Allocating Systematic and Unsystematic Risks in a Regulatory Perspective

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(First version, September 2010, revised September, 2011)

The first author gratefully acknowledges financial support of the NSERC Canada and the chair AXA/Risk Foundation : "Large Risks in Insurance". We thank M., Summer, D., Tasche for valuable feedback. The views expressed in this paper are those of the authors and do not necessarily reflect those of the Banque de France.

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Abstract

This paper discusses the contributions of financial entities to a global reserve from a regulatory perspective. We introduce axioms of decentralization, additivity and compatibility with risk ordering, which should be satisfied by the contributions of the entities and we characterize the set of contributions compatible with these axioms. Then, we explain how to disentangle the systematic and unsystematic risk components of these contributions. Finally, we discuss the usual relationship between baseline reserve and reglementary required capital, and propose alternative solutions to the question of procyclical required capital.

Keywords : Risk Measure, Allocation, Regulation, Systematic Risk, Procyclical Effect.

1 Introduction

The definition of the required capital in Basel regulation is often presented as an important reason of the development of the recent financial crisis. The following arguments are in particular invoked :

i) The regulatory capitals, which the banks were required to hold by the regulator, were not sufficiently large to cover the (extreme) risks.

ii) They did not account for comovement of financial institutions assets and liabilities, that is, for systematic risk factors.

iii) This regulation had a procyclical effect, instead of the expected counter-cyclical effect.

These possible drawbacks of the previous regulation explain the recent changes in both regulation and academic research. Examples are the introduction of additional regulators focusing on systemic risk, the various stress-testing performed in US as well as in European countries, or the new measures of systemic risk introduced in the academic literature.

The aim of our paper is to identify more precisely possible deficiencies of the regulation rules and to propose alternative methods. Let us first recall how Basel regulation defines the required capital of a given entity. In a first step each financial entity has to compute a measure of its own risk (and also such measures for each business line separately.) The standard risk measure used by these entities is the Value-at-Risk (VaR), which gives the maximum loss within a $\alpha\%$ confidence interval. The level α is fixed by the regulator and the risk of an individual entity is considered in isolation. Then, in a second step, the required capital is fixed from the observed individual history of VaR. A typical formula for required capital at day t is for instance :

$$RC_t = \max(VaR_t, k\frac{1}{60}\sum_{h=0}^{59} VaR_{t-h}), \tag{1.1}$$

where VaR_t denotes the VaR at horizon 1-year and the trigger parameter k depends on the technical level of the entity and is generally larger than 3. This nonlinear link function between the risk measure and the required capital induces two regimes : in a standard risk environment for the bank, formula (1.1) reduces to : $RC_t = k\frac{1}{60}\sum_{h=0}^{59} VaR_{t-h}$. The smoothing of risk measures over 60 opening days, i.e. 3 months, is introduced in order to avoid an erratic evolution of the reserves. Trigger coefficient k provides an additional insurance against risk. When the entity becomes suddenly very risky, that is, when the current risk measure VaR_t is larger than $\frac{k}{60}\sum_{h=0}^{59} VaR_{t-h}$, the required capital becomes equal to VaR_t.

The potential drawbacks of this kind of approach are threefold. First it seems questionable to consider each entity separately without any reference to its role in the global system. Our approach is a top-down method focusing on the way of defining the contributions of the entities to a global reserve needed for the whole system. The second drawback of the standard approach is the lack of distinction between systematic and unsystematic components of the risk, which will be separated in our approach. Third, the usual method may imply procyclicity that we try to avoid in our framework.

The paper is organized as follows. In Section 2, we explain how to allocate the global reserve among the entities. We introduce the decentralization, additivity and risk ordering axioms, which are relevant for this decomposition, and characterize the contributions satisfying the three axioms. In Section 3, we discuss the alternatives proposed in the literature to define the contributions from the regulatory perspective, such as the CoVar, Shapley values, or Euler allocation [3]. In Section 4 we derive a disaggregation formula not only in terms of entities, but also in terms of systematic and unsystematic risks, both in linear and nonlinear factor models. Section 5 explores the link between the required capital and the objective measures of systematic and unsystematic risks. In particular, we explain why the Through The Cycle (TTC) smoothing treatment of these components have to be performed separately to avoid the spurious procyclical effect of the standard regulation and we propose alternative solutions to avoid this procyclicity of the required capital. Section 6 concludes. Technical proofs are gathered in appendices.

[3]The research of an appropriate allocation of capital for a purpose internal to a bank , for instance to maximize shareholder value, achieve capital efficiency, or measure concentration risk in a portfolio [see e.g. Patrick et alii (1999), Dhaene, Goovaerts, Kaas (2003), Sherris (2007), Tasche (2008)] is clearly out of the scope of the present paper.

2 Disaggregation of a global reserve

Let us denote by $X_i, i = 1, \ldots, n$, the Loss and Profit (L&P) of the entities (banks). The global L&P, that is the L&P of the whole banking system is : $X = \sum_{i=1}^{n} X_i$. The global reserve for the system is $R(X)$. It is assumed to depend on the distribution P of the global L&P only. Thus, this global reserve can utilize the potential diversification benefits. In other words, the contribution of an entity in good health can be used to rescue another entity close to default. At this point, we do not discuss how the global reserve is computed. It can be a VaR, or a coherent risk measure, such an Expected Shortfall (ES), or something else (see Section 5). This global reserve has to be assigned to the different entities :

$$R(X) = \sum_{i=1}^{n} R(X, X_i), \tag{2.1}$$

say, where $R(X, X_i)$ denotes the contribution of entity i to the total reserve.

Moreover, we would like to decompose the individual contribution into :

$$R(X, X_i) = R_s(X, X_i) + R_u(X, X_i) + R_{s,u}(X, X_i), \tag{2.2}$$

where R_s (resp. R_u) denotes the contribution for marginal systematic (resp. unsystematic) risk and $R_{s,u}$ the contribution for the cross effects. We focus in this section and the following one on decomposition (2.1) and defer the main discussion on systematic risk to Sections 4 and 5.

2.1 A set of axioms

From an axiomatic point of view, it is important to distinguish the measure of the regulatory capital for the global risk, that is the function $R(.) : X \to R(X)$, and the contributions to the total reserve, that is, the function $R(X, .) : X_i \to R(X, X_i)$. These contributions are contingent to the total risk level and should satisfy at least a decentralization axiom, an additivity axiom and a risk ordering axiom.

i) Decentralization axiom

A1. Decentralization axiom : The individual contribution $R(X, X_i)$ of entity i depends on the joint distribution of (X, X_i), but is independent of the decomposition of $X - X_i$ into $\Sigma_{j \neq i} X_j$.

This axiom has been first introduced in Kalkbrener (2005). In a regulatory perspective, it has the advantage of allowing for a computation of $R(X, X_i)$ by entity i, while preserving a minimal confidentiality on the individual portfolios of the other entities.

More precisely, let us consider entities invested in stocks. The individual $L\&P's$ are : $X_i = Y'\gamma_i$, where Y denotes the vector of share values of the stocks, and γ_i is the portfolio composition of the loss for entity i. The reserves are usually evaluated for a crystallized portfolio, that is, with the composition γ_i^- existing at the beginning of period t (end of period $t-1$). With this practice, the regulator has to provide entity i with the type of measure $R(X, X_i)$ to consider, the past data on Y, and the sum $\sum_{j=1}^{n} \gamma_j^-$ corresponding to the global crystallized portfolio, without providing the individual information on competitors'portfolios $\gamma_j^-, \forall j \neq i$.

ii) Additivity axiom

The additivity axiom has been first introduced in Garman (1997) [see also Kalkbrener (2005), where it is called linear aggregation axiom].

A2. Additivity axiom :

$R(X) = \sum_{i=1}^{n} R(X, X_i)$, for any decomposition of X into $X = \sum_{i=1}^{n} X_i$.

Intuitively, the total reserve should not depend on the number of entities holding the risk and of their respective sizes, whenever the sum of these $L\&P's$ stays the same. The additivity axiom has several consequences.

i) For $n = 1$, we get : $R(X) = R(X, X)$.
ii) We also have :
$R(X) = R(X, X_1 + X_2) + R(X, X - X_1 - X_2)$

$= R(X, X_1) + R(X, X_2) + R(X, X - X_1 - X_2)$,
which implies :

$$R(X, X_1 + X_2) = R(X, X_1) + R(X, X_2), \forall X_1, X_2. \tag{2.3}$$

This is the additivity property of the function $R(X, .) : X_i \to R(X, X_i)$, for any given X. By imposing the additivity axiom, any merging, or demerging of entities without effect on global risk provides no spurious advantage in terms of contribution to the total reserve.

Note that standard stand-alone risk measures such as the $VaR(X_i)$ or any other coherent risk measure cannot be chosen for defining contributions. Indeed, they are subadditive [Artzner et al. (1999), Acerbi, Tasche (2002)], and thus do not satisfy the axiom of additivity. The main reason is that the standard measures of individual risk such as the VaR and the expected shortfall are not contingent to the level of global risk. According to Tasche (2008) (Remark 17.2), the choice of stand-alone risk measure values as risk contributions would punish more the banks which improve the diversification of the global regulatory portfolio.

iii) Risk ordering axiom

The contributions have also to be compatible with an appropriate notion of stochastic dominance. Intuitively, the contribution has to take into account not only the individual risk of entity i, but also its hedging potential with respect to the set of other entities. Thus, we have to introduce a directional notion of stochastic dominance valid for an individual risk X_1, say, and a given global $L\&P : X$. For this purpose, let us consider the virtual decomposition of the global portfolio into X_1 and $\tilde{X}_2 = X - X_1$ obtained by aggregating the $L\&P$ of the other entities.

Definition 2.1 : Let us consider the $L\&P$: X, X_1, X_1^*. We say that X_1^* stochastically dominates X_1 at order 2 with respect to X, if and only if :

$$E[\tilde{U}(X_1^*, X - X_1^*)|X] \geq E[\tilde{U}(X_1, X - X_1)|X],$$

for any concave function $\tilde{U}$.

This is a second-order stochastic dominance [Rothschild, Stiglitz (1970)], applied to virtual portfolios $X_1, \tilde{X}_2$, whose allocations are constrained to sum up to a given X. The directional stochastic partial ordering is denoted by $\succeq_X$.

The directional stochastic dominance can be characterized in simpler ways (see Appendix 1).

Proposition 2.2 : We have the following equivalences :
i) $X_1^* \succeq_X X_1$;
ii) $E[U(X_1^*)|X] \geq E[U(X_1)|X]$, for any concave function U;
iii) There exists a variable Z such that :

$$X_1 = X_1^* + Z, \text{ with } E(Z|X, X_1^*) = 0.$$

Proposition 2.2 shows that the directional stochastic dominance is equivalent to the standard second-order stochastic dominance applied to the conditional distribution of X_1 given X [see Rothschild, Stiglitz (1970)].

The next axiom concerns the compatibility of the contribution with the directional stochastic dominance.

Risk ordering axiom

We have $R(X, X_1^*) \leq R(X, X_1)$ for any pair of entity risks such that $X_1^* \succeq_X X_1$.

iv) Restrictions implied by the set of axioms

The decentralization and additivity axioms imply rather strong restrictions on the contributions as shown by the next Proposition.

Proposition 2.3 : Under the decentralization and additivity axioms, we have :

$$R(X, X_1 + Z) = R(X, X_1),$$

for any variable Z independent of (X, X_1) with a symmetric distribution.

Proof : We have the equalities :

$$\begin{aligned} R(X) &= R(X, X_1) + R(X, X_1) + R(X, X - 2X_1) \\ &= R(X, X_1 + Z) + R(X, X_1 - Z) + R(X, X - 2X_1). \end{aligned}$$

Since the joint distributions of the pairs $(X, X_1 + Z)$ and $(X, X_1 - Z)$ are the same under the assumptions of Proposition 2.3, we deduce :

$$R(X, X_1) = R(X, X_1 + Z).$$

QED

The result in Proposition 2.3 shows the difference between a marginal measure of risk such as a VaR and a contribution. By passing from X_1 to $X_1 + Z$, we increase marginally the risk for the second order stochastic dominance. However, due to the additivity and decentralization axioms, this increase has not been taken into account in the contribution. This is due to the compensation between X_1 and the $L\&P$ of the other entities.

The next proposition characterizes the contributions satisfying the decentralization, additivity and risk ordering axioms.

Proposition 2.4 : Let us assume that the $L\&P$: X, X_i belong to a space $L^2(Y)$, where Y is a given set of random variables. If the contribution function $R(X, .) : X_i \to R(X, X_i)$ is continuous with respect to X_i for the L^2-norm, then, the contributions satisfying Axioms A1, A2, A3 are such that $R_{\mu_P}(X, X_i) = \int E(X_i|X = x)\mu_P(dx)$, where μ_P is a measure (not necessarily a probability measure), which can depend on the distribution P of X and is such that $\int x\mu_P(dx) = R(X)$.

Proof : See Appendix 2,

The condition $X_i \in L^2(Y)$ means that the portfolios of interest are written on some basic assets Y, possibly including derivatives with nonlinear (square integrable) payoffs.

When the distribution of X is continuous, with a strictly increasing cumulative distribution function (cdf), the contributions in Proposition 2.4 can be written in terms of quantiles.

Corollary 2.5 : The contributions satisfying the three axioms are :

$$R_{\nu_P}(X, X_i) = \int E[X_i|X = q_\alpha(X)]\nu_P(d\alpha),$$

where ν_P is a measure, which can depend on the distribution P of X and is such that :

$$\int q_\alpha(X)\nu_P(d\alpha) = R(X).$$

Proof : The result is obtained by applying the change of variable $x = q_\alpha(X)$.

QED

The equality $R(X) = \int q_\alpha(X)\nu_P(d\alpha)$ implies that $R(X)$ is a weighted quantile. ν_P looks like a distortion measure, except that it can depend on the distribution of X, since the contribution is contingent to X. It will be called the **allocation distortion measure** (ADM) in the rest of the paper.

More precisely, let us assume for illustrative purpose that the global risk measure is equal to a distortion risk measure (DRM) that is, a weighted combination of VaR's [see Wang (2000), Acerbi (2002)] :

$$R(X) = DRM_H(X) = \int q_{\alpha^*}(X)H(d\alpha^*),$$

where H denotes a distortion (or spectral) probability measure on $(0, 1)$. This measure H is fixed independently of the distribution P of the global risk. Corollary 2.5 is saying that we do not have necessarily to use the fixed distortion measure H defining the DRM as the allocation distortion risk measure. Moreover, a given level of global reserve, 2 billions \$, say, can be seen as the value of a VaR as well as the value of an ES[4], and more generally as the value of an infinite number of alternative DRM, whenever $R(X) = \int q_\alpha(X)\nu_P(d\alpha)$.

Corollary 2.6 : The contributions defined in Proposition 2.4 do not depend on μ_P when $(X, X_1, \ldots, X_n)$ is Gaussian and are all equal to $E(X_i) + \frac{Cov(X_i, X)}{V(X)}[R(X) - E(X)]$.

[4]Indeed the functions $\alpha \to VaR_\alpha$ and $\alpha \to ES_\alpha$ are continuous increasing functions of the critical value α; therefore, there exist two critical values α_1, α_2 such that :

$$VaR_{\alpha_1}(X) = ES_{\alpha_2}(X) = \text{ 2 billions \$.}$$

Proof : it is a consequence of the formula $E[X_i|X = x] = E(X_i) + \frac{Cov(X_i, X)}{V(X)}(x - X_i)$.

QED

3 Related literature

There exist three streams of literature for defining risk measures contingent to the level of global risk. Some authors propose alternative sets of axioms to be satisfied by the contributions. Other authors focus on the direct introduction of contingent reserve levels $R(X, X_i)$. The CoVaR or the computation of the reserve based on the Shapley value belong to this literature. Finally, it is possible to directly introduce a decomposition formula (2.1) of the total reserve, and try to interpret ex-post the elements $R(X, X_i)$ in this decomposition. The Euler allocation provides an example of this approach. Let us briefly review these approaches.

3.1 Alternative sets of axioms

Other sets of axioms have been considered in the literature. For instance Kalkbrener (2005) [see also Hesselager, Andersson (2002), Furman, Zitikis (2008) for a similar approach] proved the uniqueness of the contribution under an additional continuity assumption and the fact that :

Diversification axiom :

$$R(X, X_i) \leq R(X, X) = R(X), \forall X_i.$$

Under these additional conditions, the global reserve $R(.) : X \to R(X) = R(X, X)$ is necessarily a subadditive function.

We do not introduce this diversification axiom in our approach. Indeed, since :

$$R(X) = R(X, X_1) + R(X, X_2), \text{ when } X_1 + X_2 = X,$$

the diversification axiom implies the nonnegativity of any contribution. It is important to leave open the possibility of a negative contribution [5], when a given entity is more prudential than deemed necessary. Typically, if the entity portfolio includes mainly a riskfree asset, the contribution will be negative, which means the authorization for an increased leverage without requiring positive reserve in liquid riskfree asset.

Moreover, as for the notion of coherent risk measure [see Artzner et al. (1999)], the axiomatic has to restrict the set of possibilities to easily interpretable allocations, not necessarily to provide a unique solution.

3.2 The CoVaR

Adrian and Brunnermeier (2009) propose to analyze the risk of the entities, when the system is in distress. More precisely, let us denote by $q_\alpha(X)$ the α-quantile corresponding to the system[6]. The CoVaR for entity i and confidence level α when the system is in distress is defined by :

$$P[X_i < CoVaR_{i|s,\alpha}(X)|X = q_\alpha(X)] = \alpha. \tag{3.1}$$

The CoVar is generally larger than the VaR of the entity. It is implicitly proposed to choose their difference, called ΔCoVar, as the contribution to systematic risk :

$$R_s(X, X_i) = CoVaR_{i|s,\alpha}(X) - q_\alpha(X_i), \tag{3.2}$$

with $R_u(X, X_i) = q_\alpha(X_i)$. In this case the contribution of entity i to the total reserve i is :

$$R(X, X_i) = CoVaR_{i|s,\alpha}(X).$$

The ΔCoVar is an interesting measure of systematic risk, but the CoVaR itself cannot be used directly for defining the contribution. Even if the decentralization axiom is satisfied, the CoVaR as the VaR based approaches are bottom-up approaches. In particular, the associated total re-

[5] See e.g. Uryasev, Theiler, Serrano (2010) for a practical example of negative contribution.

[6] A similar approach can be based on another risk measure such as the Expected Shortfall [see e.g. Kim (2010)].

serve $\sum_{i=1}^{n} CoVaR_{i|s,\alpha}(X)$ is not a function of the distribution of the total risk only and the additivity axiom is not satisfied.

3.3 The Shapley value

A Shapley value [Shapley (1953)] is a fair allocation of gains obtained by cooperation among several actors. Let us assume that all actors $i = 1, \dots, n$ accept to cooperate and introduce a superadditive value function $v(S)$, which measures the gain of this cooperation for a coalition $S \subset \{1, \dots, n\}$. The superadditivity condition :

$$v(S \cup T) \geq v(S) + v(T),$$

expresses the fact that cooperation can only be profitable.

The Shapley value is one way to distribute the total gains of the players, if they all collaborate, by demanding for each actor i a contribution

$v(S \cup \{i\}) - v(S)$ as a fair compensation to join coalition S. The Shapley value is defined as a mean of these compensations over all possible coalitions :

$$V_i = \sum_{S \subset \{1,\dots,n\} \setminus \{i\}} \left\{ \frac{|S|!(n - |S| - 1)!}{n!} [v(S \cup \{i\}) - v(S)] \right\}, \tag{3.3}$$

where $|S|$ denotes the number of actors in coalition S. Denault (2001), Koyluoglu, Stocker (2002), Tarashev et alii (2009) (with the Varying Tail Events procedure) propose the Shapley value as a fair allocation of the reserve with $v(S) = -R(\Sigma_{i \in S} X_i)$, and $R(.)$ a risk measure such as a VaR, or an expected shortfall.

In a regulatory perspective, the drawback of this approach is twofold :

(*) It assumes a total cooperation of the entities, which are in practice competitors.

(**) The Shapley allocation does not satisfy the decentralization, i.e. confidentiality, axiom which intuitively is not compatible with a total cooperation.

In a regulatory perspective, such a Shapley allocation could only be implemented by the regulator itself due to the confidentiality restriction. This would lead to a highly centralized computation of the contributions by the regulator itself, but is clearly not implementable in practice. First, the regulator does not possess the technical departments to make such computations for all entities. Second, such a centralized approach contradicts the spirit of the second Pillar of Basel 2 regulation, where the entities have to learn how to manage and control their internal risks by themselves.

3.4 Euler allocation

As noted in Litterman (1996), p28, and Garman (1997), footnote 2, if the function $R(.)$ defining the total reserve is homogenous of degree 1, that is, satisfies the condition :

$$R(\lambda X) = \lambda R(X), \forall \lambda > 0, \tag{3.4}$$

or, equivalently, $R^*(\lambda e) = \lambda R^*(e), \forall \lambda > 0$,

where $R^*(\lambda_1, \ldots, \lambda_n) = R(\sum_{i=1}^{n} \lambda_i X_i)$, we get the Euler condition :

$$R^*(e) = \sum_{i=1}^{n} \frac{\partial R^*(e)}{\partial \lambda_i}, \tag{3.5}$$

obtained by differentiating both sides of equation (3.4) with respect to λ. This provides a decomposition of the total reserve as the sum of its sensitivities corresponding to shocks performed separately on each entity. This justifies the terminology Euler allocation used in McNeil et al. (2005), Section 6.3.

Let us consider this decomposition for a global reserve defined by a distortion risk measure :

$$R(X) = DRM_H(X) = \int q_\alpha(X) H(d\alpha), \tag{3.6}$$

where H denotes a distortion (probability) measure on (0,1).

The VaR and more generally any DRM is homogenous function of degree 1. Thus, the total reserve can be decomposed into :

$$DRM_H(X) = \sum_{i=1}^{n} DRM_{H,i}, \tag{3.7}$$

where the marginal expected distortion risk measures are given by :

$$DRM_{H,i} = \int q_{\alpha,i} H(d\alpha), \tag{3.8}$$

$$\text{with } \; q_{\alpha,i} = \frac{\partial q^*_\alpha(e)}{\partial \lambda_i} \text{ and } q^*_\alpha(\lambda_1, \ldots, \lambda_n)' = q_\alpha(\sum_{i=1}^{n} \lambda_i X_i), . \tag{3.9}$$

The following result has been derived in Gourieroux, Laurent, Scaillet (2000) (see also the beginning of Appendix 3, formula a.3).

Proposition 3.1 :

$$q_{\alpha,i} = \frac{\partial q^*_\alpha(e)}{\partial \lambda_i} = E[X_i | X = q_\alpha(X)].$$

and

$$DRM_{H,i} = \int E[X_i | X = q_\alpha(X)] H(d\alpha).$$

This Euler allocation applied to a DRM satisfies the three axioms of Section 2.1. However, it is rather restrictive, since **it assumes that function** $R(.)$ **is a DRM and corresponds to the choice of an allocation distortion measure equal to the distortion measure itself.**

The Euler decomposition formula (3.7) is in particular valid for the expected shortfall, for which the distortion measure is the uniform distribution on the interval $(\alpha, 1)$ [Wang (2000), Acerbi, Tasche (2002)] :

$$ES_\alpha(X) = \frac{1}{1-\alpha} \int_\alpha^1 q_{\alpha*}(X) d\alpha^*. \tag{3.10}$$

It can be checked that the marginal expected shortfall is equal to :

$$\begin{aligned} MES_i &= \frac{1}{1-\alpha} \int_\alpha^1 E[X_i | X = q_{\alpha*}(X)] d\alpha^* \\ &= E[X_i | X > q_\alpha(X)]. \end{aligned} \tag{3.11}$$

This simplified expression of the marginal expected shortfall has been first derived by Tasche (2000) [see also Kurth, Tasche (2003) and Appendix 3].

The Euler decomposition of the VaR and the Expected Shortfall above differ essentially by the conditioning set. They stress that the additive decompositions involve both conditioning with respect to system distress [as in definition (3.4) of the CoVaR] and conditional expectations (instead of conditional quantiles as in the CoVaR approach) to ensure the additivity property.

However, the homogeneity assumption of the global risk measure $R(.)$ is questionable, especially if the regulation is used for economic policy. As an illustration, let us assume that the portfolios of interest include the different types of credits. From a macroeconomic point of view, there exists an optimal level for the global amount of credit to be distributed in the economy. The global risk measure has to be chosen as an incentive to reach this optimal level. The cost of the reserve has to be small, if the current amount of credit is below this optimal level, large, otherwise. Mathematically, we expect function $\lambda \to R(\lambda X)/\lambda$ to be increasing in λ, not constant. For instance $R_c(X) = E(X) + E[(X-c)^+]$, where c is the "optimal level" would satisfy this condition[7].

It is important to note that the condition on function $R(.)$ assumed in Corollary 2.5 does not imply the homogeneity of degree 1. More precisely, we have :

$$\begin{aligned} R(\lambda X) &= \int q_\alpha(\lambda X)\nu_{P_\lambda}(d\alpha) \\ &= \lambda \int q_\alpha(X)\nu_{P_\lambda}(d\alpha) \\ &\neq \lambda \int q_\alpha(X)\nu_{P_1}(d\alpha) = \lambda R(X), \end{aligned}$$

since the ADM can depend on the distribution of X and thus on λ.

[7]The homogeneity assumption is sometimes justified by an invariance of the risk measure with respect to a change of money unit. Function $R_c(X)$ satisfies this latter condition which involves the changes $X \to \lambda X$ and $c \to \lambda c$, since the optimal level is also written in money unit, but is not homogenous of degree 1 in X for fixed c.

4 Contribution to systematic risk

The axiomatic approach of Section 2, or the Euler allocation of Section 3.4 provide contributions of individual entities satisfying the axioms, but do not try in general to reallocate the global risk between systematic and unsystematic components. The aim of this section is to explain how the allocation principle can be applied to disentangle the systematic and unsystematic components of the risk. For expository purpose, we first consider models with linear factors driving the systematic risk which are usually considered when we focus on market risk. Then, the approach is extended to nonlinear factors. Nonlinear factor models are involved whenever options and/or credit risks are considered.

4.1 Linear factor model

Let us first consider a linear factor model. The individual $L\&P's$ can be decomposed as :

$$X_i = \sum_{k=1}^{K} \beta_{ik} f_k + \gamma_i u_i, \tag{4.1}$$

where $f_1, \ldots, f_K$ are systematic factors and $u_1, \ldots, u_n$ idiosyncratic terms with $K < n$. These factors are random at the beginning of period t, and observed at the end of this period. For instance, for fixed income derivatives, the main risk factors can be the interest rate level, slope and curvature, the spreads over T-bond rates, the exchange rates... The total L&P can be decomposed as :

$$X = \sum_{k=1}^{K} \beta_k f_k + \sum_{i=1}^{n} \gamma_i u_i, \tag{4.2}$$

with $\beta_k = \sum_{i=1}^{n} \beta_{ik}$.

i) Euler allocation of a VaR global risk measure

For expository purpose, let us first assume that the global risk measure is a VaR : $R(X) = q_\alpha(X)$.

The α-quantile of X is a function :

$$q_\alpha(X) = q_\alpha^*\left(\beta_1, \ldots, \beta_K, \gamma_1, \ldots, \gamma_n, \theta\right), \tag{4.3}$$

where θ denotes the parameters characterizing the joint distribution of $f_1, \ldots, f_K, u_1 \ldots, u_n$.

The marginal effect of an homothetic change of exposure of the entities passing from X_i to $\lambda X_i = \sum_{k=1}^{K} \lambda\beta_{ik} f_k + \lambda\gamma_i u_i, i = 1, \ldots, n$ gives :

$$\begin{aligned} q_\alpha(X) = \left[\frac{dq_\alpha(\lambda X)}{d\lambda}\right]_{\lambda=1} &= \sum_{k=1}^{K}(\sum_{i=1}^{n}\beta_{ik})\frac{\partial q_\alpha(X)}{\partial\beta_k} + \sum_{i=1}^{n}\gamma_i\frac{\partial q_\alpha(X)}{\partial\gamma_i} \quad (4.4) \\ &= \sum_{i=1}^{n} q_{\alpha,i}. \end{aligned}$$

where the composite term :

$$q_{\alpha,i} \equiv \sum_{k=1}^{K} \beta_{ik}\frac{\partial q_\alpha(X)}{\partial\beta_k} + \gamma_i\frac{\partial q_\alpha(X)}{\partial\gamma_i}, \tag{4.5}$$

shows how the VaR Euler contribution $q_{\alpha,i}$ of entity i can be decomposed in order to highlight the effects of systematic factors and idiosyncratic term.

In some sense, decomposition formulas (4.4)-(4.5) explain how to pass from Euler allocations computed by entity to Euler allocations computed by "virtual business lines" associated with the different risk factors, as summarized in Table 1. In other words, we propose to treat in a symmetric way the contributions to global risk of both risk factors and entities.

Table 1 : Euler decomposition of a global VaR

entity risk factor	1	i	n	risk contribution of the factors
f_1 $\vdots$ f_k $\vdots$ f_K		$\beta_{ik}\dfrac{\partial q_\alpha(X)}{\partial \beta_k}$		$\beta_k\dfrac{\partial q_\alpha(X)}{\partial \beta_k}$
u_1 $\vdots$ u_i $\vdots$ u_n	0	$\gamma_i\dfrac{\partial q_\alpha(X)}{\partial \gamma_i}$	0	$\gamma_i\dfrac{\partial q_\alpha(X)}{\partial \gamma_i}$
risk contribution of the entities		$q_{\alpha,i}$		$VaR(X)$

By applying the expression of the VaR sensitivity (see Proposition 3.1), we get as a by-product the Euler components associated with systematic and unsystematic risks, respectively, as :

$$\begin{aligned} R(X) &= R_s(X) + R_u(X), \\ \text{with } R_s(X) &= \sum_{i=1}^n R_s(X, X_i), R_u(X) = \sum_{i=1}^n R_u(X, X_i), \end{aligned}$$

$$R_s(X, X_i) = \sum_{k=1}^K \beta_{ik} \frac{\partial q_\alpha}{\partial \beta_k}(X) = \sum_{k=1}^K \beta_{ik} E[f_k | X = q_\alpha(X)], \tag{4.6}$$

$$R_u(X, X_i) = \gamma_i \frac{\partial q_\alpha}{\partial \gamma_i}(X) = \gamma_i E[u_i | X = q_\alpha(X)], \tag{4.7}$$

ii) General case

The example of Euler allocation of a VaR discussed in Table 1 provides a principle of allocation between systematic and unsystematic risks valid in a general framework where function $R(.)$ is not necessarily a DRM, or even an homothetic function. Indeed, whenever we have $R(X) = \int q_\alpha(X)\nu_P(d\alpha)$, we can use the contribution derived in Corollary 2.5, which takes a form similar to a contribution associated with an Euler allocation. The idea is to define the new entities by crossing the initial entity and the type of risk. The $L\&P$ for entity i and systematic risk (resp. unsystematic risk) is : $X_{s,i} = \sum_{k=1}^{K} \beta_{ik} f_k$(resp. $X_{u,i} = \gamma_i u_i$). The components of the total $L\&P$ are defined accordingly by : $X_s = \sum_{i=1}^{n} X_{s,i}$ and $X_u = \sum_{k=1}^{n} X_{u,i}$. Then, for a given ADM ν_P, the allocations are defined by :

$$
\begin{aligned}
R_{\nu_{P,s}}(X, X_i) &= R_{\nu_P}(X, X_{s,i}), \\
R_{\nu_{P,u}}(X, X_i) &= R_{\nu_P}(X, X_{u,i}), \\
R_{\nu_P}(X, X_i) &= R_{\nu_{P,s}}(X, X_i) + R_{\nu_{P,u}}(X, X_i), \\
R_{\nu_{P,s}}(X) &= \sum_{i=1}^{n} R_{\nu_{P,s}}(X, X_i) = R_{\nu_P}(X, X_s), \\
R_{\nu_{P,u}}(X) &= \sum_{i=1}^{n} R_{\nu_{P,u}}(X, X_i) = R_{\nu_P}(X, X_u).
\end{aligned}
$$

with $R_{\nu_P}(X, .) = \int E[.|X = q_\alpha(X)]\nu_P(d\alpha)$ (see Corollary 2.5).

These contributions correspond to the axiomatic of Section 3 applied to departments specialized in systematic (resp. unsystematic) risk components. It is justified whenever there exist on the market financial products introduced to hedge the systematic factor.

4.2 Nonlinear factor model

The allocation of the global reserve among the entities can be done for both linear and nonlinear factor models. However, the allocation between systematic and unsystematic components is less than obvious if :

$$X_i = g_i(f, u_i), \tag{4.8}$$

where the (multidimensional) factor f and the idiosyncratic term u_i are independent, but function g_i is nonlinear, due to the presence of cross effects.

Nevertheless, it is possible to decompose the individual $L\&P$ as :

$$\begin{aligned} X_i &= E(X_i|f) + [E(X_i|u_i) - E(X_i)] + [X_i - E(X_i|f) - E[X_i|u_i] + E(X_i)] \\ &\equiv X_{s,i} + X_{u,i} + X_{u,s,i}, \text{say}, \end{aligned} \tag{4.9}$$

where $X_{s,i}, X_{u,i}, X_{s,u,i}$ are the marginal systematic and unsystematic effects, and the cross effect, respectively [8]. Even if the systematic factor f and the idiosyncratic terms u_i are independent, interaction effects will appear in the risk contributions due to the nonadditive decomposition.

Then, the total contribution of entity i can be decomposed as :

$$R_{\nu_P}(X, X_i) = R_{\nu_{P,s}}(X, X_i) + R_{\nu_{P,u}}(X, X_i) + R_{\nu_{P,s,u}}(X, X_i), \tag{4.10}$$

where :

$$\begin{aligned} \text{with} \quad R_{\nu_{P,s}}(X, X_i) &= R_{\nu_P}(X, X_{s,i}) \\ R_{\nu_{P,u}}(X, X_i) &= R_{\nu P}(X, X_{u,i}) \\ R_{\nu_{P,s,u}}(X, X_i) &= R_{\nu_P}(X, X_{u,s,i}) \\ \text{and} \quad R_{\nu_p}(X, .) &= \int E[.|X = q_\alpha(X)]\nu_P(d\alpha) \end{aligned}$$

[8]This type of decomposition can also be used to distinguish the effects of dependent systematic factors [see Rosen, Saunders (2010), Section 4.5]

This decomposition does not depend on the selected representations of the factor and the idiosyncratic term, that is, the decomposition is invariant when either f, or u_i is transformed by a one-to-one transformation.

As an illustration, let us consider a model with stochastic drift and volatility driven by a same factor f :

$$X_i = m_i(f) + \sigma_i(f)u_i, i = 1, \ldots, n,$$

where f is independent of $u = (u_1, \ldots, u_n)'$, and the errors are iid zero mean. We have, using $R(X) = q_\alpha(X)$:

$$R_s(X, X_i) = E[m_i(f)|X = q_\alpha(X)],$$

$$R_u(X, X_i) = E[E\{\sigma_i(f)\}u_i|X = q_\alpha(X)],$$

$$R_{s,u}(X, X_i) = E[(\sigma_i(f) - E\{\sigma_i(f)\})u_i|X = q_\alpha(X)].$$

This example shows that in a nonlinear model, the effect of the systematic factor is captured by both R_s and $R_{s,u}$. Their relative magnitude can be highly different for the different entities. For instance in a basic stochastic volatility model, where $m_i(f) = 0$, only the cross effect matters.

4.3 Large number of entitites

Finally, let us discuss the case of a large number n of similar entities. If n is large, and the entities of similar sizes, we deduce from the Law of Large Numbers (LLN) that the idiosyncratic terms can be diversified, whereas the systematic factors cannot be. For expository purpose, let us consider a single linear factor model with $\gamma_i = 1, \forall i$. We have :

$$X = (\sum_{i=1}^{n} \beta_i)f + \sum_{i=1}^{n} u_i.$$

Let us assume that the beta coefficients are i.i.d. with a positive mean $E(\beta) > 0$, and are independent of factor f and idiosyncratic errors $u_i, i = 1, \ldots, n$. Let us also assume that these errors are independent with zero mean $E(u_i) = 0$. The contributions are equal to :

$$\begin{aligned} R_{\nu_P}(X, X_i) &= \int E[X_i|X = q_\alpha(X)]\nu_P(d\alpha) \\ &= \int E[X_i|X/n = q_\alpha(X/n)]\nu_P(d\alpha), \end{aligned}$$

since the quantile function is homogenous of degree 1[9].

By the LLN, we deduce that :

$$\lim_{n\to\infty}(X/n) = E(\beta)f.$$

Thus, the idiosyncratic part has been diversified, whereas the effect of systematic risk persists asymptotically.

When $n = \infty$, we get :

$$\begin{aligned} \lim_{n\to\infty} E[X_i|X = q_\alpha(X)] &= \lim_{n\to\infty} E[X_i|X/n = q_\alpha(X/n)] \\ &= E[X_i|E(\beta)f = q_\alpha[E(\beta)f]] \\ &= E[\beta_i f + u_i|E(\beta)f = q_\alpha[E(\beta)f] \\ &= E[\beta_i f|E(\beta)f = q_\alpha[E(\beta)f] \\ &= \lim_{n\to\infty} E[\beta_i f|X = q_\alpha(X)] \\ &= \lim_{n\to\infty} E[X_{s,i}|X = q_\alpha(X)]. \end{aligned}$$

In this limiting case, the contribution for entity i and the contribution for its systematic component coincide, that is, $R_{\nu_P}(X, X_i) = R_{\nu_{P,s}}(X, X_i)$. Moreover, it is equivalent to condition either on X, or on the factor summary $E(\beta)f$, or on factor f itself.

The derivation above helps to understand the contribution for systematic risk used in Acharya et alii (2010), Brownlees, Engle (2010), which underlies

[9]The distribution of X depends on size n, but this dependence is not indicated for expository purpose.

the daily updated systematic risk ranking published by NYU Stern's Volatility Lab. [www.systemicrisisranking.stern.nyu.edu], defined by [10] $E[X_i|X = q_\alpha(X)]$. This definition is valid under the assumptions above, i.e. when the unsystematic component can be diversified and when factor f can be identified to the market portfolio.

For n large, but finite, it is possible by applying granularity theory [Gagliardini, Gourieroux (2010)], to evaluate at order $1/n$ the difference between $E[X_{s,i}|X = q_\alpha(X)] = E[\beta_i f|X = q_\alpha(X)]$, and $E[X_i|X = q_\alpha(X)]$.

5 Required Capital

As mentioned in the introduction, the current regulation defines the required capital in two steps. A dated measure of individual risk is first computed. This measure generally features an erratic behavior. Then a partial smoothing is applied to derive the required capital.

This practice does not distinguish the systematic and unsystematic components of the risk. We consider this question in Section 5.1, when the dated risk measures are time independent Euler allocations of a DRM global risk measure. In particular, we explain why the systematic and unsystematic components have to be smoothed differently to avoid spurious procyclical effects. Then in Section 5.2 we discuss how a part of this effect might be avoided by considering an appropriate path dependent global risk measure.

5.1 Change of link function

The link function [see equation (1.1) for a typical example] is a crucial element of the regulatory policy. Whereas the trigger parameter is a control variate for more or less prudential policy, the smoothing can be used to decrease, or increase the effects of cycles.

The recent financial crisis revealed important drawbacks of a link function like (1.1) :
i) The trigger parameter k has been fixed, independently of the market environment. We would have expected a reduced value during a liquidity crisis.
ii) As already mentioned in the introduction, the link function implies two regimes that are a smoothed and an unsmoothed regime, the latter one ap-

[10] Their analysis concerns stock market. X_i is the capitalization in stock i, whereas X is the market portfolio value.

pearing with a large increase of the risk of entity i. However, the consequences are not the same if this risk increase is due to an idiosyncratic shock, or to a shock on a systematic factor. In the first situation, there is an additional demand of liquid asset by entity i, which can be easily satisfied by the market. In the second situation, there is the demand for liquid asset by several entities together, which may force financial institutions to deliver at fire-sale prices, creates the deleveraging spiral, that is, selling assets to reduce the debt, and accentuates the cycles and the crisis.

Clearly the drawback of the link function (1.1) is the lack of distinction between systematic and unsystematic risk, that is, of the micro and macro prudential approaches.

Let us assume that the dated individual risk measures are the Euler allocations of a time independent global DRM. Then functions $R(.)$ and $R(X,.)$ are time independent and the values $R(X_t, X_{i,t})$ are Point-In-Time (PIT) measures of risk. Instead of a formula of the type :

$$RC_{i,t} = \max[R(X_t, X_{i,t}), k_t \frac{1}{60} \sum_{h=0}^{59} R(X_{t-h}, X_{i,t-h})], \tag{5.1}$$

an improved formula has to separate the two types of risks, in order to take into account the fact that the systematic components are highly dependent, and has to apply the two regime formula to the unsystematic component only [11]. An alternative to formula (5.1) might be :

$$\begin{aligned} RC_{i,t} &= \max[R(X_t, X_{i,u,t}), k_{u,t} \frac{1}{60} \sum_{h=0}^{59} R(X_{t-h}, X_{i,u,t-h})] \\ &+ k_{s,t} \frac{1}{H} \sum_{h=0}^{H-1} R(X_{t-h}, X_{s,i,t-h}) \equiv RC^u_{i,t} + RC^s_{i,t}. \end{aligned} \tag{5.2}$$

The first component concerning the unsystematic risk is sufficient to penalize the risky investments specific to a given entity, while avoiding a too volatile evolution of the associated required capital. The second component is linear to account for the strong serial dependence between the systematic components. The smoothing window H is introduced for another purpose,

[11] To simplify the discussion, we assume a linear factor model, therefore a zero cross term : $R_{s,u} = 0$.

not for avoiding a too volatile required capital, but for obtaining a counter-cyclical effect. In this respect, the smoothing window for systematic risk has too be much larger than the usual 3-month (i.e. $H = 60$), and able to cover a significant part of the cycle (one year for instance). Finally, the parameter $k_{s,t}$ should be dependent of the position within the cycle, that is smaller in the bottom of the cycle, larger in its top, to avoid the spurious creation of a liquidity gap.

The required capital $RC_{i,t}$ has to be provided to the regulators in liquid, high rated assets. Intuitively the components $RC^s_{i,t}$ and $RC^u_{i,t}$ have to be included in different accounts of the Central Bank : the unsystematic component should be in an account specific to entity i, but all contributions for systematic risk might be mutualized at least at the country level. Indeed, this account will serve to insure the global system and a mutualization is usual for an unfrequent catastrophic event. If an entity is close to failure due to a systematic effect, the total reserve for systematic component should be used to avoid the failure of the entity and also some potential failures of the other ones by contagion.

Even if the application of different link functions to the systematic and unsystematic risk components seems relevant, its implementation will encounter the same difficulties than the stress testing. Indeed, the systematic factors have to be defined in a same way for all the entities by the regulators and there are many common risk factors which can be considered.

5.2 Change of global risk measure

The usual regulatory approach distinguishes the underlying baseline allocations and the required capital. The aim is to pass from PIT measures of the type $R(X_t, X_{i,t})$, which depend on the dated distribution of $(X_t, X_{i,t})$, to the Through The Cycle (TTC) measures $RC_{i,t}$, which depend on the current and lagged dated distributions of risks. However, this two step approach lacks coherency. For instance, the additivity property satisfied by the baseline contribution is not satisfied by the required capital, since the link function is nonlinear. Is it possible to develop a one-step TTC coherent approach ?

A possible answer is to define more precisely the global risk function $R_t(X_t)$ of date t in a regulatory perspective. Let us consider the framework of Section 4 with underlying factors driving the systematic risk. These factors have to be known and observed ex-post by the regulators. Thus, the regulators have an augmented information set including both the current and lagged values of the global $L\&P, X$, and factors F. Their risk measures should not only take into account the current level of risk X, but also the comparison of this level with the position in the cycle, function of F. In other words, the assumption, that the global reserve $R(.)$ is time independent function of the current distribution of risk X_t only, is likely not appropriate in a regulatory perspective. This global measure should depend on the joint distribution of F_t, X_t, and a better notation would be $R(F, X)$. We have seen that the allocation problem is an hedging problem w.r.t. the global $L\&P$ X. Similarly, the choice of an appropriate level of global reserve is also an hedging problem, but w.r.t. to a real economic benchmark and not the problem of controlling the stand alone risk of X only. Typically, for a mortgage portfolio the optimal amount of mortgage to be distributed should depend on the real estate cycle. The global reserve might be a quantile $q_{\alpha(Q)}(X)$, with a critical value $\alpha(Q)$ function of the distribution Q of factor F, or $R_c(F, X) = E(X) + E([X - c(F)]^+)$, that is, the global risk measure could change along the real estate cycle.

A similar remark can be done for the contributions. The result of Corollary 2.5 is still valid, but with an increased information set. More precisely, the ADM ν_P should now be replaced by an ADM ν_{P*}, where P^* denotes the joint distribution of (X, F).

6 Concluding remarks

The aim of this paper was to survey and complete the current literature on capital allocation in a regulatory perspective and with special attention to systematic risk. The main message of this paper is to avoid a crude use of a coherent risk measure such as a VaR, or an ES for computing the reserve both at the individual and global levels. More precisely,

i) An allocation problem is different from a risk measurement problem. The contributions are contingent to the level of global risk and have to satisfy some basic axioms.

ii) The axioms are not sufficient to define a unique contribution, and it could be important to distinguish the distortion risk measure underlying the measure of global risk from the allocation distortion measure explaining how to allocate the global reserve.

iii) The allocation by ADM seems applicable to any type of global risk measure and is also able to disentangle marginal systematic and unsystematic risk components and cross effects.

iv) If the regulation has a purpose of economic policy, that is, if the monetary policy is not enough for the Central Bank, a time independent coherent risk measure for the global risk is likely not appropriate. The subadditivity or homogeneity axiom may be unappropriate for regulatory purpose. The global risk measure has to be chosen in relation with the economic environment, especially with the position in the business or real estate cycle. The regulator faces an hedging problem, not the problem of managing the stand alone global risk.

The main questions that are still to be solved are the following ones :

i) What is (are) the objective(s) of the regulator ? In particular, is he/she partly in charge of economic policy ? What is seen in our paper is that the definition of the required capital is an important instrument to control the quantity of credits and its distribution among firms, households, real estates, but also the leveraging among banks... Several Central Banks have for official objective the control of inflation by means of a prime rate. It seems important to debate of the control of the credit distribution and leveraging by means of the required capital [see e.g. Hellwig (2010) for a polemical discussion of the role of regulation].

ii) Once the objective is well-defined, how to choose the global level of reserve $R_t(F_t, X_t)$, which will be likely different from a global VaR, or from the sum of VaR's of the different entities ?

iii) Once the objective and global level of reserve are well-defined, how to choose the Allocation Distortion Measure, that is, the way of allocating the global reserve between the entities, between systematic, unsystematic and cross components?

To summarize, at a time where the databases, the statistical tools, the marginal and hedging risk measures, are almost in place, the different possible objectives of the regulation have now to be debated and their consequences on the real economy and on the required capitals to be evaluated and compared.

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A P P E N D I X 1

Proof of Proposition 2.2

Equivalence between i) and ii)

First note that the function :

$$y \to \tilde{U}(y, x - y),$$

where $\tilde{U}$ is concave is itself concave. Therefore ii) implies i).

Conversely, i) implies ii). Indeed let us denote by a a value for which the one-dimensional concave function U is maximal. If a is finite, U can be written as :

$$\begin{aligned} U(y) &= U_1(y) + U_2(y) + U(a), \\ \text{with} \quad U_1(y) &= 0, \text{ if } y \leq a, \\ &\quad U(y) - U(a), \text{otherwise}, \\ U_2(y) &= U(y) - U(a), \text{ if } y \leq a, \\ &\quad 0, \text{ otherwise}, \end{aligned}$$

U_1 (resp. U_2) is a decreasing concave (resp. increasing concave) function. The result is deduced by noting that the function :

$$\tilde{U}(y_1, y_2) = U_1(y_1) + U_2(x - y_2) + U(a),$$

is concave, and that

$$U(y) = \tilde{U}(y, x - y)$$

If $a = +\infty$ [resp. $-\infty$], the result is deduced by noting that the function : $(y_1, y_2) \to U(y_1)$ [resp. $(y_1, y_2) \to U(x - y_2)$] is concave.

Equivalence between ii) and iii)

This equivalence is given in Rothschild, Stiglitz (1970), Theorem 2.

QED

A P P E N D I X 2

Proof of Proposition 2.4

The proof is based on two Lemmas.

Lemma A.1 : The contributions : $R_{\mu_P}(X, X_i) = \int E[X_i|X = x]\mu_P(dx)$ satisfy the three axioms, if $\int x d\mu_P(dx) = R(X)$.

Proof : The decentralization axiom is clearly satisfied and we consider below the two other axioms. i) We have :

$$\begin{aligned}
\sum_{i=1}^{n} R_{\mu_P}(X, X_i) &= \sum_{i=1}^{n} \int E[X_i|X = x]\mu_P(dx) \\
&= \int x d\mu_P(dx), \\
&= R(X),
\end{aligned}$$

which proves the additivity.

ii) Moreover if $X_1^* \succeq_X X_1$, we have :

$$\begin{aligned}
& E[X_1|X = x] \\
= \;& E[X_1^* + Z|X = x], \text{ with } E(Z|X_1^*, X) = 0, \text{ by Proposition 3.2 iii)}, \\
= \;& E\{E[X_1^* + Z|X_1^*, X]|X = x\} \\
= \;& E[X_1^*|X = x], .
\end{aligned}$$

Thus, $R_{\mu_P}(X, X_1^*) = R_{\mu_P}(X, X_1)$.

We deduce the compatibility with the risk preordering, at least in a wide sense.

QED

Lemma A.2 : Under the conditions of Proposition 2.4, all the contributions are necessarily of the form given in Lemma A.1.

Proof : In the Hilbert space $L^2(Y)$, the continuous linear form are necessarily of the type :

$$R(X, X_i) = E[a(Y)X_i],$$

where $a(Y) \in L^2(Y)$ [see e.g. Rudin (1966), Chapter 4]. The decentralization and additivity axioms imply that $a(Y)$ can be written $a_P(X)$.

Moreover, we have :

$$\begin{aligned} E[a_P(X)X_i] &= E[a_P(X)E(X_i|X)] \\ &= \int E[X_i|X = x]a_P(x)P(dx), \end{aligned}$$

where $P(dx)$ is the distribution of global risk X. This expression is of the form given in Lemma A.1.

QED

Since a_P is a measure density, $R(X, X_i)$ is simply a weighted risk allocation in the terminology of Furman, Zitikis (2008). When a_P is positive, the contribution can be interpreted as the value of X_i obtained by applying the pricing operator a_P function of the distribution of X. This corresponds to the premium calculation principle introduced in Gerber (1979).

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Explicit expression of the marginal expected shortfall

The proof is based on a succession of Lemmas

We have to prove that :

$$\frac{\partial E[\beta X+Y|\beta X+Y>q_\alpha(\beta)]}{\partial\beta} = E[X|\beta X+Y>q_\alpha(\beta)], \quad \text{(a.1)}$$

$$\text{where } P[\beta X+Y>q_\alpha(\beta)] = 1-\alpha, \forall\beta, \quad \text{(a.2)}$$

and $q_\alpha(\beta)=q_\alpha(\beta X+Y)$, say.

Let us assume that the joint distribution of (X,Y) is continuous with probability density function [12] $f(x,y)$. Equality (a.2) can be written as :

$$\int\left[\int_{-\beta x+q_\alpha(\beta)}^{\infty} f(x,y)dy\right]dx = 1-\alpha, \forall\beta.$$

Thus, by differentiating with respect to β, we get :

$$\int[x-\frac{\partial q_\alpha(\beta)}{\partial\beta}]f[x,q_\alpha(\beta)-\beta x]dx = 0, \forall\beta, \quad \text{(a.3)}$$

which implies $\frac{\partial q_\alpha(\beta)}{\partial\beta}=E[X|\beta X+Y=q_\alpha(\beta)]$.

The expected shortfall for $\beta X+Y$ is :

$$ES(\beta)$$

$$=E[\beta X+Y|\beta X+Y>q_\alpha(\beta)]$$

$$=\frac{1}{1-\alpha}\int\left[\int_{-\beta x+q_\alpha(\beta)}^{\infty}(\beta x+y)f(x,y)dy\right]dx$$

Its derivative with respect to β is equal to :

[12] The general proof valid for any type of joint distribution for (X,Y) has been given in Tasche (2000), Lemma (5.6).

$$
\begin{aligned}
\frac{\partial ES(\beta)}{\partial \beta} &= \frac{1}{1-\alpha}\int\left[\int_{-\beta x+q_\alpha(\beta)}^{\infty} x f(x,y)dy\right]dx \\
&+ \frac{1}{1-\alpha}\int\left[x-\frac{\partial q_\alpha(\beta)}{\partial \beta}\right] q_\alpha(\beta) f[x, q_\alpha(\beta)-\beta x]dx \\
&= \frac{1}{1-\alpha}\int\left[\int_{-\beta x+q_\alpha(\beta)}^{\infty} x f(x,y)dy\right]dx \quad \text{[from (a.3)]} \\
&= E[X|\beta X + Y > q_\alpha(\beta)],
\end{aligned}
$$

with is equation (a.1).