} + \theta_M} \kappa\sigma_A\sigma_M + \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\frac{1 - e^{-2\theta_M}}{2\theta_M} - \frac{1 - e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1} + \theta_M} \right) \sigma_M^2$$

2. $\int_t^{t+1} \left(a^2\sigma_A^2 + 2ab\kappa\sigma_A\sigma_M + b^2\sigma_M^2 \right) du$

$$\int_t^{t+1} a^2 du = \int_t^{t+1} e^{-2\theta_{A1}(t+1-u)} du = - \left. \frac{e^{-2\theta_{A1}(t+1-u)}}{2\theta_{A1}} \right|_t^{t+1} = \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}}$$

$$\int_t^{t+1} ab du = \int_t^{t+1} \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(e^{-\theta_M (t+1-u)} - e^{-\theta_{A1}(t+1-u)} \right) du$$

$$= \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\int_t^{t+1} e^{-(\theta_{A1}+\theta_M)(t+1-u)} du - \int_t^{t+1} e^{-2\theta_{A1}(t+1-u)} du \right)$$

$$= \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\left(\left. \frac{e^{-(\theta_{A1}+\theta_M)(t+1-u)}}{\theta_{A1} + \theta_M} \right|_t^{t+1} \right) - \left(\left. \frac{e^{-2\theta_{A1}(t+1-u)}}{2\theta_{A1}} \right|_t^{t+1} \right) \right)$$

$$= \frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(\frac{1 - e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1} + \theta_M} - \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}} \right)$$

$$\int_0^t b^2 du = \int_0^t \left(\frac{\theta_{A2}}{\theta_{A1} - \theta_M} \left(e^{-\theta_M (t+1-u)} - e^{-\theta_{A1}(t+1-u)} \right) \right)^2 du$$

$$= \left(\frac{\theta_{A2}}{\theta_{A1} - \theta_M} \right)^2 \int_0^t \left(e^{-2\theta_M (t+1-u)} + e^{-2\theta_{A1}(t+1-u)} - 2e^{-(\theta_{A1}+\theta_M)(t+1-u)} \right) du$$

$$= \left(\frac{\theta_{A2}}{\theta_{A1} - \theta_M} \right)^2 \left(\frac{1 - e^{-2\theta_M}}{2\theta_M} + \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}} - \frac{2\left(1 - e^{-(\theta_{A1}+\theta_M)}\right)}{\theta_{A1} + \theta_M} \right)$$

$$\int_t^{t+1} \left(a^2\sigma_A^2 + 2ab\kappa\sigma_A\sigma_M + b^2\sigma_M^2 \right) du$$

$$= \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}} \sigma_A^2 + \frac{2\theta_{A2}}{\theta_{A1} - \theta_M} \left(\frac{1 - e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1} + \theta_M} - \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}} \right) \kappa\sigma_A\sigma_M$$

$$+ \left(\frac{\theta_{A2}}{\theta_{A1} - \theta_M} \right)^2 \left(\frac{1 - e^{-2\theta_M}}{2\theta_M} + \frac{1 - e^{-2\theta_{A1}}}{2\theta_{A1}} - \frac{2\left(1 - e^{-(\theta_{A1}+\theta_M)}\right)}{\theta_{A1} + \theta_M} \right) \sigma_M^2$$

3. $\int_t^{t+1} c^2\sigma_M^2 du = \sigma_M^2 \int_t^{t+1} e^{-2\theta_M(t+1-u)} du = \sigma_M^2 \left. \frac{e^{-2\theta_M(t+1-u)}}{2\theta_M} \right|_t^{t+1} = \frac{1-e^{-2\theta_M}}{2\theta_M}\sigma_M^2$

$$\Omega = \begin{bmatrix} \begin{pmatrix} \frac{1-e^{-2\theta_{A1}}}{2\theta_{A1}}\sigma_A^2 + \frac{2\theta_{A2}\kappa\sigma_A\sigma_M}{\theta_{A1}-\theta_M}\left(\frac{1-e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1}+\theta_M} - \frac{1-e^{-2\theta_{A1}}}{2\theta_{A1}}\right) \\ +\left(\frac{\theta_{A2}}{\theta_{A1}-\theta_M}\right)^2\left(\frac{1-e^{-2\theta_M}}{2\theta_M} + \frac{1-e^{-2\theta_{A1}}}{2\theta_{A1}} - \frac{2\left(1-e^{-(\theta_{A1}+\theta_M)}\right)}{\theta_{A1}+\theta_M}\right)\sigma_M^2 \end{pmatrix} & \begin{pmatrix} \frac{1-e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1}+\theta_M}\kappa\sigma_A\sigma_M + \\ \frac{\theta_{A2}}{\theta_{A1}-\theta_M}\left(\frac{1-e^{-2\theta_M}}{2\theta_M} - \frac{1-e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1}+\theta_M}\right)\sigma_M^2 \end{pmatrix} \\ \begin{pmatrix} \frac{1-e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1}+\theta_M}\kappa\sigma_A\sigma_M + \\ \frac{\theta_{A2}}{\theta_{A1}-\theta_M}\left(\frac{1-e^{-2\theta_M}}{2\theta_M} - \frac{1-e^{-(\theta_{A1}+\theta_M)}}{\theta_{A1}+\theta_M}\right)\sigma_M^2 \end{pmatrix} & \left(\frac{1-e^{-2\theta_M}}{2\theta_M}\sigma_M^2\right) \end{bmatrix}$$

Once the covariance matrix of the residuals, Ω, is estimated, we can compute parameters σ_A, σ_M and κ .

Estimate (A12) with OLS, estimate the matrix A and the residual covariance matrix Ω. From $\hat{A}$, σ_A, σ_M and κ are calculated. Then multiply $\Omega^{-\frac{1}{2}}$ to the both sides of the equation (A12),

$$\Omega^{-\frac{1}{2}}\begin{bmatrix} q_A(t+h) \\ q_M(t+h) \end{bmatrix} = \left(\Omega^{-\frac{1}{2}}A\Omega^{\frac{1}{2}}\right)\Omega^{-\frac{1}{2}}\begin{bmatrix} q_A(t) \\ q_M(t) \end{bmatrix} + \Omega^{-\frac{1}{2}}\begin{bmatrix} u_A(t) \\ u_M(t) \end{bmatrix} \quad \text{(A13)}$$

$A_\Omega := \left(\Omega^{-\frac{1}{2}}A\Omega^{\frac{1}{2}}\right)$ can be estimated from (A15). Then calculate parameters θ_M, θ_{A1} and θ_{A2} from $\hat{A}_\Omega$.

This ensures we have residual vector $\begin{bmatrix} \varepsilon_A(t) \\ \varepsilon_M(t) \end{bmatrix} = \Omega^{-\frac{1}{2}}\begin{bmatrix} u_A(t) \\ u_M(t) \end{bmatrix} \sim (0, I)$

The detailed steps of the estimation method are as follows.

1. Get OLS estimate on Q(t) and Q(t-1)
2. Using the OLS estimate, compute estimated Q(t) and Q(t-1) and obtain residual
3. From residual from step 2, get GLS estimate of parameters
4. Using the GLS estimate, obtain residual
5. Run the first iterate GLS based on the residuals from step 4

K. Derivation of Proposition 5

ES is an average expected loss larger than VaR therefore is the weighted average of tail return distribution below VaR. The average of the worst $100(1-q)\%$ of losses can be expressed as

$$ES_q^P = \frac{1}{1-q}\int_q^1 q\,dp$$

Under normality assumption, its analytical form can be expressed as

$$ES_q^P = -\mu_P + \frac{\phi(\lambda_q)}{1-q}\sigma_P$$

Applying conditional moments, analytical CoES is

$$\begin{aligned} CoES_q^P &= -\mu_{P|M} + \frac{\phi(\lambda_q)}{1-q}\sigma_{P|M} \\ &= -\mu_{P|M} - \rho_{P,M}\frac{\sigma_P}{\sigma_M}\left(VaR_q^M + \mu_M\right) + \frac{\phi(\lambda_q)}{1-q}\sqrt{\left(1-\rho_{P,M}^2\right)}\sigma_P \end{aligned}$$

9. Tables and Figures

Table 1: Log Return Correlation against S&P500

	GS	MS	JPM	BAC	UBS	CS	BCS	DB
Pre	0.69	0.68	0.71	0.67	0.62	0.57	0.48	0.67
Crisis	0.76	0.76	0.76	0.70	0.76	0.80	0.62	0.81

GS: Goldman Sachs, MS: Morgan Stanley, JPM: JP Morgan, BAC: Bank of America, CS: Credit Suisse, BCS: Barclays and DB: Deutsche Bank

Table 2: VaR and CoVaR Exceedances for 1/n Portfolio, first 50 days

Date	muM	muA	sigM	sigA	rho	Mkt VaR	Port VaR	Port CoVaR	Portfolio Return	VaR Breach	CoVaR Breach
2007-07-24	0.09%	0.12%	0.72%	0.81%	0.9489	-1.10%	-1.22%	-1.57%	-2.06%	Yes	Yes
2007-07-25	0.08%	0.11%	0.75%	0.84%	0.9523	-1.16%	-1.28%	-1.63%	0.44%		
2007-07-26	0.09%	0.12%	0.74%	0.83%	0.9512	-1.13%	-1.25%	-1.60%	-1.77%	Yes	Yes
2007-07-27	0.08%	0.11%	0.78%	0.85%	0.9497	-1.20%	-1.28%	-1.65%	-1.42%	Yes	
2007-07-30	0.05%	0.08%	0.78%	0.84%	0.9493	-1.23%	-1.31%	-1.67%	1.36%		
2007-07-31	0.06%	0.10%	0.78%	0.85%	0.9499	-1.23%	-1.31%	-1.68%	-1.90%		
2007-08-01	0.04%	0.07%	0.79%	0.87%	0.9497	-1.26%	-1.37%	-1.74%	0.67%		
2007-08-02	0.05%	0.07%	0.80%	0.87%	0.9498	-1.26%	-1.36%	-1.74%	0.62%		
2007-08-03	0.05%	0.07%	0.80%	0.87%	0.9508	-1.26%	-1.36%	-1.74%	-2.73%	Yes	Yes
2007-08-06	0.04%	0.06%	0.82%	0.90%	0.9537	-1.30%	-1.41%	-1.79%	2.59%		
2007-08-07	0.06%	0.08%	0.85%	0.93%	0.9570	-1.33%	-1.45%	-1.82%	0.31%		
2007-08-08	0.06%	0.08%	0.85%	0.93%	0.9562	-1.33%	-1.45%	-1.83%	1.47%		
2007-08-09	0.08%	0.10%	0.86%	0.94%	0.9571	-1.33%	-1.44%	-1.82%	-2.55%	Yes	Yes
2007-08-10	0.04%	0.07%	0.90%	0.97%	0.9589	-1.45%	-1.53%	-1.92%	-0.62%		
2007-08-13	0.03%	0.05%	0.90%	0.97%	0.9567	-1.45%	-1.55%	-1.94%	0.09%		
2007-08-14	0.02%	0.03%	0.89%	0.95%	0.9550	-1.44%	-1.53%	-1.93%	-1.77%	Yes	
2007-08-15	0.00%	0.01%	0.90%	0.97%	0.9566	-1.49%	-1.58%	-1.97%	-1.77%		
2007-08-16	-0.02%	-0.01%	0.92%	0.98%	0.9576	-1.52%	-1.62%	-2.02%	-0.30%		
2007-08-17	-0.01%	-0.01%	0.92%	0.98%	0.9556	-1.52%	-1.63%	-2.03%	3.56%		
2007-08-20	0.02%	0.03%	0.95%	1.04%	0.9561	-1.54%	-1.69%	-2.12%	-0.33%		
2007-08-21	0.02%	0.04%	0.94%	1.04%	0.9554	-1.53%	-1.68%	-2.10%	-0.08%		
2007-08-22	0.02%	0.04%	0.94%	1.04%	0.9566	-1.53%	-1.67%	-2.10%	1.29%		
2007-08-23	0.03%	0.05%	0.95%	1.05%	0.9574	-1.53%	-1.67%	-2.10%	-0.44%		
2007-08-24	0.03%	0.04%	0.95%	1.05%	0.9570	-1.53%	-1.68%	-2.11%	1.05%		
2007-08-27	0.03%	0.04%	0.95%	1.05%	0.9570	-1.53%	-1.68%	-2.11%	-1.34%		
2007-08-28	0.02%	0.02%	0.96%	1.05%	0.9570	-1.55%	-1.71%	-2.14%	-2.80%	Yes	Yes
2007-08-29	0.00%	-0.01%	0.98%	1.09%	0.9596	-1.62%	-1.80%	-2.24%	2.25%		
2007-08-30	0.02%	0.02%	1.01%	1.11%	0.9614	-1.64%	-1.82%	-2.25%	-0.54%		
2007-08-31	0.01%	0.01%	1.01%	1.11%	0.9617	-1.65%	-1.83%	-2.26%	1.34%		
2007-09-04	0.03%	0.03%	1.01%	1.12%	0.9621	-1.64%	-1.81%	-2.25%	0.88%		
2007-09-05	0.03%	0.03%	1.02%	1.12%	0.9623	-1.64%	-1.81%	-2.25%	-1.05%		
2007-09-06	0.02%	0.02%	1.02%	1.13%	0.9636	-1.66%	-1.83%	-2.26%	0.40%		
2007-09-07	0.01%	0.02%	1.02%	1.12%	0.9638	-1.66%	-1.83%	-2.26%	-2.21%		
2007-09-10	-0.01%	0.00%	1.03%	1.15%	0.9646	-1.70%	-1.89%	-2.32%	-0.35%		
2007-09-11	-0.01%	-0.01%	1.03%	1.15%	0.9644	-1.71%	-1.90%	-2.33%	1.30%		
2007-09-12	0.01%	0.00%	1.04%	1.15%	0.9648	-1.71%	-1.89%	-2.33%	-0.19%		
2007-09-13	0.00%	0.00%	1.04%	1.15%	0.9648	-1.71%	-1.90%	-2.33%	0.39%		
2007-09-14	0.01%	0.00%	1.04%	1.15%	0.9638	-1.70%	-1.89%	-2.33%	0.54%		
2007-09-17	0.01%	0.00%	1.04%	1.15%	0.9650	-1.70%	-1.89%	-2.32%	-0.40%		
2007-09-18	-0.01%	-0.02%	1.04%	1.15%	0.9646	-1.71%	-1.90%	-2.33%	3.26%		
2007-09-19	0.02%	0.01%	1.08%	1.19%	0.9703	-1.75%	-1.95%	-2.36%	0.81%		
2007-09-20	0.03%	0.02%	1.08%	1.19%	0.9704	-1.74%	-1.94%	-2.36%	-0.71%		
2007-09-21	0.03%	0.02%	1.08%	1.19%	0.9705	-1.74%	-1.93%	-2.35%	0.57%		
2007-09-24	0.03%	0.02%	1.08%	1.19%	0.9706	-1.74%	-1.93%	-2.35%	-0.88%		
2007-09-25	0.02%	0.00%	1.08%	1.19%	0.9709	-1.75%	-1.95%	-2.36%	0.07%		
2007-09-26	0.02%	0.00%	1.08%	1.19%	0.9709	-1.76%	-1.95%	-2.36%	0.59%		
2007-09-27	0.02%	0.00%	1.08%	1.19%	0.9710	-1.75%	-1.95%	-2.36%	0.40%		
2007-09-28	0.02%	0.00%	1.08%	1.19%	0.9713	-1.75%	-1.95%	-2.36%	-0.38%		
2007-10-01	0.02%	0.00%	1.08%	1.19%	0.9714	-1.76%	-1.96%	-2.36%	1.25%		
2007-10-02	0.03%	0.01%	1.09%	1.19%	0.9716	-1.76%	-1.96%	-2.37%	0.04%		

Table 3: Result of Failure Frequency Test

	Number of Breaches (x)	Expected Breaches at 5% Confidence Level	The LR test statistic	Chi Square Critical Value	Test Results
VaR	50	40	2.447	3.84	Pass
CoVaR	29	40	3.507	3.84	Pass

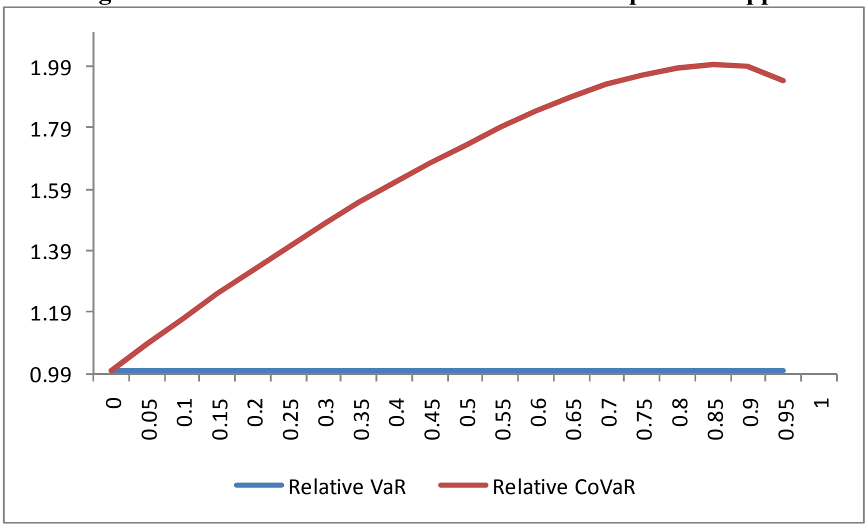
Table 4: The Estimates of the Probability

	n00	n01	n10	n11	pi_hat_01	pi_hat_11	pi_hat_2
VaR	704	45	46	4	0.06	0.08	0.06
CoVaR	743	28	29	0	0.04	0.00	0.04

Table 5: Result of Conditional Backtest

	LR_uc	LR_ind	LR_cc	Chi2(1)	Chi2(2)	UC Test	Ind Test	CC Test
VaR	2.447	5.018	12.277	3.84	5.99	Pass	Fail	Fail
CoVaR	3.507	N/A	N/A	3.84	5.99	Pass	Pass	Pass

Figure 1: Relative VaR and CoVaR Size with respect to Kappa



x – axis: κ
y – axis: Relative magnitude of VaR and CoVaR when κ = 0