Affine Modelling of Credit Risk, Pricing of Credit Events and Contagion

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Abstract

We propose a new discrete-time affine pricing model for defaultable securities breaking down the most restrictive assumptions made in existing frameworks. Specifically, our model simultaneously allows for (i) the presence of systemic entities by departing from the no-jump condition on the factors' conditional distribution, (ii) contagion effects, (iii) the pricing of credit events and (iv) the presence of stochastic recovery rates. Our affine framework delivers explicit pricing formulas for default-sensitive securities like bonds and credit default swaps. A first application shows how this framework can be exploited to estimate sovereign credit risk premiums in an equilibrium model. In a second application, we jointly model term structures of sovereign CDS denominated in different currencies and extract market-implied probabilities of depreciations at default. A third application illustrates the ability of the model to replicate the behavior of banks' CDS spreads that was observed in the aftermath of the Lehman Brothers' bankruptcy.

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Key-words: affine credit risk model, recursive affine processes, Gamma-zero distribution, no-jump condition, contagion, credit-event risk, credit spread puzzle.

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1 Introduction

The specification of no-arbitrage asset pricing models is concerned with the formulation of empirically realistic assumptions while maintaining a large degree of tractability. This trade-off is particularly problematic in credit risk models, which require the modeling of the joint dynamics of common factors (y_t), of entity specific factors (x_t) ($w_t^* = (y_t', x_t')'$), and of entity default indicators (d_t), along with their interplay reflecting financial and economic linkages between entities. In the tradition of Duffie and Singleton (1999), closed-form or semi closed-form pricing formulas for defaultable securities can be obtained in an affine intensity-based framework. In this class of models the vector w_t^* is an affine process (see e.g. Darolles, Gourieroux, and Jasiak (2006)) in the risk-neutral world and both the default intensities and the risk-free short rate are affine functions of these factors. However, in the existing standard frameworks, tractability comes at the cost of several restrictive assumptions that are at odds with either theoretical or empirical evidence.

The main assumptions usually considered are the following. First, the dynamics of w_t^* does not depend on the vector of default indicators d_t; more precisely, d_t does not cause w_t^* in the Granger sense. This assumption – usually referred to as the *no-jump* condition – is made in particular in the popular doubly-stochastic Cox process framework used by e.g. Jarrow and Turnbull (1995), Lando (1998) or Duffie (2005).[1] If w_t^* contains macroeconomic variables, this condition may be seen as the assumption that the modeled entities are not systemic. While this is reasonable when the entities are firms of small size, it is less realistic when large banks, insurance companies, or supranational and sovereign entities are considered. Second, the default probabilities of different entities are independent given the path of w_t^*, hence there are no lagged or instantaneous contagion effects. In contrast, economic and financial linkages imply a significant amount of default clustering and dynamic contagion effects (see Jarrow and Yu (2001), Ait-Sahalia, Laeven, and Pelizzon (2014), Bai, Collin-Dufresne, Goldstein, and Helwege (2015), Benzoni, Collin-Dufresne, Goldstein, and Helwege (2015) and Azizpour, Giesecke, and Schwenkler (2017)). Third, the default event of any entity is usually not priced, that is d_t is absent from the representative investor's stochastic discount factor (SDF). However, the pricing of credit surprises have been shown to be an important driver of corporate bond returns risk premia and to be a possible explanation of the so-called credit spread puzzle (see Driessen (2005) and Gourieroux, Monfort, and Renne (2014), respectively).[2] Furthermore, as illustrated by one of our applications, the need for incorporating default events in the SDF arises in an endowment-economy asset-pricing model where the consumption of the representative agent is affected by the defaults of central institutions such as governments. Fourth, the default event is an absorbing state, thus associated phenomena like post-default financial distress ending through firm solvency or sovereign debt restructuring episodes are ignored (see Guo, Jarrow, and Zeng (2009) and Asonuma and Trebesch (2016)). Fifth, the recovery payment in case of default is typically defined as a constant or predetermined fraction (recovery rate)

[1] A discussion of the no-jump assumption can be found in e.g. Collin-Dufresne, Goldstein, and Hugonnier (2004), Duffie, Schroder, and Skiadas (1996) and Duffie and Singleton (1999).

[2] The credit spread puzzle corresponds to the observation that corporate and sovereign bond spreads are seemingly higher than warranted by historical default rates (see e.g. D'Amato (2003), Almeida and Philippon (2007), Gabaix (2012) and Giesecke, Longstaff, Schaefer, and Strebulaev (2011)).

of an exposure-at-default given by the zero-coupon bond price that would have prevailed in case of no default; it is the *recovery of market value* (RMV) convention of Duffie and Singleton (1999).[3] However, several studies point to the existence of stochastic recovery rates (e.g. Altman, Brady, Resti, and Sironi (2005), Das (2009)).

This paper introduces a general discrete-time affine credit-risk modelling framework able to relax simultaneously all the critical assumptions listed above while maintaining tractability of the pricing formulas of defaultable securities. The asset pricing model we specify is based on the class of Vector Autoregressive Gamma processes introduced by Monfort, Pegoraro, Renne, and Roussellet (2016) and generalizing the ARG process introduced by Gourieroux and Jasiak (2006). These non-negative processes belong to the affine class and some of their components can stay at zero for prolonged periods of time. In our model, the default event of each entity e is described by such a component, called credit event variable and denoted by $\delta_t^{(e)}$. In particular, the default date of any entity is defined as the first date at which $\delta_t^{(e)}$ becomes strictly positive. Common and entity-specific pricing factors $w_t^* = (y_t', x_t')'$ are the other components of the multivariate process.

In our approach different channels of default contagions can be represented. First, we introduce feedbacks between the δ_t's so as to capture direct contagion effects or cross-excitation effects. Second, the δ_t's may also depend contemporaneously on the common factor y_t, thus allowing for the simultaneous impact of a global shock on all the credit event variables (so-called "frailty" effect described by Duffie, Eckner, Horel, and Saita (2009)). Third, δ_t may Granger-cause w_t^*, breaking down the no-jump condition, therefore allowing for an indirect contagion through the frailty effect.

Our SDF has a standard exponential-affine formulation but the new feature is that it depends not only on the factors w_t^*, but also on the credit event variables $\delta_t^{(e)}$ (see Gourieroux, Monfort, and Renne (2014) for a related approach). This formulation allows to price credit events while preserving the affine structure of our multivariate process under the associated change of probability measure. Importantly, we show that the relationship between risk-neutral and physical parameters driving the dynamics of our system is available in closed-form.

We close the model specification of default-sensitive securities' payoffs by assuming, for any entity, a date-t stochastic recovery rate given by an exponential-affine function of $\left(y_t, x_t^{(e)}, \delta_t^{(e)}\right)$, $x_t^{(e)}$ denoting the factors specific to entity e. The recovery payoff is then defined as the product of this recovery rate and of the exposure-at-default. The three usual types of exposures-at-default are considered: recovery of market value (RMV), recovery of face value (RFV) and recovery of Treasury (RT) (see Brennan and Schwartz (1980), Duffie (1998), Jarrow and Turnbull (1995), Longstaff and Schwartz (1995) and Duffie and Singleton (1999)).

Our affine credit-risk pricing model proves to be particularly tractable. The price of defaultable bonds of any entity can be computed from closed-form recursions for any maturity, and has the classical exponential-affine form when the RMV convention is adopted. Importantly, we show that

[3] If the RMV convention is replaced by the recovery of face value (RFV) or the recovery of Treasury (RT) conventions, tractable pricing formulas can still be obtained for defaultable bonds and credit default swaps. In the recovery of face value (RFV), the exposure-at-default is the face value of the considered bond and it is the value of the otherwise equivalent default-free bond in the recovery of Treasury (RT) case.

the possible alternative recovery value assumptions (RFV and RT) change the pricing formulas but do not alter their tractability. Similarly, CDS spreads are easily computable with closed-form recursions under the RFV convention. This result is reached thanks to a key lemma providing explicit formulas for truncated multi-horizon Laplace transforms associated with multivariate positive affine processes, typically characterizing the pricing of defaultable securities.

We provide three applications of our framework. The first application deals with the pricing of sovereign credit risk in the context of an endowment-economy asset-pricing model.[4] We jointly model the fluctuations of the sovereign CDS term structures of the four largest euro-area countries – France, Germany, Italy and Spain – over the period 2008-2016. The four credit-event intensities are driven by a common factor and country-specific factors, as in Ang (2013). However, contrary to this latter paper, we make the SDF explicit, relating it to the macroeconomic context. The SDF specification is deduced from the following two assumptions: there exists a representative agent featuring Epstein and Zin (1989) preferences and the agent's consumption process is affected by sovereign defaults, which are interpreted as forms of disasters.[5] In this equilibrium model, default events appear in the SDF, which cannot be done in standard credit-risk models. The model is used to derive the term structures of credit-risk premiums, defined as the difference between observed CDS spreads and those that would prevail if agents were not risk-averse. These risk premiums are sizeable, accounting for about half of sovereign CDS spreads across the maturity spectrum. Our application contribute to the literature on the credit spread puzzle and consider that it can be – at least partly – solved by the the fact that default events are significantly priced.

In the second application, we show how our framework offers the possibility to study quanto CDS, which are the deviations between spreads of CDS on the same entity but denominated in different currencies (see e.g. Ehlers and Schonbucher (2004)). Our stochastic-recovery-rate pricing formulas are easily adjusted to price a CDS whose payoffs are denominated in one or another currency. To do so, it suffices to augment the model with the appropriate exchange rate. Hence, we introduce the €-$ exchange rate in our previous euro-area model and allow for depreciationary effects of sovereign defaults. The magnitude of these effects is estimated by optimizing the model fit of the observed quanto CDS. We find that market prices are consistent with the fact that sovereign defaults in France, Germany, Italy and Spain would be followed by average euro depreciations of, respectively, 17%, 20%, 9% and 12%. This simple extension of the model, which does not involve a novel latent factor, accounts for more than 50% of the variances of the 16 considered quanto CDS series. Moreover, we show that observed levels of quanto CDS observations could not be matched by postulating that, conditionally on w_t^*, the exchange rate does not correlate with the default events. In other words, our results suggest that the correlation between the exchange rate and the default per se (and not the correlation between

[4] Examples of interest-rate term-structure models based on endowment economies can be found in Piazzesi and Schneider (2007), Eraker (2008), Eraker and Shaliastovich (2008), Bansal and Shaliastovich (2013) or Doh (2013). Recently, Augustin and Tedongap (2016) and Chernov (2016) have considered the pricing of CDS in that kind of context. In the latter two studies however, and contrary to what is allowed by our framework, there is no influence of the sovereign default on consumption conditional on the state variable. The sovereign default *event* is therefore not priced.

[5] Several papers of the disaster-risk literature consider that sovereign defaults may be induced by disasters. More precisely, in these models, it is often assumed that once a disaster happens, there is a fixed probability that the government defaults (see e.g. Barro (2006), Gourio (2012) or Tsai and Wachter (2015)).

the exchange rate and the conditional default probability) is key to explain the fluctuation of quanto CDS. This is in line with the findings of Ehlers and Schonbucher (2004) and of Brigo, Pede, and Petrelli (2015).[6]

Our third application illustrates the ability of the model to replicate the behavior of banks' CDS spreads that was observed in the aftermath of the Lehman bankruptcy. Following Lehman's default, the CDS spreads of multinational investment banks experienced sharp increases. We show how such effects arise in a specification of the model featuring contagion effects.[7]

The remainder of the paper is organized as follows. In Section 2, we present the standard assumptions allowing the discrete-time classical credit risk model to provide tractable pricing formulas for defaultable bonds. We formally show that tractability is lost in the classical model as soon as one of these assumptions is not satisfied. Section 3 provides the general affine credit risk framework delivering explicit pricing formulas for defaultable bonds and multi-currency CDS. Section 4 presents the positive affine credit risk model, where the vector of state variables w_t is assumed to follow a Vector Autoregressive Gamma process under both the historical and risk-neutral dynamics. Section 5 presents three applications of this positive affine framework. Section 6 concludes. An appendix provides details about the calibration and solution of our sovereign credit-risk model. An online appendix gathers proofs and technical results.

2 Pricing Defaultable Securities: Standard Assumptions

2.1 A Classical Credit Risk Model: Historical and Risk-Neutral Dynamics

To isolate precisely the main restrictions present in credit risk modelling, we first consider a very simple model. The information of the investor at date t is $\underline{w_t} = (w_t', w_{t-1}', \dots, w_1')'$, where $w_t = (y_t', d_t')'$, y_t being a vector of common factors and $d_t = (d_{1,t}, d_{2,t})'$ a vector of two binary variables representing the possible default of entities $e \in \{1, 2\}$, $d_{e,t} = 1$ ($d_{e,t} = 0$) meaning that entity e has (has not) defaulted at time t. For ease of presentation we do not consider entity-specific factors x_t (say). We assume that the process $\{w_t\}$ is Markov and that its historical ($\mathbb{P}$, say) dynamics is defined by the conditional densities $f(y_t \,|\, w_{t-1})$ and $p(d_t \,|\, y_t, w_{t-1})$. We also introduce a one-period positive SDF between $t-1$ and t, denoted $M_{t-1,t}(w_t, w_{t-1})$.

In a standard credit risk model, the following assumptions are typically made:

Assumption S.1 *$\{d_t\}$ does not Granger cause $\{y_t\}$:*

$$f(y_t \,|\, w_{t-1}) = f(y_t \,|\, y_{t-1})\,.$$

In other words, y_t is *exogenous* or *autonomous*. When y_t is a vector of macroeconomic variables,

[6] The former paper is based on CDS data for Japanese multinational corporations, the latter exploits Italian sovereign CDS data.

[7] See e.g. Acharya, Philippon, Richardson, and Roubini (2009), Jorion and Zhang (2012), Yang and Zhou (2013) or Helwege and Zhang (2016) for discussions of the contagion channels at play during the Lehman episode.

this assumption can be seen as considering the entities as *non-systemic* since their default does not impact the state of the economy. It is also the discrete-time equivalent of the continuous-time *no-jump condition* (discussed in Collin-Dufresne, Goldstein, and Hugonnier (2004), Duffie, Schroder, and Skiadas (1996) and Duffie and Singleton (1999)), since there is no jump in the conditional distribution of y_t when an entity defaults.

Assumption S.2 *There is no instantaneous or lagged contagion between entities, i.e. (with obvious notations):*

$$p(d_t \,|\, y_t, w_{t-1}) = p_1(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1}) \;\times\; p_2(d_{2,t} \,|\, y_t, y_{t-1}, d_{2,t-1})\,.$$

Assumption S.3 *The default events (or credit events) are not priced, in the sense that:*

$$M_{t-1,t}(w_t, w_{t-1}) = M_{t-1,t}(y_t, y_{t-1})\,.$$

The risk-free short rate, between $t-1$ and t, known at $t-1$ is defined by:

$$r_{t-1} = r_{t-1}(w_{t-1}) = -\log \mathbb{E}\left[M_{t-1,t}(w_t, w_{t-1}) \,|\, w_{t-1}\right]\,.$$

Thus, under S.1 *and* S.3, r_{t-1} is function of y_{t-1} only.

Assumption S.4 *The default event is an absorbing state:*

$$p_e(0 \,|\, y_t, y_{t-1}, 1) = 0\,, \quad e \in \{1, 2\}\,.$$

The survival probability is defined as:

$$p_e(0 \,|\, y_t, y_{t-1}, 0) = \exp\left[-\lambda_e\left(y_t, y_{t-1}\right)\right]\,,$$

where $\lambda_{e,t} := \lambda_e\left(y_t, y_{t-1}\right)$ is a non-negative function of (y_t, y_{t-1}), called the *default intensity.*

Proposition 2.1 *Under assumptions S.1 to S.4, the risk-neutral ($\mathbb{Q}$, say) dynamics is such that:*

$$f^{\mathbb{Q}}(y_t \,|\, w_{t-1}) = f^{\mathbb{Q}}(y_t \,|\, y_{t-1}) \propto M_{t-1,t}(y_t, y_{t-1})\, f(y_t \,|\, y_{t-1})$$

$$p_t^{\mathbb{Q}}(d_t \,|\, y_t, w_{t-1}) = p_1(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1}) \;\times\; p_2(d_{2,t} \,|\, y_t, y_{t-1}, d_{2,t-1})\,,$$

and therefore $\lambda_e^{\mathbb{Q}}\left(y_t, y_{t-1}\right) = \lambda_e\left(y_t, y_{t-1}\right)$*, where* $\lambda_e^{\mathbb{Q}}\left(y_t, y_{t-1}\right) = -\log\left[p_e^{\mathbb{Q}}(0 \,|\, y_t, y_{t-1}, 0)\right]$ *is the risk-neutral default intensity of entity* e*. In other words, the exogeneity of* y_t *is preserved in the risk-neutral world as well as the lack of contagion, and the default intensities are the same under* $\mathbb{P}$ *and* $\mathbb{Q}$*.*

Proof The risk-neutral conditional density of w_t given w_{t-1}, namely $f^{\mathbb{Q}}(y_t \,|\, w_{t-1})\, p^{\mathbb{Q}}(d_t \,|\, y_t, w_{t-1})$, is proportional to:

$$M_{t-1,t}(w_t, w_{t-1})\, f(y_t \,|\, w_{t-1})\, p(d_t \,|\, y_t, w_{t-1})\,,$$

and thus proportional to:

$$M_{t-1,t}(y_t, y_{t-1})\, f(y_t \,|\, y_{t-1})\, p_1(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1})\, p_2(d_{2,t} \,|\, y_t, y_{t-1}, d_{2,t-1})\,.$$

The result follows immediately. ∎

It is important to stress that, although the historical and risk-neutral intensities are the same functions of y_t and y_{t-1}, their historical and risk-neutral dynamics are in general different since these dynamics are different for the process y_t.

2.2 Pricing a Defaultable Bond in the Classical Credit Risk Model

Let us denote by $B_e(t,h)$ ($e \in \{1,2\}$) the random variable equal to the price at date t of a zero-coupon bond issued by entity e with residual maturity h ($h \geq 0$) if entity e is alive at time t, and equal to zero (by convention) otherwise. It is useful to introduce:

(i) the *recovery rate* $RR_t^{(e)} := RR_t^{(e)}(y_t, y_{t-1})$ such that $RR_t^{(e)}(y_t, y_{t-1}) \in (0,1)$;

(ii) the *pseudo-intensity*:

$$\widetilde{\lambda}_{e,t} = \widetilde{\lambda}_{e,t}\,(y_t, y_{t-1}) := -\log\left\{\exp\left[-\lambda_e\,(y_t, y_{t-1})\right] + \Big(1 - \exp\left[-\lambda_e\,(y_t, y_{t-1})\right]\Big)\, RR_t^{(e)}(y_t, y_{t-1})\right\}$$

(iii) an associated *pseudo-price* for any date t and maturity h:

$$\widetilde{B}_e(t,h) = \mathbb{E}_t^{\mathbb{Q}}\left\{\exp\left[-\sum_{i=1}^{h}\Big(r_{t+i-1} + \widetilde{\lambda}_{e,t+i}\Big)\right]\right\},$$

with $\widetilde{B}_e(t,0) = 1$, and where $\mathbb{E}_t^{\mathbb{Q}}(\cdot)$ denotes the expectation under $\mathbb{Q}$, conditional on the available information $\underline{w_t}$.

In general, $\widetilde{B}_e(t,h)$ is a function of y_t and d_t but, under assumptions S.1 and S.3, it becomes a function of y_t only. For $\lambda_e\,(y_t, y_{t-1}) \approx 0$, we have $\widetilde{\lambda}_{e,t} \approx \lambda_e\,(y_t, y_{t-1}) \times LGD_t^{(e)}$ with $LGD_t^{(e)} = \Big(1 - RR_t^{(e)}\Big)$, that is, the pseudo-intensity is a Loss-Given-Default fraction of the intensity.

Assumption S.5 *In case of default at time $t+i$ ($i \leq h$) the recovery payment takes place at the same date and is given by:*

$$RR_{t+i}^{(e)}(y_{t+i}, y_{t+i-1})\, \widetilde{B}_e(t+i, h-i)\,.$$

In this case, we have:

Proposition 2.2 *Under assumptions S.1 to S.5:*

$$\begin{aligned} B_e(t,h) &= (1 - d_{e,t})\; \widetilde{B}_e(t,h)\,,\ \textit{for any pair } (t,h) \\ &= (1 - d_{e,t})\; \mathbb{E}_t^{\mathbb{Q}}\left\{\exp\left[-\sum_{i=1}^{h}\Big(r_{t+i-1} + \widetilde{\lambda}_{e,t+i}\Big)\right]\right\}. \end{aligned} \quad (1)$$

In particular, if entity e has not defaulted at time t, the price of the zero-coupon bond with unitary face value and maturing at $t+h$ is $\widetilde{B}_e(t,h)$.

Proof Relation (1) is true for $B_e(t+h,0)$ since $\widetilde{B}_e(t+h,0)=1$ and $B_e(t+h,0)=1-d_{e,t+h}$. Assuming that $B_e(t+i+1,h-i-1)=(1-d_{e,t+i+1})\,\widetilde{B}_e(t+i+1,h-i-1)$, we get:

$$\begin{aligned}B_e(t+i,h-i) \;&=\; (1-d_{e,t+i})\,\mathbb{E}^{\mathbb{Q}}_{t+i}\Big\{\exp(-r_{t+i})\,\Big[(1-d_{e,t+i+1})\,\widetilde{B}_e(t+i+1,h-i-1)\\ &\qquad + d_{e,t+i+1}\,RR^{(e)}_{t+i+1}\,\widetilde{B}_e(t+i+1,h-i-1)\Big]\Big\}\,,\end{aligned}$$

since, in case of no default at date $t+i$, the value of the bond at date $t+i+1$ is either $RR^{(e)}_{t+i+1}\,B_e(t+i+1,h-i-1)=RR^{(e)}_{t+i+1}\,\widetilde{B}_e(t+i+1,h-i-1)$ if default happens, and $B_e(t+i+1,h-i-1)=\widetilde{B}_e(t+i+1,h-i-1)$ otherwise. Assumptions S.1 and S.3 imply that $\widetilde{B}_e(t+i+1,h-i-1)$ does not depend on d_{t+i+1} and taking first the conditional expectation given w_{t+i} and y_{t+i+1} we obtain:

$$\begin{aligned}B_e(t+i,h-i) \;&=\; (1-d_{e,t+i})\,\mathbb{E}^{\mathbb{Q}}_{t+i}\Big\{\exp(-r_{t+i})\,\widetilde{B}_e(t+i+1,h-i-1)\times\\ &\qquad \mathbb{E}^{\mathbb{Q}}\Big[(1-d_{e,t+i+1})\,+\,d_{e,t+i+1}\,RR^{(e)}_{t+i+1}\,|\,y_{t+i+1},w_{t+i}\Big]\Big\}\,.\end{aligned} \tag{2}$$

Using assumptions S.2 and S.4 and the definition of $\widetilde{\lambda}_{e,t}$ we get:

$$\begin{aligned}B_e(t+i,h-i) \;&=\; (1-d_{e,t+i})\,\mathbb{E}^{\mathbb{Q}}_{t+i}\Big\{\exp\Big(-r_{t+i}-\widetilde{\lambda}_{e,t+i+1}\Big)\,\widetilde{B}_e(t+i+1,h-i-1)\Big\}\\ &=\; (1-d_{e,t+i})\,\widetilde{B}_e(t+i,h-i)\,.\end{aligned}$$

By a recursive argument, relation (1) is true for any pair (t,h). ■

Proposition 2.2 considers a recovery rate and a pseudo-intensity both stochastic (non-predetermined) and it implies that the recovery payment in case of default at time $t+i$, namely $RR^{(e)}_{t+i}\,\widetilde{B}_e(t+i,h-i)$, can be interpreted as the fraction $RR^{(e)}_{t+i}$ of the zero-coupon bond value that would have prevailed at date $t+i$ in case of no default. This assumption is related to the Recovery of Market Value (RMV) convention of Duffie and Singleton (1999). However, in the presentation of the discrete-time motivation of their continuous-time pricing framework, the authors make two assumptions; first, $RR^{(e)}_t(y_t,y_{t-1})=RR^{(e)}_t(y_{t-1})$, and second $\lambda_e\,(y_t,y_{t-1})=\lambda_e\,(y_{t-1})$. This means that the recovery rate at t is known at $t-1$ and that there is no instantaneous causality between $\{y_t\}$ and $\{d_{e,t}\}$. In this case, $\widetilde{\lambda}_{e,t}$ is also function of y_{t-1} only and we get:

$$B_e(t,h) \;=\; (1-d_{e,t})\,\mathbb{E}^{\mathbb{Q}}_t\left\{\exp\left[-\sum_{i=1}^{h}\widetilde{r}_{e,t+i-1}\right]\right\}\,, \tag{3}$$

where $\widetilde{r}_{e,t}:=r_t+\widetilde{\lambda}_{e,t+1}\,(y_t)$ is a default-adjusted interest rate between t and $t+1$ (known at t).

Let us now consider the following assumption:

Assumption S.6 *The process $\{y_t\}$ is affine under the $\mathbb{Q}$ measure, r_t is an affine function of y_t and $\widetilde{\lambda}_{e,t}$ is affine in (y_t, y_{t-1}).*

Under assumption S.6, the computation of the defaultable zero-coupon bond price provided by Proposition 2.2:

$$B_e(t,h) \quad = \quad (1-d_{e,t})\, \mathbb{E}_t^{\mathbb{Q}} \left\{ \exp\left[-\sum_{i=1}^{h} \left(r_{t+i-1} + \widetilde{\lambda}_{e,t+i} \right) \right] \right\}, \tag{4}$$

is straightforward and the pricing formula takes the usual exponential-affine form (see Proposition 3.1 for a description of the multi-horizon Laplace transform formula in an affine framework). Formula (4) and its tractability is a key feature of several models (see Duffie and Singleton (1999), Duffie, Pan, and Singleton (2000), Monfort and Renne (2013), Monfort and Renne (2014) and Gourieroux, Monfort, Pegoraro, and Renne (2014)). However, we may lose the high degree of tractability of this pricing formula when some of the previous assumptions characterizing classical credit risk models are relaxed. More precisely, we will see in Sections 2.3 to 2.5 that, if either Assumptions S.1, S.2 or S.3 are not satisfied (and even if Assumption S.6 still holds true), then the pricing formula is not tractable anymore. On the contrary, if the RMV convention (Assumption S.5) is replaced by the recovery of face value (RFV) or the recovery of Treasury (RT) conventions, under Assumptions S.1 to S.4 and S.6 we still have a tractable pricing formula, even if it is not exponential-affine anymore (see Section 2.6).

2.3 Pricing in presence of a Systemic Entity

We assume that assumption S.1 is replaced by:

Assumption G.1

$$f(y_t \,|\, w_{t-1}) = f(y_t \,|\, y_{t-1}, d_{1,t-1}),$$

meaning that entity $e = 1$ could be seen as systemic.

We also assume that the other assumptions (S.2 to S.6) of the classical model are maintained. Under the risk-neutral probability we have:

$$f^{\mathbb{Q}}(y_t \,|\, w_{t-1}) = f^{\mathbb{Q}}(y_t \,|\, y_{t-1}, d_{1,t-1}),$$

therefore, the pseudo-price:

$$\widetilde{B}_e(t,h) = \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left[-\sum_{i=1}^{h} \left(r_{t+i-1} + \widetilde{\lambda}_{e,t+i} \right) \right] \Big|\, y_t, d_{1,t} \right\},$$

depends on both y_t and $d_{1,t}$. In this case, the result of Proposition 2.2 is not valid for entity $e = 1$, even if it remains valid for entity $e = 2$. It can easily be seen by replacing the first conditioning of Equation (2) in the proof of Proposition 2.2 by a conditioning given w_{t+i}, y_{t+i+1} and $d_{1,t+i+1}$, and using the fact that $\widetilde{B}_2(t+i+1, h-i-1)$ only depends on this conditioning information and not on $d_{2,t+i+1}$. However, the computation of $\widetilde{B}_2(t,h)$ is not straightforward since the process $\{y_t\}$ is not autonomous and the autonomous process $\{y_t, d_{1,t}\}$ is not affine because of $d_{1,t}$ (even if the conditional Laplace transform of y_t, given $y_{t-1}, d_{1,t-1}$, is exponential-affine in $y_{t-1}, d_{1,t-1}$).

In the riskless (non-defaultable) term structure, r_t does not depend on $d_{1,t}$ (because of S.6) but the other risk-free rates $R(t,h)$ (say), for $h \geq 2$, depend on both y_t and $d_{1,t}$ since the price of the riskless zero-coupon bond $\mathbb{E}_t^{\mathbb{Q}}\left[\exp(-r_t - \ldots - r_{t+h-1})\right]$ depends on $d_{1,t}$ (because of G.1). The presence of a systemic entity hence opens the way to a *flight-to-quality* effect since the default of entity $e = 1$ may have an impact on the risk-free term structure (see Chang and Sundaresan (2005), Longstaff (2004) and Collin-Dufresne, Goldstein, and Hugonnier (2004)).[8]

2.4 Defaultable Bond Pricing and Contagion

Let us now assume that assumption S.2 is replaced by:

Assumption G.2 *There is a contagion effect of the default of entity 1 towards entity 2, that is:*

$$p(d_t \,|\, y_t, w_{t-1}) = p_1(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1}) \times p_2(d_{2,t} \,|\, y_t, y_{t-1}, d_{1,t}, d_{1,t-1}, d_{2,t-1}),$$

which implies that the default intensity of the first entity, namely $\lambda_{1,t}(y_t, y_{t-1})$, remains function of (y_t, y_{t-1}) only, whereas the default intensity of the second entity becomes function of $(y_t, y_{t-1}, d_{1,t}, d_{1,t-1})$, that is $\lambda_{2,t}(y_t, y_{t-1}, d_{1,t}, d_{1,t-1})$.

If the other assumptions S.1, S.3 and S.4 are maintained, we still have $\lambda_{e,t}^{\mathbb{Q}} = \lambda_{e,t}$. The result of Proposition 2.2 is still valid for both entities, but $\widetilde{\lambda}_{2,t}$ is now a function of $(y_t, y_{t-1}, d_{1,t}, d_{1,t-1})$.

If $\{y_t\}$ is affine under the $\mathbb{Q}$ measure, r_t is an affine function of y_t, and $\widetilde{\lambda}_{1,t}$ is an affine function of (y_t, y_{t-1}), the computation of $\widetilde{B}_1(t,h)$ and, therefore, of $B_1(t,h)$ is straightforward. However, the pricing formulas for $\widetilde{B}_2(t,h)$ and $B_2(t,h)$ are not explicit anymore even if $\widetilde{\lambda}_{2,t}$ is affine in $(y_t, y_{t-1}, d_{1,t}, d_{1,t-1})$. Also note that, since the exogeneity of the process y_t still holds (under both measures), the riskless term structure does not feature flight-to-quality effects.

2.5 Pricing Default Events

Let us suppose now that assumptions S.1, S.2 and S.4 are satisfied while assumption S.3 is replaced by:

Assumption G.3 *The default event of the second entity ($d_{2,t}$) is a source of risk that is not priced, whereas ($d_{1,t}$) is priced. Formally, we assume:*

$$M_{t-1,t}(w_t, w_{t-1}) = M_{t-1,t}(y_t, y_{t-1}, d_{1,t}, d_{1,t-1}).$$

Then, we have the following:

[8] Note that we could constrain r_{t-1} to be always function of y_{t-1} only by specifying $M_{t-1,t}$ as $\exp\left[-r_{t-1}(y_{t-1}) + g(y_t, y_{t-1})\right]$, where $\mathbb{E}_{t-1}\left[\exp g(y_t, y_{t-1})\right] = 1$, but if y_t is not exogenous under $\mathbb{Q}$, then $R(t,h)$ would depend on y_t and $d_{1,t}$ for $h \geq 2$, implying a flight-to-quality effect for these maturities.

Proposition 2.3 *Under assumptions S.1, S.2, S.4 and G.3, the risk-neutral dynamics is such that:*

a) $f^{\mathbb{Q}}(w_t \,|\, w_{t-1}) \propto M_{t-1,t}(y_t, y_{t-1}, d_{1,t}, d_{1,t-1})\, f(y_t \,|\, y_{t-1})\, p_1(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1})\, p_2(d_{2,t} \,|\, y_t, y_{t-1}, d_{2,t-1})$,

b) $p_2^{\mathbb{Q}}(d_{2,t} \,|\, d_{1,t}, y_t, w_{t-1}) = p_2^{\mathbb{Q}}(d_{2,t} \,|\, y_t, y_{t-1}, d_{2,t-1}) = p_2(d_{2,t} \,|\, y_t, y_{t-1}, d_{2,t-1})$,

c) $\lambda_{2,t}^{\mathbb{Q}} = \lambda_{2,t}$,

d) $f^{\mathbb{Q}}(y_t\,, d_{1,t} \,|\, w_{t-1}) \propto M_{t-1,t}(y_t, y_{t-1},\, d_{1,t}, d_{1,t-1})\, f(y_t \,|\, y_{t-1})\, p_1(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1})$,

e) $p_1^{\mathbb{Q}}(d_{1,t} \,|\, y_t, w_{t-1}) = p_1^{\mathbb{Q}}(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1}) \propto M_{t-1,t}(y_t, y_{t-1}, d_{1,t}, d_{1,t-1})\, p_1(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1})$,

f) $d_{1,t} = 1$ *is absorbing under* $\mathbb{Q}$,

g) $p_1^{\mathbb{Q}}(0 \,|\, y_t, y_{t-1}, 0)$

$$= \frac{M_{t-1,t}(y_t, y_{t-1}, 0, 0) \exp\left[-\lambda_{1,t}(y_t, y_{t-1})\right]}{M_{t-1,t}(y_t, y_{t-1}, 0, 0) \exp\left[-\lambda_{1,t}(y_t, y_{t-1})\right] + M_{t-1,t}(y_t, y_{t-1}, 1, 0) \left\{1 - \exp\left[-\lambda_{1,t}(y_t, y_{t-1})\right]\right\}}$$

$$= \exp\left[-\lambda_{1,t}^{\mathbb{Q}}(y_t, y_{t-1})\right] ,$$

or, equivalently:

$$\lambda_{1,t}^{\mathbb{Q}}(y_t, y_{t-1}) = \lambda_{1,t}(y_t, y_{t-1}) - \log M_{t-1,t}(y_t, y_{t-1}, 0, 0) +$$
$$\log\left\{M_{t-1,t}(y_t, y_{t-1}, 0, 0) \exp\left[-\lambda_{1,t}(y_t, y_{t-1})\right] + M_{t-1,t}(y_t, y_{t-1}, 1, 0) \left\{1 - \exp\left[-\lambda_{1,t}(y_t, y_{t-1})\right]\right\}\right\}.$$

Proof *See Online Appendix A.3.* ■

Proposition 2.3 shows that the default intensities of the first entity in the risk-neutral and historical world are now *different* functions of (y_t, y_{t-1}). Moreover, the risk-neutral conditional distribution of y_t given the past is proportional to the sum of $M_{t-1,t}(y_t, y_{t-1}, d_{1,t}, d_{1,t-1})\, f(y_t \,|\, y_{t-1})\, p_1(d_{1,t} \,|\, y_t, y_{t-1}, d_{1,t-1})$ over $d_{1,t}$ (see *d*)) and therefore depends on $d_{1,t-1}$. This means that the exogeneity of $\{y_t\}$ is *no longer preserved* under the $\mathbb{Q}$ probability measure even if it was assumed under the $\mathbb{P}$ probability measure. A direct consequence is that the result of Proposition 2.2 is no longer valid for entity $e = 1$. The result remains valid for entity 2 but the computation of $\widetilde{B}_2(t, h)$ is not straightforward because, even if y_t is autonomous in the historical world, this is not true in the risk-neutral one. For the same reason, the riskless rates $R_t(h)$ depend on $d_{1,t}$ (for $h \geq 2$) and there is a flight-to-quality effect through a different channel than the one implied by the systemic entity assumption of Section 2.3. In some sense, the pricing of the default event of an entity is another way to make that entity systemic.

2.6 Pricing under Alternative Recovery Conventions

In Section 2.2 we have assumed (assumption S.5) that, in case of default of entity e at $t + i$, the recovery payment is given by a fraction $RR_{t+i}^{(e)}$ of the pseudo-price $\widetilde{B}_e(t + i, h - i)$. In addition, under assumptions S.1 to S.4, this pseudo-price is equal to the zero-coupon bond value that would have prevailed in case of no default at the same date.

We now assume that assumption S.5 is replaced by one of the following two alternative assumptions about the recovery payment:

Assumption G.5.i *In case of default at $t+i$, the recovery payment is given by a fraction $RR^{(e)}_{t+i}$ (function of $\underline{y_{t+i}}$ only) of the (unitary) face value (it is the Recovery of Face Value (RFV) convention adopted by Brennan and Schwartz (1980) and Duffie (1998)).*

Assumption G.5.ii *In case of default at date $t+i$, the recovery payment is $RR^{(e)}_{t+i}\,B(t+i,h-i)$, where $B(t,h)=\mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{h} r_{t+j-1}\right)\mid \underline{w_t}\right\}$ is the date-t market price of a default-free zero-coupon bond maturing at $t+h$ (it is the Recovery of Treasury (RT) convention, introduced by Jarrow and Turnbull (1995) and Longstaff and Schwartz (1995), stating that the creditor receives a recovery payoff equal to the fraction of the present value of the principal).*

It is easy to see that, if S.5 is replaced by G.5.i or G.5.ii, the result of Proposition 2.2 is no longer valid. Nevertheless, tractable pricing formulas might still be reached if assumptions S.1 to S.4 remain satisfied.

Proposition 2.4 *Under assumptions S.1 to S.4, the price at date t of a defaultable zero-coupon bond with unitary face value and maturing at $t+h$ is, in case of no default at t, given by:*

$$\begin{aligned} B_e(t,h) &= \sum_{i=1}^{h}\mathbb{E}^{\mathbb{Q}}_t\left\{\exp\Big(-r_t-\ldots-r_{t+i-1}\Big)\mathcal{P}^{(e)}_{t+i,h-i}\left[\exp\left(-\sum_{j=1}^{i-1}\lambda_{e,t+j}\right)\Big(1-\exp(-\lambda_{e,t+i})\Big)\right]\right\} \\ &\quad +\mathbb{E}^{\mathbb{Q}}_t\left\{\exp\left[-\sum_{j=1}^{h}\Big(r_{t+j-1}+\lambda_{e,t+j}\Big)\right]\right\}, \end{aligned} \tag{5}$$

where the recovery payment $\mathcal{P}^{(e)}_{t+i,h-i}$ is given by $RR^{(e)}_{t+i}(y_{t+i},y_{t+i-1})$ under RFV convention, and by $RR^{(e)}_{t+i}(y_{t+i},y_{t+i-1})\,B(t+i,h-i)$ under the RT convention.

Proof *See Online Appendix A.3.* ■

Finally, let us assume:

Assumption G.6 *The process $\{y_t\}$ is affine under the $\mathbb{Q}$ measure, r_t is an affine function of y_t, $\lambda_{e,t}$ is an affine function of (y_t,y_{t-1}) and the recovery payment at time t, $\mathcal{P}^{(e)}_{t,h}$, is an exponential-affine function of (y_t,y_{t-1}).*

Then, from Equality (5) we see that $B_e(t,h)$ is a sum of exponential-affine (in y_t and y_{t-1}) conditional Laplace transforms. Under assumptions S.1 to S.4, G.5.i or G.5.ii (RFV or RT conventions, respectively) and G.6, an explicit (tractable) pricing formula is also easily obtained for a Credit Default Swap (CDS).

3 Generalized Affine Credit Risk Modelling Framework

In the previous section we have seen that the Classical Credit Risk Model delivers tractable pricing formulas for defaultable bonds (and credit default swaps) under assumptions S.1 to S.6, with the last two assumptions possibly replaced by G.5.i (or G.5.ii) and G.6 if the RFV (or the RT) convention is considered instead of the RMV convention. The purpose of this section is to build a new affine credit risk modelling framework able to provide tractable pricing formulas *even if*: *i*) the exogeneity (or no-jump) condition is not satisfied (presence of a systemic entity, i.e. S.1 is not satisfied), *ii*) contagion is allowed (S.2 is not satisfied), and *iii*) credit-event risk is priced (S.3 is relaxed).

The key ingredients opening the way to *tractable* pricing formulas in this general setting are the following hypotheses replacing the standard ones presented in Section 2:

H.1) The default date of any entity is the first date at which a non-negative credit-event variable becomes strictly positive. This variable can stay at zero for prolonged periods of time, can leave zero and return there afterwards.

H.2) The credit-event variables, along with a given set of common and entity-specific factors characterizing the economy, form a multivariate affine process under the historical world. The vector of date-t state variables is denoted w_t.

H.3) The stochastic discount factor between $t-1$ and t is an exponential-affine function of w_t and the risk-free short rate r_{t-1} is an affine function of w_{t-1}.

H.4) The date-t stochastic recovery rate of any defaultable entity is a given exponential-affine function of w_t.

3.1 Default Time Modelling

Let us consider E (risky) entities (e.g. firms or countries) indexed by $e \in \{1, \dots, E\}$. Let us denote by $d_t^{(e)}$ the indicator of default of entity e: $d_t^{(e)} = 1$ if entity e is in default at time t (or before), and $d_t^{(e)} = 0$ otherwise.

Assumption H.1 *The default date $\tau^{(e)}$ (say) of any entity e is defined as:*

$$\tau^{(e)} = \inf \left\{ t > 0 : \delta_t^{(e)} > 0 \right\} , \tag{6}$$

where $\delta_t^{(e)}$ is a non-negative variable called credit-event variable. The default indicator function can be equivalently written as $d_t^{(e)} = \mathbb{1}_{\{\tau^{(e)} \le t\}}$ or $d_t^{(e)} = 1 - \mathbb{1}_{\{\underline{\delta_t^{(e)}}' \mathbf{1} = 0\}}$, with $\underline{\delta_t^{(e)}} = (\delta_t^{(e)}, \dots, \delta_1^{(e)})$ and where $\mathbf{1} = (1, \dots, 1)'$ with conformable dimension.

The default of entity e occurs at the first date where $\delta_t^{(e)}$ is strictly positive. As will be seen, the variables $\delta_t^{(e)}$ are embedded in a multivariate setting where their conditional distribution given their past and given the present and past values of the other factors is a Gamma-zero distribution (see

Section 4.1). This assumption allows any non-negative credit-event variable $\delta_t^{(e)}$ to stay at zero for prolonged periods of time, signaling the absence of credit events, and then to leave zero at the default date.

Observe that, contrary to Assumption S.4 where the default event is an absorbing state of $d_{e,t}$, we assume here that after the default date $\tau^{(e)}$, i.e. the date of the credit event, $\delta_t^{(e)}$ might move back and be stuck at zero for an extended period of time before jumping again. Indeed, as highlighted by Guo, Jarrow, and Zeng (2009), the credit event affecting a firm triggers in reality a period of resolution that can possibly lead this entity to insolvency after the default date (see also Kraft and Steffensen (2007)). In comparison, the standard credit risk setting of Section 2 implicitly assumes that entities stop operating at the default date. Thanks to Assumption H.1, after the default date $\tau^{(e)}$, the entity continues to operate in the economy; if the financial distress is resolved and the firm thus remains solvent, then this is captured by a positive $\delta_t^{(e)}$ moving back to zero; if the financial distress induced by the default is followed by insolvency, then $\delta_t^{(e)}$ remains positive.

For instance, in the case of sovereign debts, Asonuma and Trebesch (2016) find that 62% of debt restructuring episodes observed between 1978 and 2010 occurred post-default with an average duration of five years.[9] This phenomenon is properly described by a credit event variable taking positive values after the default event and moving back to zero at the end of the restructuring period.

3.2 Discrete-Time Affine Historical and Risk-Neutral Dynamics

The purpose of this section is to specify the vector of state variables w_t and to define its historical and risk-neutral dynamics. The date-t state variables are stacked in a N-dimensional vector $w_t = (w_t^{*'}, \delta_t')'$, with $w_t^* = (y_t', x_t')'$; y_t is a N_y-dimensional vector of factors common to all the entities in the economy, each $x_t^{(e)}$ is a N_e-dimensional vector of factors specific to entity e and the vector $x_t = (x_t^{(1)'}, \dots, x_t^{(E)'})'$ is of dimension $N_x = \sum_e N_e$. The vector $\delta_t = (\delta_t^{(1)}, \dots, \delta_t^{(E)})'$ collecting the credit-event processes is of dimension $N_\delta = E$. The information available at date t is given by $\underline{w_t} = (\underline{y_t}, \underline{x_t}, \underline{\delta_t})$, where $\underline{\ell_t} = (\ell_t', \dots, \ell_1')'$, $\ell \in \{y, x, \delta\}$.

The following two assumptions state that the vector w_t is a multivariate affine process under the historical probability (Assumption H.2), and that the SDF is exponential-affine (Assumption H.3). Together, these assumptions preserve the affine property of w_t under the risk-neutral probability. We will see in Section 4 that it is possible to find an affine representation of the process $w_t = (y_t', x_t', \delta_t')'$ such that, conditional on a proper information set, the vector of credit event variables δ_t is Gamma-zero-distributed, thus satisfying Assumption H.1 (see Chen and Filipovic (2007) for a continuous-time approach).

Assumption H.2 *The stochastic process $\{w_t\}$ is affine under the historical probability measure $\mathbb{P}$.*

[9] The remaining 38% of restructuring episodes are preemptive, that is with the restructuring implemented prior to a credit event. The associated average duration is of one year.

The historical Laplace transform of w_t*, conditional* $\underline{w_{t-1}}$*, is denoted by:*

$$\varphi^{\mathbb{P}}_{w,t-1}(u_w) = \mathbb{E}\left[\exp(u_w' w_t) \,|\, \underline{w_{t-1}}\right] = \exp\left[A_w(u_w)' w_{t-1} + B_w(u_w)\right] , \tag{7}$$

with $u_w = (u_y', u_x', u_\delta')'$.

Assumption H.3 *The one-period positive SDF* $M_{t-1,t}$ *(say) in our discrete-time economy is given by:*

$$M_{t-1,t} = \exp\left[-r_{t-1}(\underline{w_{t-1}}) + \theta_w'\, w_t - \psi^{\mathbb{P}}_{w,t-1}(\theta_w)\right] \tag{8}$$

where the risk-free short rate (between $t-1$ *and* t*) is given by the following affine function of the factors:*

$$r_{t-1}(\underline{w_{t-1}}) = \xi_0 + \xi_1' w_{t-1} \,. \tag{9}$$

$\theta_w = (\theta_y', \theta_x', \theta_\delta')'$ *is the vector of risk-correction parameters and* $\psi^{\mathbb{P}}_{w,t-1}(u_w) = \log \varphi^{\mathbb{P}}_{w,t-1}(u_w)$ *is the log-Laplace transform of* w_t*, given* $\underline{w_{t-1}}$*, which is also an affine function of* w_{t-1}*.*[10]

We see from (8) that the condition $\mathbb{E}_{t-1}\left[M_{t-1,t}\right] = \exp[-r_{t-1}(\underline{w_{t-1}})]$ is automatically satisfied, and since $\dfrac{M_{t-1,t}}{\mathbb{E}_{t-1}\left[M_{t-1,t}\right]} = \exp\left[\theta_w'\, w_t - \psi^{\mathbb{P}}_{w,t-1}(\theta_w)\right]$, the risk-neutral ($\mathbb{Q}$, say) p.d.f. and log-Laplace transform of the conditional distribution of w_t, given $\underline{w_{t-1}}$, are respectively given by:

$$\begin{aligned} f^{\mathbb{Q}}(w_t \,|\, \underline{w_{t-1}}) &= f^{\mathbb{P}}(w_t \,|\, \underline{w_{t-1}}) \exp[\theta_w'\, w_t - \psi^{\mathbb{P}}_{w,t-1}(\theta_w)] \\ \psi^{\mathbb{Q}}_{w,t-1}(u_w) &= \psi^{\mathbb{P}}_{w,t-1}(u_w + \theta_w) - \psi^{\mathbb{P}}_{w,t-1}(\theta_w) \,, \end{aligned} \tag{10}$$

($f^{\mathbb{Q}}$ is a Esscher transform of $f^{\mathbb{P}}$). It is important to preserve the affine nature of the entire state vector under the $\mathbb{Q}$ measure in order to guarantee the tractability of our pricing formulas. From relation (10) and Assumption H.2 we immediately find that the risk-neutral Laplace transform of w_t, conditional on $\underline{w_{t-1}}$, is given by:

$$\varphi^{\mathbb{Q}}_{w,t-1}(u_w) = \mathbb{E}^{\mathbb{Q}}\left[\exp(u_w' w_t) \,|\, \underline{w_{t-1}}\right] = \exp\left[\widetilde{A}_w(u_w)' w_{t-1} + \widetilde{B}_w(u_w)\right] , \tag{11}$$

where $\widetilde{A}_w(u_w) = A_w(u_w + \theta_w) - A_w(\theta_w)$ and $\widetilde{B}_w(u_w) = B_w(u_w + \theta_w) - B_w(\theta_w)$. In other words, the process $\{w_t\}$ remains affine under $\mathbb{Q}$.

3.3 Defaultable Bond Pricing

3.3.1 Recovery Payment, Recovery Rate and Recovery Value

Let us consider the problem of pricing at date $t < \tau^{(e)}$ a defaultable zero-coupon bond (ZCB) issued by entity e, maturing at date $T = t + h$ and with unitary face value. Its payoff $\pi^{(e)}_{t+i,h-i}$ (say) at date

[10] Although it is not the main objective of this paper, it would be easy to allow r_t to reach a given lower bound $\underline{\xi}_0$ (say) and to stay at that lower bound for some time, by setting $\xi_0 = \underline{\xi}_0$ and by assuming that r_t is linear function of some components of y_t and x_t with degree of freedom parameters assumed to be equal to zero.

$t+i$ is given by:

$$\pi^{(e)}_{t+i,h-i} = \begin{cases} \mathcal{P}^{(e)}_{t+i,h-i}\, \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i-1}\mathbf{1}=0\}}\, \mathbb{1}_{\{\delta^{(e)}_{t+i}>0\}} & \text{for } i \in \{1,\ldots,h-1\} \\ \mathcal{P}^{(e)}_{t+h,0}\, \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+h-1}\mathbf{1}=0\}}\, \mathbb{1}_{\{\delta^{(e)}_{t+h}>0\}} + \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+h}\mathbf{1}=0\}} & \text{for } i = h\,, \end{cases}$$

where $\mathcal{P}^{(e)}_{t+i,h-i}$ denotes the recovery payment at $t+i$ in case of default at the same date and where, for $i \in \{1,\ldots,h\}$, $\delta^{(e)}_{t:t+i} = (\delta^{(e)}_t,\ldots,\delta^{(e)}_{t+i})'$ and $\mathbf{1} = (1,\ldots,1)'$ with conformable dimension. The no-arbitrage price at date $t < \tau^{(e)}$ of a defaultable zero-coupon bond with unitary face value, issued by an entity $e \in \{1,\ldots,E\}$ and maturing in h periods is given by:

$$B_e(t,h) \;=\; \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{ \exp\left(-\sum_{j=1}^{i} r_{t+j-1}\right) \pi^{(e)}_{t+i,h-i} \,|\, \underline{w_t} \right\}. \tag{12}$$

with $\underline{\delta}^{(e)}_t = 0$ in $\underline{w}_t$. Using the identity $\mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i-1}\mathbf{1}=0\}}\, \mathbb{1}_{\{\delta^{(e)}_{t+i}>0\}} = \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i-1}\mathbf{1}=0\}} - \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i}\mathbf{1}=0\}}$ we get:

$$\begin{aligned} B_e(t,h) \;=\;& \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{ \exp\left(-\sum_{j=1}^{i} r_{t+j-1}\right) \mathcal{P}^{(e)}_{t+i,h-i}\, \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i-1}\mathbf{1}=0\}} \,|\, \underline{w_t} \right\} \\ & - \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{ \exp\left(-\sum_{j=1}^{i} r_{t+j-1}\right) \mathcal{P}^{(e)}_{t+i,h-i}\, \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i}\mathbf{1}=0\}} \,|\, \underline{w_t} \right\} \\ & + \mathbb{E}^{\mathbb{Q}}\left\{ \exp\left(-\sum_{j=1}^{h} r_{t+j-1}\right) \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+h}\mathbf{1}=0\}} \,|\, \underline{w_t} \right\}. \end{aligned} \tag{13}$$

The recovery payment $\mathcal{P}^{(e)}_{t+i,h-i}$ is defined as $\mathcal{P}^{(e)}_{t+i,h-i} = RR^{(e)}_{t+i}\, \mathcal{V}^{(e)}_{t+i,h-i}$, namely the product of a Recovery Rate $RR^{(e)}_{t+i}$ and a Recovery Value $\mathcal{V}^{(e)}_{t+i,h-i}$ (say) and different specifications of these two variables will be proposed. More formally, if we denote by $\delta^{(-e)}_t$ the vector δ_t without $\delta^{(e)}_t$, we have:

Assumption H.4 *The recovery payment $\mathcal{P}^{(e)}_{t+i,h-i}$ at date $t+i = \tau^{(e)}$ of the defaultable zero-coupon bond issued by entity $e \in \{1,\ldots,E\}$ and maturing at date $T = t+h > \tau^{(e)}$ is given by:*

$$\mathcal{P}^{(e)}_{t+i,h-i} = RR^{(e)}_{t+i}\, \mathcal{V}^{(e)}_{t+i,h-i}\left(\underline{\widetilde{w}^{(e)}_{t+i}},\, \underline{\delta^{(e)}_{t+i-1}}, \delta^{(e)}_{t+i}\right), \tag{14}$$

with $\widetilde{w}^{(e)}_t = \left(y_t, x_t, \delta^{(-e)}_t\right)'$. The recovery rate $RR^{(e)}_{t+i}$ triggered by the default event is specified as:

$$RR^{(e)}_{t+i} = \exp\left(-a_e - a'_y\, y_{t+i} - a'_{x,e}\, x^{(e)}_{t+i} - a_{\delta,e}\, \delta^{(e)}_{t+i}\right) = \exp\left(-a_e - a'_{w,e}\, w_{t+i}\right), \tag{15}$$

where a_e and $a_{\delta,e}$ are non-negative scalars, a_y and $a_{x,e}$ are vectors of positive components with conformable dimensions, and where $\mathcal{V}^{(e)}_{t+i,h-i}\left(\underline{w_{t+i}}\right) = \mathcal{V}^{(e)}_{t+i,h-i}\left(\underline{\widetilde{w}^{(e)}_{t+i}},\, \underline{\delta^{(e)}_{t+i-1}}, \delta^{(e)}_{t+i}\right)$ denotes the Recovery

Value (Exposure-at-Default) at $t+i$, characterizing the recovery convention of entity $e \in \{1, \dots, E\}$, and such that $\mathcal{V}^{(e)}_{T,0}\left(\underline{\widetilde{w}^{(e)}_T}, \ \underline{\delta^{(e)}_{T-1}} = 0, \delta^{(e)}_T = 0\right) = 1$.

Relation (15) specifies a positive stochastic recovery rate whose time-varying magnitude depends not only on the size of the credit-event process but also on common and entity specific factors. The assumption of positive state variables characterizing the asset pricing model of Section 4 will guarantee to have a recovery rate bounded by one. Observe in addition that, the particular specification:

$$RR^{(e)}_{t+i} = \exp\left(-\delta^{(e)}_{t+i}\right), \tag{16}$$

delivers another interesting interpretation of the process $\delta^{(e)}_t$. Relation (16) formalizes the idea of a stochastic recovery rate equal to one as long as $\delta^{(e)}_t = 0$, and leaving the unitary upper bound at the default time $\tau^{(e)}$ with a reduction whose *magnitude* depends on the credit-event process variation $\delta^{(e)}_{\tau^{(e)}}$. In other words, $\delta^{(e)}_{\tau^{(e)}}$ measures the strength of the default event. For this reason, under this specification, $\delta^{(e)}_t$ will be identified both as a credit-event *and a loss process*.

In sections 3.3.3 and 3.3.4 we show that under Assumption H.4 our modeling framework delivers *tractable* pricing formulas for the usual forms of recovery value, namely the (discrete-time) recovery of market value (RMV), recovery of face value (RFV) and recovery of Treasury (RT). We focus in particular on the RMV and RFV conventions given that, in these cases, our pricing formulas are not only explicit but can also be implemented through a *computationally fast* algorithm (the RT case is presented in the Online Appendix A.5). In the former case we will consider the recovery rate specification of Equation (16), while the latter one opens the way to tractable pricing formula under the more general recovery rate specification of Equation (15). The key mathematical tool behind this feature, presented in the next section, is the family of multi-horizon conditional Laplace Transforms at various horizons featuring a reverse order structure, also called Reverse-Order Multi-Horizon Laplace Transform.

3.3.2 The Reverse-Order Multi-Horizon Laplace Transform

Proposition 3.1 *If the conditional risk-neutral Laplace transform of (w_t) is given by (11) and if the associated family of multi-horizon conditional Laplace transforms:*

$$\varphi^{\mathbb{Q}}_{w,t,i}\left(u^{(i)}_{1:i}\right) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(u^{(i)\prime}_1 w_{t+1} + \ldots + u^{(i)\prime}_i w_{t+i}\right) \mid \underline{w_t}\right], \ i \in \{1, \ldots, I\}, \ t \in \{1, \ldots, T\}, \tag{17}$$

has variables featuring a reverse order structure, namely if there is a sequence $u_1, \ldots, u_I$ such that:

$$\left(u^{(i)}_1 \ldots, u^{(i)}_i\right) = (u_i, \ldots, u_1), \ \forall\, i \in \{1, \ldots, I\}, \tag{18}$$

then:

$$\varphi^{\mathbb{Q}}_{w,t,i}(u_i, \ldots, u_1) = \exp\left(\mathcal{A}'_i w_t + \mathcal{B}_i\right), \tag{19}$$

where the $\mathcal{A}_i$ and $\mathcal{B}_i$ loadings are obtained from the i^{th} step of the recursive system:

$$\begin{cases} \mathcal{A}_0 &= 0\,, \;\; \mathcal{B}_0 = 0\,, \\ \mathcal{A}_j &= \widetilde{A}_w\left(u_j + \mathcal{A}_{j-1}\right)\,, \\ \mathcal{B}_j &= \widetilde{B}_w\left(u_j + \mathcal{A}_{j-1}\right) + \mathcal{B}_{j-1}\,, \end{cases} \tag{20}$$

for $j = 1, \ldots, i$, and that has to be used only once *regardless the date $t \in \{1, \ldots, T\}$ and the horizon $i \in \{1, \ldots, I\}$ of interest.*

Proof *See Online Appendix A.4.* ■

Given our state process w_t, $t \in \{1, \ldots, T\}$ and for a given maximum horizon of interest I, the I-step recursion (20) automatically generates the *entire family* of multi-horizon conditional Laplace transforms $\left\{\varphi^{\mathbb{Q}}_{w,t,i}\left(u_i, \ldots, u_1\right), i \in \{1, \ldots, I\}, t \in \{1, \ldots, T\}\right\}$. This means that, once we have run the I-step recursion (20), we can retrieve any pair of interest $(\mathcal{A}_i, \mathcal{B}_i)$ (with $i < I$). By contrast, if the reverse-order condition (18) does not hold, for any pair of horizons $i \neq \kappa$ (both smaller than I), $(\mathcal{A}_\kappa, \mathcal{B}_\kappa)$ and $(\mathcal{A}_i, \mathcal{B}_i)$ have to calculated separately, running for each of them a i-step and a κ-step recursion (see also Online Appendix A.4 and A.5).

In the following sections, the conditional multi-horizon Laplace transforms that we have to calculate in order to determine the asset prices of interest, feature a particular reverse-order structure completely characterized by two variables u_1 and u_2 only, condition (18) being:

$$\left(u^{(i)}_i, \ldots, u^{(i)}_2, u^{(i)}_1\right) = \left(u_2, \ldots, u_2, u_1\right).$$

In these cases, for ease of presentation, we will adopt the notation $\varphi^{\mathbb{Q}}_{w,t,i}\left(u_2, \ldots, u_2, u_1\right) = \varphi^{\mathbb{Q}}_{w,t,i}\left(u_2, u_1\right)$.

3.3.3 Recovery of Market Value

Proposition 3.2 *Let us assume that the Recovery Value at $t + i$ is given by:*

$$\mathcal{V}^{(e)}_{t+i,h-i}\left(\underline{\widetilde{w}^{(e)}_{t+i}}, \underline{\delta^{(e)}_{t+i-1}}, \delta^{(e)}_{t+i}\right) = \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=i}^{h-1}\left(r_{t+j} + \delta^{(e)}_{t+j+1}\right)\right] \mid \underline{w_{t+i}}\right\}. \tag{21}$$

and $RR^{(e)}_t = \exp\left(-\delta^{(e)}_t\right)$. Then, under Assumption H.4 and given Relation (9) specifying

$$r_t = \xi_0 + \xi_1' w_t\,,$$

the no-arbitrage price $B_e(t, h)$, $h \in \{1, \ldots, H\}$ at date $t < \tau^{(e)}$ is given by:

$$B_e(t, h) = \mathcal{V}^{(e)}_{t,h}\left(\underline{\widetilde{w}^{(e)}_t}, 0, 0\right)\,, \;\; h \in \{1, \ldots, H\}\,, \tag{22}$$

where:

$$\begin{aligned}\mathcal{V}^{(e)}_{t,h}\left(\underline{\widetilde{w}^{(e)}_t}, \underline{\delta^{(e)}_{t-1}}, \delta^{(e)}_t\right) &= \exp\left[-\xi_0\, h - \xi_1' w_t\right] \varphi^{\mathbb{Q}}_{w,t,h}\left(u_2, u_1\right) \\ &= \exp\left[\left(\mathcal{A}_h - \xi_1\right)' w_t + \left(\mathcal{B}_h - h\,\xi_0\right)\right] .\end{aligned} \tag{23}$$

with $u_1 = -\widetilde{e}_\delta$*,* $u_2 = -(\widetilde{e}_\delta + \xi_1)$*, where* $\widetilde{e}_\delta = (0', e_\delta')'$ *is a* N*-dimensional vector, and where* e_δ *is the* e^{th} *column of the* (N_δ, N_δ)*-dimensional identity matrix.*

Proof *See Online Appendix A.4.* ■

Equation (22) is a key result of the paper. Indeed, it shows that:

$$B_e(t,h) = \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{i=1}^{h}\left(r_{t+i-1} + \delta^{(e)}_{t+i}\right)\right] \mid \underline{w_t}\right\} \tag{24}$$

(where $\underline{w}_t$ includes $\underline{\delta}^{(e)}_t = 0$). In other words, we get the familiar no-arbitrage bond pricing formula based on a *ex-post default-adjusted* short rate $(r_{t+i-1} + \delta^{(e)}_{t+i})$, in spite of the fact that the *credit events are sources of risk that are priced.* This result can be seen as a (discrete-time) generalization of the recovery of market value (RMV) setting proposed by Duffie and Singleton (1999), where the credit-event risk is not priced and where the Recovery Value at $t+i$ is given by $\mathcal{V}^{(e)}_{t+i,h-i}(\underline{\widetilde{w}^{(e)}_{t+i}}, 0, 0)$, that is the defaultable zero-coupon bond price that we would observe in case of no default at that date.

3.3.4 Recovery of Face Value

The purpose of this section is to determine defaultable bond pricing formulas when the recovery of face value convention is considered. In this case (and in the CDS pricing case presented in Section 3.4) the following Lemma proves useful and can be seen as a generalization of Lemma 2.1 in Monfort, Pegoraro, Renne, and Roussellet (2016).[11]

Lemma 3.1 *Let* Z_1 *be a random variable valued in* $\mathbb{R}^d$ $(d \geq 1)$ *and* Z_2 *be a random variable valued in* $\mathbb{R}^+ = [0, +\infty)$*. Suppose that* $\mathbb{E}\left[\exp\left(u_1' Z_1 + u_2 Z_2\right)\right]$ *exists for a given* u_1 *and* $u_2 \leq 0$*. Then, we have:*

$$\mathbb{E}\left[\exp(u_1' Z_1)\, \mathbb{1}_{\{Z_2 = 0\}}\right] = \lim_{u_2 \to -\infty} \mathbb{E}\left[\exp(u_1' Z_1 + u_2\, Z_2)\right] . \tag{25}$$

Proof *See Online Appendix A.4.* ■

The recovery of face value (RFV) case, adopted by Brennan and Schwartz (1980) and Duffie (1998), assumes that the creditor receives in case of default at $\tau^{(e)} = t+i$ the recovery payment $\mathcal{P}^{(e)}_{t+i,h-i} = RR^{(e)}_{t+i} = \exp(-a_e - a_{w,e}' w_{t+i})$, that is $\mathcal{V}^{(e)}_{t+i,h-i}\left(\underline{w_{t+i}}\right) = 1$ for any $i \in \{1, \ldots, h\}$. Given the previous assumptions H.1 to H.4, given Equations (13), (14) and (15) and Lemma 3.1, we have the following proposition.

[11] See also Chen and Filipovic (2007).

Proposition 3.3 *Under the RFV convention, the no-arbitrage price at date $t < \tau^{(e)}$ of a defaultable zero-coupon bond issued by an entity $e \in \{1, \ldots, E\}$ and maturing in h periods is given by:*

$$B_e(t,h) = \textstyle\sum_{i=1}^{h} \left(\Lambda^{\mathbb{Q}}_{(1,t,i)} - \Lambda^{\mathbb{Q}}_{(2,t,i)} \right) + \Lambda^{\mathbb{Q}}_{(3,t,h)}, \tag{26}$$

where:

$$\begin{aligned} \Lambda^{\mathbb{Q}}_{(1,t,i)} &:= \lim_{u \to -\infty} \Psi^{\mathbb{Q}}_{(t,i)}(-a_e, u\widetilde{e}_\delta - \xi_1, -a_{w,e}) \\ \Lambda^{\mathbb{Q}}_{(2,t,i)} &:= \lim_{u \to -\infty} \Psi^{\mathbb{Q}}_{(t,i)}(-a_e, u\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta - a_{w,e}) \\ \Lambda^{\mathbb{Q}}_{(3,t,i)} &:= \lim_{u \to -\infty} \Psi^{\mathbb{Q}}_{(t,i)}(0, u\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta) \end{aligned} \tag{27}$$

with $u \in \mathbb{R}$ and where, for a given $\kappa \in \mathbb{R}$:

$$\Psi^{\mathbb{Q}}_{(t,i)}(\kappa, u_2, u_1) \quad := \quad \exp\left[-i\xi_0 + \kappa + u_2' w_t\right] \varphi^{\mathbb{Q}}_{w,t,i}(u_2, u_1) \tag{28}$$

Proof *See Online Appendix A.4.* ∎

3.4 Multi Currency Credit Default Swap Pricing

Let us consider now the problem of pricing at date $t < \tau^{(e)}$ a credit default swap (CDS) issued to provide protection against a credit event (e.g. default) of an entity e. The currency of denomination of a CDS is not necessarily the domestic one (i.e. the currency in which the assets of the reference entity are denominated). For instance, a CDS protection on sovereign bonds is frequently available in a foreign and in the domestic currency. The reason behind the development of foreign-currency-denominated CDS is the protection they provide not only against the credit event but also against the associated potential depreciation of the domestic currency with respect to the foreign one (see Section 5.2 for an application to European CDS denominated in dollars and in euros). So, for the sake of generality, we consider here a CDS denominated in a foreign currency. We denote by s_t the log of the exchange rate between the domestic and the foreign currency, where the exchange rate is defined as the price in the domestic currency of one unit of foreign currency. That is, an increase in s_t corresponds to a depreciation of the domestic currency. Let us denote by $\mathcal{S}^{(e)f}_{t,t+h}$ this CDS spread, set at date t with maturity $t+h$.

Both the notional and the premium payments of a CDS are expressed in the currency of denomination. We assume in the following that the notional of the CDS is equal to one unit of the foreign currency (i.e. to $\exp(s_t)$ units of the domestic currency). The CDS spread is such that the present value of the payments made by the protection buyer (the fixed leg) is equal to present value of the payment made by the protection seller in case of default (the floating leg).

As far as the fixed leg is concerned, if entity e has not defaulted at date $t+i$ ($\leq t+h$), the cash flow on this date, expressed in the domestic currency, is: $\mathcal{S}^{(e)f}_{t,t+h} \exp(s_{t+i})$. The present value of the fixed-leg payments, expressed in the domestic currency ($PB^{(e)f}_{t,t+h}$, say), is thus given by:

$$PB^{(e)f}_{t,t+h} = \mathcal{S}^{(e)f}_{t,t+h} \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}} \left[\exp\left(s_{t+i} - \sum_{j=1}^{i} r_{t+j-1} \right) \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i}\mathbf{1}=0\}} \,|\, \underline{w_t} \right] , \tag{29}$$

Under the RFV convention, the protection seller will make a payment of $(1 - RR^{(e)}_{t+i}) \exp(s_{t+i})$ (the Loss-Given-Default) at date $t+i$ in case of default over the time interval $(t+i-1, t+i]$. Observe that the recovery rate $RR^{(e)}_{t+i}$ is the same as for a CDS denominated in the domestic currency. The present value of this promised payment, expressed in the domestic currency, is:

$$PS^{(e)f}_{t,t+h} = \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}} \left[\exp\left(s_{t+i} - \sum_{j=1}^{i} r_{t+j-1} \right) (1 - RR^{(e)}_{t+i}) \left(\mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i-1}\mathbf{1}=0\}} - \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i}\mathbf{1}=0\}} \right) |\, \underline{w_t} \right] , \tag{30}$$

and the date-t CDS spread $\mathcal{S}^{(e)f}_{t,t+h}$ is set such that $PB^{(e)f}_{t,t+h} = PS^{(e)f}_{t,t+h}$. Under the assumptions H.1 to H.4, given Lemma 3.1, assuming $RR^{(e)}_t = \exp\left(-a_e - a'_{w,e}\, w_t\right)$ and:

$$s_t = \chi + u'_s w_t \, , \tag{31}$$

we have the following:

Proposition 3.4 *The no-arbitrage CDS spread $\mathcal{S}^{(e)f}_{t,t+h}$ at date $t < \tau^{(e)}$ associated to a credit default swap (CDS) maturing in h periods and denominated in the foreign currency f having an exchange rate with respect to the domestic currency of $\exp(s_t)$ (Equation (31)), is given by:*

$$\mathcal{S}^{(e)f}_{t,t+h} = \frac{\sum_{i=1}^{h} \Lambda^{\mathbb{Q}}_{(t,i)}}{\sum_{i=1}^{h} \lim_{u \to -\infty} \Psi^{\mathbb{Q}}_{(t,i)}\left(\chi, u\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta + u_s\right)} , \tag{32}$$

where:

$$\begin{aligned} \Lambda^{\mathbb{Q}}_{(t,i)} \quad := \quad \lim_{u \to -\infty} \Big[& \Psi^{\mathbb{Q}}_{(t,i)}\left(\chi, u\widetilde{e}_\delta - \xi_1, u_s\right) - \Psi^{\mathbb{Q}}_{(t,i)}\left(\chi, u\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta + u_s\right) \\ & - \Psi^{\mathbb{Q}}_{(t,i)}\left(\chi - a_e, u\widetilde{e}_\delta - \xi_1, u_s - a_w\right) + \Psi^{\mathbb{Q}}_{(t,i)}\left(\chi - a_e, u\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta + u_s - a_w\right) \Big] \end{aligned} \tag{33}$$

and where $\Psi^{\mathbb{Q}}_{(t,i)}\left(\kappa, u_2, u_1\right)$ is given in Proposition 3.3.

Proof *See Online Appendix A.4.* ■

It is clear from Proposition 3.4 that the premium $\mathcal{S}^{(e)}_{t,t+h}$ of a CDS denominated in the domestic currency is easily obtained from Equation (32) by assuming $\chi = 0$ and $u_s = 0$ in Equation (33).

4 The Positive Affine Credit Risk Model

In this section we make precise assumptions about the distribution of the state vector $w_t = (y_t', x_t', \delta_t')'$, in such a way to specify explicit pricing formulas that will be adopted in the empirical applications of Section 5. The conditional distributions of y_t and x_t are assumed to be non-central Gamma while, in line with Assumption H.1, the vector of credit event variables δ_t is assumed to be conditionally Gamma-zero distributed. More precisely, we assume that w_t is a Vector Autoregressive Gamma (VARG) process with a recursive structure under the historical probability, thus satisfying Assumption H.2 (Sections 4.1, 4.2 and 4.3). Under Assumption H.3, the SDF-implied change of probability measure makes the state process w_t VARG also under the risk-neutral probability, thus guaranteeing tractable pricing formulas once the recovery rate assumption H.4 is properly taken into account (Section 4.4).

4.1 The Gamma-zero distribution

The purpose of this section is to review the univariate Gamma-zero distribution introduced by Monfort, Pegoraro, Renne, and Roussellet (2016). This random variable is the basic building block of the multivariate affine process we adopt in the following sections to characterize the dynamics of the relevant state vector (containing common factors, entity-specific factors and credit-event variables). Let us first remind briefly the definition of non-central Gamma distribution.

Definition 4.1 *Let X be a positive random variable. We say that X follows a non-central Gamma distribution with degree of freedom parameter $\nu > 0$, intensity parameter $\lambda > 0$ and scale parameter $\mu > 0$, denoted $X \sim \gamma_\nu(\lambda, \mu)$, if its conditional distribution given $Z \sim \mathcal{P}(\lambda)$ is:*

$$X \,|\, Z \sim \gamma_{\nu+Z}(\mu)\,. \tag{34}$$

The Laplace transform of X is given by:

$$\varphi_X(u) = \mathbb{E}[\exp(uX)] = \exp\left[-\nu\log(1-u\,\mu) + \lambda\frac{u\,\mu}{1-u\,\mu}\right]\,, \qquad \textit{for} \quad u < \frac{1}{\mu}\,.$$

Given that the $\gamma_\nu(\mu)$ converges in distribution to the Dirac distribution at zero when ν goes to zero, the $\gamma_\nu(\lambda, \mu)$ distribution is also defined for $\nu = 0$ if $\gamma_0(\mu)$ is considered as the Dirac distribution at zero. We thus obtain the non-negative Gamma-zero distribution $\gamma_0(\lambda, \mu)$ characterized by a point mass at zero.

Definition 4.2 *Let X be a non-negative random variable. We say that X follows a Gamma-zero distribution with parameters $\lambda > 0$ and $\mu > 0$, denoted $X \sim \gamma_0(\lambda, \mu)$, if its conditional distribution given $Z \sim \mathcal{P}(\lambda)$ is:*

$$X \,|\, Z \sim \gamma_Z(\mu)\,. \tag{35}$$

The p.d.f. and the Laplace transform of X, respectively $f_X(x\,;\,\lambda,\mu)$ and $\varphi_X(u\,;\,\lambda,\mu)$, are given by:

$$\begin{aligned} f_X(x\,;\,\lambda,\mu) &= \sum_{z=1}^{+\infty}\left[\frac{\exp(-x/\mu)\,x^{z-1}}{(z-1)!\,\mu^z}\times\frac{\exp(-\lambda)\lambda^z}{z\,!}\right]\mathbb{1}_{\{x>0\}}+\exp(-\lambda)\mathbb{1}_{\{x=0\}} \\ \varphi_X(u\,;\,\lambda,\mu) &= \exp\left[\lambda\frac{u\mu}{(1-u\mu)}\right] \quad \textit{for} \quad u<\frac{1}{\mu}\,. \end{aligned} \tag{36}$$

Although the probability density function (36) looks complicated, its Laplace transform is extremely simple. Equation (36) shows other key features of the Gamma-zero distribution: it has a point-mass located at $x=0$, and $\mathbb{P}(X=0)=\mathbb{P}(Z=0)=\exp(-\lambda)$.

4.2 Positive Affine Historical Dynamics

The date-t state variables are stacked in a N-dimensional vector $w_t=(w_t^{*'},\delta_t')'$, with $w_t^*=(y_t',x_t')'$; y_t is a N_y-dimensional vector of factors common to all the entities in the economy, each $x_t^{(e)}$ is a N_e-dimensional vector of factors specific to entity e and the vector $x_t=(x_t^{(1)'},\ldots,x_t^{(E)'})'$ is of dimension $N_x=\sum_e N_e$. The vector $\delta_t=(\delta_t^{(1)},\ldots,\delta_t^{(E)})'$ collecting the credit-event processes is of dimension $N_\delta=E$. The information available at date t is given by $\underline{w_t}=(\underline{y_t},\underline{x_t},\underline{\delta_t})$, where $\underline{\ell_t}=(\ell_t',\ldots,\ell_1')'$, $\ell\in\{y,x,\delta\}$.

Proposition 4.1 *The historical distribution of $w_t=(y_t',x_t',\delta_t')'$, conditional on $\underline{w_{t-1}}$, is defined recursively by the following conditional distributions:*

$$\begin{aligned} &\left(y_t\,|\,\underline{w_{t-1}}\right)\overset{\mathbb{P}}{\sim}\textstyle\bigotimes_{j=1}^{N_y}\gamma_{\nu_j^{(y)}}\left(\alpha_j^{(y)}+\beta_{j,y}^{(y)}\,y_{t-1}+\beta_{j,\delta}^{(y)}\,\delta_{t-1},\mu_j^{(y)}\right)\,, \\ &\left(x_t\,|\,y_t,\underline{w_{t-1}}\right)\overset{\mathbb{P}}{\sim}\textstyle\bigotimes_{e=1}^{E}\bigotimes_{k=1}^{N_e}\gamma_{\nu_k^{(e)}}\left(\alpha_k^{(e)}+\beta_{k,y}^{(e)}\,y_t+\beta_{k,x}^{(e)}\,x_{t-1}^{(e)}+\beta_{k,\delta}^{(e)}\,\delta_{t-1},\mu_k^{(e)}\right)\,, \\ &\left(\delta_t\,|\,y_t,x_t,\underline{w_{t-1}}\right)\overset{\mathbb{P}}{\sim}\textstyle\bigotimes_{e=1}^{E}\gamma_0\left(\alpha_e^{(\delta)}+\beta_{e,y}^{(\delta)}\,y_t+\beta_{e,x}^{(\delta)}\,x_t^{(e)}+\beta_{e,\delta}^{(\delta)}\,\delta_{t-1},\mu_e^{(\delta)}\right)\,, \end{aligned} \tag{37}$$

where $\alpha_j^{(y)}$, $\alpha_k^{(e)}$, $\alpha_e^{(\delta)}$, $\nu_j^{(y)}$ and $\nu_k^{(e)}$ are non-negative scalars, while $\mu_j^{(y)}$, $\mu_k^{(e)}$ and $\mu_e^{(\delta)}$ are strictly positive parameters; the loadings in the row vectors βs are non negative and of conformable dimensions.

The associated conditional Laplace transforms are respectively given by:

$$\begin{aligned} &\mathbb{E}\left[\exp(u_{j,y}y_{j,t})\,|\,\underline{w_{t-1}}\right]=\exp\left[\frac{u_{j,y}\,\mu_j^{(y)}}{1-u_{j,y}\,\mu_j^{(y)}}\left(\alpha_j^{(y)}+\beta_{j,y}^{(y)}\,y_{t-1}+\beta_{j,\delta}^{(y)}\,\delta_{t-1}\right)-\nu_j^{(y)}\log\left(1-u_{j,y}\,\mu_j^{(y)}\right)\right] \\ &\mathbb{E}\left[\exp(u_{k,x}^{(e)}x_{k,t}^{(e)})\,|\,y_t,\underline{w_{t-1}}\right]=\exp\left[\frac{u_{k,x}^{(e)}\,\mu_k^{(e)}}{1-u_{k,x}^{(e)}\,\mu_k^{(e)}}\left(\alpha_k^{(e)}+\beta_{k,y}^{(e)}\,y_t+\beta_{k,x}^{(e)}\,x_{t-1}^{(e)}+\beta_{k,\delta}^{(e)}\,\delta_{t-1}\right)-\nu_k^{(e)}\log\left(1-u_{k,x}^{(e)}\,\mu_k^{(e)}\right)\right] \\ &\mathbb{E}\left[\exp(u_\delta^{(e)}\delta_t^{(e)})\,|\,y_t,x_t,\underline{w_{t-1}}\right]=\exp\left[\frac{u_\delta^{(e)}\,\mu_e^{(\delta)}}{1-u_\delta^{(e)}\,\mu_e^{(\delta)}}\left(\alpha_e^{(\delta)}+\beta_{e,y}^{(\delta)}\,y_t+\beta_{e,x}^{(\delta)}\,x_t^{(e)}+\beta_{e,\delta}^{(\delta)}\,\delta_{t-1}\right)\right]\,, \end{aligned} \tag{38}$$

where $u_{j,y}$, $u_{k,x}^{(e)}$ and $u_\delta^{(e)}$ are scalars.

Under the historical measure, conditional on $\underline{w_{t-1}}$, any component $(y_{j,t})$ has a non-central Gamma distribution with Poisson mixing variable (featuring a time-varying intensity affine function of y_{t-1} and δ_{t-1}) introduced in Section 4.1. Our formulation allows in particular that the credit-event process δ_t *Granger-causes* (GC, henceforth) the vector of common factors y_t, that is $\{y_t\}$ *is not exogenous*, i.e. the no-jump condition S.1 is relaxed. Any entity-specific factor $(x^{(e)}_{k,t})$ has a probability density function (p.d.f.), given $\underline{w_{t-1}}$ and the contemporaneous value of common factors y_t, which is still non-central Gamma with a Poisson intensity given by an affine function of y_t, $x^{(e)}_{t-1}$ and δ_{t-1}.

Contrary to these first two groups of state variables, which may not reach zero if the parameters $\nu^{(y)}_j$ and $\nu^{(e)}_k$ are strictly positive, the historical distribution of any $(\delta^{(e)}_t)$, given $(y_t, x_t, \underline{w_{t-1}})$, is Gamma-zero, thus the credit-event variables are non-negative and can stay at zero for prolonged periods of time. Here, the Poisson mixing variable has a time-varying intensity which is an affine function of y_t, $x^{(e)}_t$ and δ_{t-1}. Since all past credit-event variables can GC $(\delta^{(e)}_t)$ *contagion is allowed*, implying that our model can depart from assumption S.2.

The dynamics of the state process (w_t) under the historical ($\mathbb{P}$, say) probability is thus assumed to be a Vector Autoregressive Gamma (VARG) process, with conditionally independent components within each group of state variables (namely y_t, x_t and δ_t). It can be represented in a more compact way as follows:

$$\begin{aligned}
\left(y_t \,|\, \underline{w_{t-1}}\right) &\overset{\mathbb{P}}{\sim} \gamma_{\nu^{(y)}}\left(\alpha^{(y)} + \beta^{(y)}_y\, y_{t-1} + \beta^{(y)}_\delta\, \delta_{t-1}, \mu^{(y)}\right) \\
\left(x_t \,|\, y_t, \underline{w_{t-1}}\right) &\overset{\mathbb{P}}{\sim} \gamma_{\nu^{(x)}}\left(\alpha^{(x)} + \beta^{(x)}_y\, y_t + \beta^{(x)}_x\, x_{t-1} + \beta^{(x)}_\delta\, \delta_{t-1}, \mu^{(x)}\right) \\
\left(\delta_t \,|\, y_t, x_t, \underline{w_{t-1}}\right) &\overset{\mathbb{P}}{\sim} \gamma_0\left(\alpha^{(\delta)} + \beta^{(\delta)}_y\, y_t + \beta^{(\delta)}_x\, x_t + \beta^{(\delta)}_\delta\, \delta_{t-1}, \mu^{(\delta)}\right),
\end{aligned} \tag{39}$$

where, for $\ell \in \{y, x, \delta\}$, $\alpha^{(\ell)} = (\alpha^{(\ell)}_1, \ldots, \alpha^{(\ell)}_{N_\ell})'$ and $\nu^{(\ell)} = (\nu^{(\ell)}_1, \ldots, \nu^{(\ell)}_{N_\ell})'$ are N_ℓ-dimensional vectors of non-negative components, while $\mu^{(\ell)} = (\mu^{(\ell)}_1, \ldots, \mu^{(\ell)}_{N_\ell})'$ is a N_ℓ-dimensional vector with strictly positive entries. Any matrix $\beta^{(\ell)}_m$, m and $\ell \in \{y, x, \delta\}$, is of dimension (N_ℓ, N_m) and with non-negative elements ; $\beta^{(x)}_x$ is a (N_x, N_x)-dimensional block-diagonal matrix with (N_e, N_e)-dimensional matrices $\beta^{(e)}_x = (\beta^{(e)'}_{1,x}, \ldots, \beta^{(e)'}_{N_e,x})'$ in the main diagonal, while $\beta^{(\delta)}_x$ is a (N_δ, N_x)-dimensional matrix where any row $e \in \{1, \ldots, E\}$ is given by the row vector $\beta^{(\delta)}_{e,x}$ and by blocks of zeros in line with Proposition 4.1.

Since each group of state variables has an exponential-affine conditional Laplace transform, (w_t) belongs to the class of *Recursive discrete-time affine processes* (see Online Appendix A.6.1) and therefore is an affine process. More precisely, we have the following proposition.

Proposition 4.2 *Under Proposition 4.1, the conditional Laplace transforms are respectively given by:*

$$\begin{aligned}
\varphi^{\mathbb{P}}_{y,t-1}(u_y) &= \mathbb{E}\left[\exp(u_y' y_t) \,|\, \underline{w_{t-1}}\right] = \exp\left[a^{(y)}_y(u_y)' y_{t-1} + a^{(y)}_\delta(u_y)'\delta_{t-1} + b^{(y)}(u_y)\right] \\
\varphi^{\mathbb{P}}_{x,t-1}(u_x) &= \mathbb{E}\left[\exp(u_x' x_t) \,|\, y_t, \underline{w_{t-1}}\right] = \exp\left[c^{(x)}_y(u_x)' y_t + a^{(x)}_x(u_x)' x_{t-1} + a^{(x)}_\delta(u_x)'\delta_{t-1} + b^{(x)}(u_x)\right] \\
\varphi^{\mathbb{P}}_{\delta,t-1}(u_\delta) &= \mathbb{E}\left[\exp(u_\delta' \delta_t) \,|\, y_t, x_t, \underline{w_{t-1}}\right] = \exp\left[c^{(\delta)}_y(u_\delta)' y_t + c^{(\delta)}_x(u_\delta)' x_t + a^{(\delta)}_\delta(u_\delta)'\delta_{t-1} + b^{(\delta)}(u_\delta)\right]
\end{aligned} \tag{40}$$

with:

$$\begin{aligned}
c^{(\ell)}_m(u_\ell) &= a^{(\ell)}_m(u_\ell) = \left(\beta^{(\ell)}_m\right)' \left(\frac{u_\ell \odot \mu^{(\ell)}}{1 - u_\ell \odot \mu^{(\ell)}}\right), \\
b^{(\ell)}(u_\ell) &= (\alpha^{(\ell)})' \left(\frac{u_\ell \odot \mu^{(\ell)}}{1 - u_\ell \odot \mu^{(\ell)}}\right) - \nu^{(\ell)\prime} \log\left(1 - u_\ell \odot \mu^{(\ell)}\right), \ m \text{ and } \ell \in \{y, x, \delta\},
\end{aligned} \tag{41}$$

where $\odot$ denotes the Hadamard (element-wise) product and where, with abuse of notations, the division and log operators work element-by-element when applied to vectors ($\nu^{(\delta)}$ is a vector of zeros).

The Laplace transform of w_t, conditional on $\underline{w_{t-1}}$, is thus given by:

$$\begin{aligned}
\varphi^{\mathbb{P}}_{w,t-1}(u_w) &= \mathbb{E}\left[\exp(u_w' w_t) \,|\, \underline{w_{t-1}}\right] = \mathbb{E}\left[\exp(u_y' y_t + u_x' x_t + u_\delta' \delta_t) \,|\, \underline{w_{t-1}}\right] \\
&= \exp\left[A_y(u_w)' y_{t-1} + A_x(u_w)' x_{t-1} + A_\delta(u_w)'\delta_{t-1} + B(u_w)\right] \\
&= \exp\left[A_w(u_w)' w_{t-1} + B_w(u_w)\right],
\end{aligned} \tag{42}$$

with $u_w = (u_y', u_x', u_\delta')'$ and where:

$$\begin{aligned}
A_y(u_w) &= a^{(y)}_y \left[u_y + c^{(x)}_y \left(u_x + c^{(\delta)}_x(u_\delta)\right) + c^{(\delta)}_y(u_\delta)\right] \\
A_x(u_w) &= a^{(x)}_x \left[u_x + c^{(\delta)}_x(u_\delta)\right] \\
A_\delta(u_w) &= a^{(y)}_\delta \left[u_y + c^{(x)}_y \left(u_x + c^{(\delta)}_x(u_\delta)\right) + c^{(\delta)}_y(u_\delta)\right] + a^{(x)}_\delta \left[u_x + c^{(\delta)}_x(u_\delta)\right] + a^{(\delta)}_\delta(u_\delta) \\
B(u_w) &= b^{(y)} \left[u_y + c^{(x)}_y \left(u_x + c^{(\delta)}_x(u_\delta)\right) + c^{(\delta)}_y(u_\delta)\right] + b^{(x)} \left[u_x + c^{(\delta)}_x(u_\delta)\right] + b^{(\delta)}(u_\delta).
\end{aligned} \tag{43}$$

Proof *From Proposition a.3 and Corollary a.3.1 with $w_t = (w_{1,t}', w_{2,t}', w_{3,t}')' = (y_t', x_t', \delta_t')'$.*

Observe that, as presented in Online Appendix A.6.2, this class of processes has very convenient properties in term of moments, VAR representation, stationarity conditions and predictions.

4.3 Historical Stochastic Intensities and Mutual Excitation

It is important to highlight that any credit-event variable $\delta^{(e)}_t$ can be seen also as a *jump-like* variable intervening into the dynamics of the other variables. Indeed, the distribution of $\delta^{(e)}_t$, given $y_t, x_t, \underline{w_{t-1}}$, is a mixture of $\gamma_{z_t^{(e)}}(\mu_e^{(\delta)})$ distributions, where the mixing variable $z_t^{(e)}$ is Poisson:

$$z_t^{(e)} \,|\, y_t, x_t, \underline{w_{t-1}} \overset{\mathbb{P}}{\sim} \mathcal{P}\left(\lambda_{e,t}^{\mathbb{P}}\right) , \tag{44}$$

with:

$$\mathbb{P}\left(\delta_t^{(e)} = 0 \,|\, y_t, x_t, \underline{w_{t-1}}\right) = \mathbb{P}\left(z_t^{(e)} = 0 \,|\, y_t, x_t, \underline{w_{t-1}}\right) = \exp\left(-\lambda_{e,t}^{\mathbb{P}}\right) , \tag{45}$$

and where $\lambda_{e,t}^{\mathbb{P}} = \alpha_e^{(\delta)} + \beta_{e,y}^{(\delta)} y_t + \beta_{e,x}^{(\delta)} x_t + \beta_{e,\delta}^{(\delta)} \delta_{t-1}$ denotes the *stochastic* (jump) $\mathbb{P}$-*intensity* of $\delta_t^{(e)}$. When the Poisson draws of $z_t^{(e)}$ are equal to zero, then $\delta_t^{(e)}$ is equal to zero as well, and there is no lagged impact of $\delta_t^{(e)}$ on the other variables. Nevertheless, when the Poisson draw is strictly positive, then $\delta_t^{(e)}$ takes a strictly positive value (the jump amplitude) that will provide not only a lagged impact on y_t and x_t (via the e^{th} column vector of the matrix $\beta_\delta^{(y)}$ and $\beta_\delta^{(x)}$, respectively) but it will also increase the subsequent jump intensities of the other credit-event components, namely $\lambda_{i,t+1}^{\mathbb{P}}$ with $i \neq e$, through the loadings $\beta_{i,y}^{(\delta)}$, $\beta_{i,x}^{(\delta)}$ and $\beta_{i,\delta}^{(\delta)}$ (cross-excitation or contagion).

Proposition 4.1 hence specifies a multivariate process featuring mutual excitation between the credit-event (jump-like) components while preserving the theoretical and empirical (econometric) tractability of the affine framework (see also Ait-Sahalia, Laeven, and Pelizzon (2014) and Ait-Sahalia, Cacho-Diaz, and Laeven (2015)).

4.4 Positive Affine Risk-Neutral Dynamics and Credit Event Pricing

The following Proposition shows that, under the risk-neutral measure, the process w_t remains not only affine, but also Recursive Vector Autoregressive Gamma.

Proposition 4.3 *Let us assume that the historical dynamics of the state vector $w_t = (y_t', x_t', \delta_t')'$ is described by the VARG process of Equation (39) and that the one-period stochastic discount factor is given by Equation (8), where $\theta_w = (\theta_y', \theta_x', \theta_\delta')'$. Then, the risk-neutral conditional Laplace transforms of the factors are respectively given by:*

$$\begin{aligned}
\varphi_{y,t-1}^{\mathbb{Q}}(u_y) &= \mathbb{E}^{\mathbb{Q}}\left[\exp(u_y' y_t) \,|\, \underline{w_{t-1}}\right] = \exp\left[\widetilde{a}_y^{(y)}(u_y)' y_{t-1} + \widetilde{a}_\delta^{(y)}(u_y)'\delta_{t-1} + \widetilde{b}^{(y)}(u_y)\right] \\
\varphi_{x,t-1}^{\mathbb{Q}}(u_x) &= \mathbb{E}^{\mathbb{Q}}\left[\exp(u_x' x_t) \,|\, y_t, \underline{w_{t-1}}\right] = \exp\left[\widetilde{c}_y^{(x)}(u_x)' y_t + \widetilde{a}_x^{(x)}(u_x)' x_{t-1} + \widetilde{a}_\delta^{(x)}(u_x)'\delta_{t-1} + \widetilde{b}^{(x)}(u_x)\right] \\
\varphi_{\delta,t-1}^{\mathbb{Q}}(u_\delta) &= \mathbb{E}^{\mathbb{Q}}\left[\exp(u_\delta' \delta_t) \,|\, y_t, x_t, \underline{w_{t-1}}\right] = \exp\left[\widetilde{c}_y^{(\delta)}(u_\delta)' y_t + \widetilde{c}_x^{(\delta)}(u_\delta)' x_t + \widetilde{a}_\delta^{(\delta)}(u_\delta)'\delta_{t-1} + \widetilde{b}^{(\delta)}(u_\delta)\right]
\end{aligned} \tag{46}$$

where:

$$\begin{aligned}
\widetilde{c}_m^{(\ell)}(u_\ell) &= \widetilde{a}_m^{(\ell)}(u_\ell) = \left(\widetilde{\beta}_m^{(\ell)}\right)' \left(\frac{u_\ell \odot \widetilde{\mu}^{(\ell)}}{1 - u_\ell \odot \widetilde{\mu}^{(\ell)}}\right) , \\
\widetilde{b}^{(\ell)}(u_\ell) &= (\widetilde{\alpha}^{(\ell)})' \left(\frac{u_\ell \odot \widetilde{\mu}^{(\ell)}}{1 - u_\ell \odot \widetilde{\mu}^{(\ell)}}\right) - \nu^{(\ell)\prime} \log\left(1 - u_\ell \odot \widetilde{\mu}^{(\ell)}\right) , \ m \text{ and } \ell \in \{y, x, \delta\} ,
\end{aligned} \tag{47}$$

and where, for any $j \in \{1, \ldots, N_\ell\}$ and m and $\ell \in \{y, x, \delta\}$:

$$\widetilde{\beta}_{j,m}^{(\ell)} = \frac{1}{1 - \widetilde{\theta}_\ell \, \mu_j^{(\ell)}} \, \beta_{j,m}^{(\ell)} \,, \;\; \widetilde{\mu}_j^{(\ell)} = \frac{\mu_j^{(\ell)}}{1 - \widetilde{\theta}_\ell \, \mu_j^{(\ell)}} \,, \;\; \widetilde{\alpha}_j^{(\ell)} = \frac{\alpha_j^{(\ell)}}{1 - \widetilde{\theta}_\ell \, \mu_j^{(\ell)}} \tag{48}$$

with

$$\widetilde{\theta}_\delta \;\; = \;\; \theta_\delta \,, \;\; \widetilde{\theta}_x = \theta_x + c_x^{(\delta)}(\widetilde{\theta}_\delta) \,, \;\; \widetilde{\theta}_y = \theta_y + c_y^{(\delta)}(\widetilde{\theta}_\delta) + c_y^{(x)}(\widetilde{\theta}_x) \,. \tag{49}$$

The risk-neutral Laplace transform of w_t, conditionally on $\underline{w_{t-1}}$, is given by:

$$\begin{aligned} \varphi_{w,t-1}^{\mathbb{Q}}(u_w) &= \mathbb{E}^{\mathbb{Q}}\left[\exp(u_w' w_t) \,|\, \underline{w_{t-1}}\right] \\ &= \exp\left[\widetilde{A}_y(u_w)' y_{t-1} + \widetilde{A}_x(u_w)' x_{t-1} + \widetilde{A}_\delta(u_w)' \delta_{t-1} + \widetilde{B}(u_w)\right] \\ &= \exp\left[\widetilde{A}_w(u_w)' w_{t-1} + \widetilde{B}_w(u_w)\right] \,, \end{aligned} \tag{50}$$

where:

$$\begin{aligned} \widetilde{A}_y(u_w) &= \widetilde{a}_y^{(y)} \left[u_y + \widetilde{c}_y^{(x)} \left(u_x + \widetilde{c}_x^{(\delta)}(u_\delta)\right) + \widetilde{c}_y^{(\delta)}(u_\delta)\right] \\ \widetilde{A}_x(u_w) &= \widetilde{a}_x^{(x)} \left[u_x + \widetilde{c}_x^{(\delta)}(u_\delta)\right] \\ \widetilde{A}_\delta(u_w) &= \widetilde{a}_\delta^{(y)} \left[u_y + \widetilde{c}_y^{(x)} \left(u_x + \widetilde{c}_x^{(\delta)}(u_\delta)\right) + \widetilde{c}_y^{(\delta)}(u_\delta)\right] + \widetilde{a}_\delta^{(x)} \left[u_x + \widetilde{c}_x^{(\delta)}(u_\delta)\right] + \widetilde{a}_\delta^{(\delta)}(u_\delta) \\ \widetilde{B}(u_w) &= \widetilde{b}^{(y)} \left[u_y + \widetilde{c}_y^{(x)} \left(u_x + \widetilde{c}_x^{(\delta)}(u_\delta)\right) + \widetilde{c}_y^{(\delta)}(u_\delta)\right] + \widetilde{b}^{(x)} \left[u_x + \widetilde{c}_x^{(\delta)}(u_\delta)\right] + \widetilde{b}^{(\delta)}(u_\delta) \,. \end{aligned} \tag{51}$$

Proof *See Online Appendix A.6, Proposition a.6 and Corollary a.3.1 with $w_t = (w_{1,t}', w_{2,t}', w_{3,t}')' = (y_t', x_t', \delta_t')'$.*

Proposition 4.3 shows that the risk-neutral conditional Laplace transform of any group of variables, given the proper conditioning information sets, belongs to the same parametric family as the historical ones (Equations (40)) but with *modified parameters*. The state vector w_t thus remains a Recursive Vector Autoregressive Gamma process with the same kind of specification as in Equation (39) and with risk-neutral parameters given by (48) and (49). Note that in the present framework, not only the common and entity-specific factors are priced sources of risk, but the *credit event risks are also priced whenever $\theta_\delta \neq 0$* (see Equation (8)). Assumption S.3 is thus relaxed. This feature also allows to formally derive a risk-neutral intensity function $\lambda_{e,t}^{\mathbb{Q}}$ (say) proportional to the historical one as in Jarrow, Lando, and Yu (2005) and Driessen (2005) (see also Duffie (2005) and references therein).

Corollary 4.3.1 *For any $e \in \{1, \ldots, E\}$ and under the affine $\mathbb{Q}$-dynamics of Proposition 4.3, the credit-event process $\delta_t^{(e)}$ has a stochastic $\mathbb{Q}$-intensity given by:*

$$\lambda_{e,t}^{\mathbb{Q}} \;\; = \;\; \widetilde{\alpha}_e^{(\delta)} + \widetilde{\beta}_{e,y}^{(\delta)} \, y_t + \widetilde{\beta}_{e,x}^{(\delta)} \, x_t + \widetilde{\beta}_{e,\delta}^{(\delta)} \, \delta_{t-1} = \left(\frac{1}{1 - \theta_{e,\delta} \, \mu_e^{(\delta)}}\right) \lambda_{e,t}^{\mathbb{P}} \,. \tag{52}$$

In particular we see that $\lambda_{e,t}^{\mathbb{Q}} = \lambda_{e,t}^{\mathbb{P}}$ if $\theta_{e,\delta} = 0$, i.e. if the credit event variable of entity e is not a *priced source of risk.*

5 Applications

5.1 Sovereign credit risk

In this section, we exploit the framework presented above to study the pricing of sovereign credit risk. We focus on the four main economies of the euro area: France, Germany, Italy and Spain. In spite of the high credit quality of these four countries, the associated sovereign CDS premiums have reached relatively high levels over the last ten years, especially during the so-called euro-area sovereign debt crisis initiated in late 2009.

For the sake of parameter parsimony, we consider a representative agent of a reduced euro area formed by the four previously-mentioned countries, which account for about 75% of the 19-country euro area GDP. This agent, who features Epstein-Zin preferences, consumes and can participate in financial markets where the four sovereign credit risks are traded. Once the relationship between consumption and the state vector w_t has been specified, one can price any asset whose payoffs depends on the state variables. In particular, in this context, sovereign CDS can be priced. Importantly, our framework allows for explicit interactions between the consumption process and sovereign defaults through the credit-event variables δ_t (that are among the components of w_t, as indicated in Section 4.2).

5.1.1 The model

State variables dynamics. The state variables are as in Section 4. The vector $w_t^* = (y_t, x_t')'$ is of dimension $E+1$ (with $E = 4$ here), where y_t is a one-dimensional common latent factor and $x_t = \left(x_t^{(1)}, \dots, x_t^{(E)}\right)$ contains the (scalar) country-specific latent variables. In line with Proposition 4.1, the common factor Granger-causes the country specific factors but the reverse is not true. We also assume that the credit-event processes $\delta_t^{(e)}$, $e \in \{1, \dots, E\}$, do not cause vector w_t^*. Therefore, y_t features the following autonomous dynamics:

$$\left(y_t|\underline{w_{t-1}}\right) \sim \gamma_0\left(\alpha^{(y)} + \beta_y^{(y)} y_{t-1}, 1\right). \tag{53}$$

The conditional distribution of the country-specific factors $x_t^{(e)}$, $e \in \{1, \dots, E\}$, are of the form:

$$\left(x_t^{(e)}|\underline{w_{t-1}}\right) \sim \gamma_0\left(\alpha^{(e)} + \beta_y^{(e)} y_{t-1} + \beta_x^{(e)} x_{t-1}^{(e)}, 1\right). \tag{54}$$

Turning to the credit-event processes $\delta_t^{(e)}$, $e \in \{1, \dots, E\}$, we assume that:

$$\left(\delta_t^{(e)}|\underline{w_{t-1}}, y_t, x_t\right) \sim \gamma_0\left(\beta_{e,x}^{(\delta)} x_t^{(e)}, \mu^{(\delta)}\right). \tag{55}$$

In this context, w_t is of dimension $2E+1$: $w_t = \left(y_t, x_t^{(1)}, \dots, x_t^{(E)}, \delta_t^{(1)}, \dots, \delta_t^{(E)}\right)'$.

Stochastic discount factor and consumption process. The representative European agent features Epstein and Zin (1989) preferences. Denoting by C_t the agent's consumption at date t, her utility is given by the following recursion:

$$U_t = \left((1-\delta^*)C_t^{1-\rho^*} + \delta^*\mathbb{E}(U_{t+1}^{1-\gamma^*}|\underline{w_t})^{\frac{1-\rho^*}{1-\gamma^*}}\right)^{\frac{1}{1-\rho^*}}$$

where γ^* is the coefficient of relative risk aversion, $\psi^* = 1/\rho^*$ is the elasticity of intertemporal substitution (EIS) and δ^* is the rate of time preference.

Epstein and Zin (1989) show that, with such preferences, the real log-SDF is given by:

$$\log(M^*_{t,t+1}) = \theta^* \log\delta^* - \frac{\theta^*}{\psi^*}\Delta c_{t+1} - (1-\theta^*)r_{a,t+1}, \quad (56)$$

where $c_t = \log(C_t)$ and

$$\theta^* = \frac{1-\gamma^*}{1-\rho^*},$$

and where $r_{a,t+1}$ is the log-return on a (virtual) asset that delivers aggregate consumption as dividends on each time period. This asset is called the wealth portfolio.

Consumption growth Δc_t is assumed to be affine in w_t:

$$\Delta c_t = \mu_c + \theta_c' w_t. \quad (57)$$

Solving the model. In order to be able to apply the pricing formulas presented in Section 3, one needs to solve the model. Solving the model consists in finding how the vector of risk corrections θ_w (see Equation (8)) and how ξ_0 and ξ_1, which define the short-term rate (see Equation (9)), depend on the structural parameters introduced above. We follow the methodology proposed by Eraker (2008) and Eraker and Shaliastovich (2008).[12] The first step is to find the relationship between the return of the wealth portfolio $r_{a,t}$ and the state vector w_t. This relationship has to be such that, as any other asset, the returns of the wealth portfolio satisfies the Euler condition:

$$1 = \mathbb{E}[M^*_{t,t+1}\exp(r_{a,t+1})|\underline{w_t}]$$

or

$$1 = \mathbb{E}\left[\exp\left(\theta^*\log\delta^* - \frac{\theta^*}{\psi^*}\Delta c_{t+1} + \theta^* r_{a,t+1}\right)|\underline{w_t}\right]. \quad (58)$$

The previous equation has to be satisfied for any possible value of the state vector w_t. In order to find an approximate solution for this problem, we follow Bansal and Yaron (2004) and introduce the log price-consumption ratio $z_t = \log(P_{a,t}/C_t)$, where $P_{a,t}$ denotes the real price of the wealth portfolio. Assuming that the fluctuations of this ratio around its average value, denoted by $\bar{z}$, are small, we

[12] See Le and Singleton (2010) for an alternative approach.

obtain the first-order Taylor expansion of $r_{a,t+1}$:

$$r_{a,t+1} \approx \kappa_0 + \kappa_1 z_{t+1} - z_t + \Delta c_{t+1}, \tag{59}$$

where

$$\kappa_1 = \exp(\bar{z})/[1+\exp(\bar{z})] \quad \text{and} \quad \kappa_0 = \log(1+\exp(\bar{z})) - \kappa_1 \bar{z}. \tag{60}$$

If the approximation used in Equation (59) is satisfying, the model is solved by positing an affine specification of z_t:

$$z_t = A_0 + A_1' w_t,$$

and by looking for values of A_0 and A_1 guaranteeing that Equation (58) holds for any state w_t (see Appendix A.1). Equation (56) then becomes:

$$\begin{aligned}\log(M^*_{t,t+1}) &= \theta^* \log \delta^* - \frac{\theta^*}{\psi^*}\mu_c - (1-\theta^*)(\kappa_0 + A_0(\kappa_1 - 1) + \mu_c) \\ &\quad - \left\{\frac{\theta^*}{\psi^*}\theta_c + (1-\theta^*)(\kappa_1 A_1 + \theta_c)\right\}' w_{t+1} + (1-\theta^*)A_1' w_t. \end{aligned} \tag{61}$$

The previous formula implies in particular that the vector of risk-correction parameters θ_w (Equation (8)) is then given by:

$$\theta_w = -\frac{\theta^*}{\psi^*}\theta_c - (1-\theta^*)(\kappa_1 A_1 + \theta_c).$$

Because $M^*_{t,t+1}$ is exponential-affine in (w_{t+1}, w_t), which is itself an affine process, $\mathbb{E}(M^*_{t,t+1}|\underline{w_t})$ is exponential-affine in w_t. Computing the previous conditional expectation then provides us with the following specification of the real short-term rate: $\tilde{\xi}_0 + \tilde{\xi}_1' w_t = -\log(\mathbb{E}(M^*_{t,t+1}|\underline{w_t}))$. For the sake of simplicity, we assume that inflation is constant and equal to π. The nominal short-term rate is then given by $\xi_0 + \xi_1' w_t \equiv \tilde{\xi}_0 + \pi + \tilde{\xi}_1' w_t$.

We proceed under the recovery of face value (RFV) assumption. All in all, the model is parameterized by: $\alpha^{(y)}$, $\beta_y^{(y)}$, $\alpha^{(e)}$, $\beta_y^{(e)}$, $\beta_x^{(e)}$, $\beta_{e,x}^{(\delta)}$, $\mu^{(\delta)}$, δ^*, ρ^*, γ^*, μ_c and θ_c. Once these parameters are known, the general affine asset pricing setting presented in Section 3 can then be used to price credit instruments for any possible value of the state w_t.

5.1.2 Estimation

Given the latent nature of factors w_t, the estimation of the model relies on filtering techniques.[13] Part of the model parameters are calibrated, the remaining ones are estimated by maximizing an approximate likelihood function whose computation is based on the Extended Kalman filter. These points are developed in the following.

[13] Note that in this case the credit-event process δ_t is observed. Indeed, because none of the four considered sovereigns have defaulted over the period in consideration, we have $\delta_t = 0$ for all t in our sample.

State-space model. The state-space form of the model is as follows:

$$\begin{cases} CDS_t^o &= f(w_t) + \eta_{cds,t} \\ c_t^o &= c_t + \eta_{c,t} \end{cases} \quad (62)$$

$$\begin{cases} w_t &= \mu_w + \Phi_w w_{t-1} + \Sigma(w_{t-1})\varepsilon_t \\ c_t &= c_{t-1} + \mu_c + \theta_c' w_t, \end{cases} \quad (63)$$

where CDS_t^o is the vector of observed CDS premiums and $f(w_t)$ denotes its model-implied counterpart, c_t^o is the observation of per capita real consumption (in logs), and where $\eta_{cds,t}$ and $\eta_{c,t}$ are independent normally-distributed measurement errors. While the measurement equations of the state-space model are given by (62), the transition equations are defined by (63). Because of its affine property, the process w_t possesses the semi-strong VAR representation shown in Equation (63), where ε_t is a martingale difference sequence with covariance matrix, conditional on $\underline{w_{t-1}}$, equal to the identity matrix and where $\Sigma(w_{t-1})$ is such that $\Sigma(w_{t-1})\Sigma(w_{t-1})'$ is an affine function of w_{t-1} (see Monfort, Pegoraro, Renne, and Roussellet (2016)).

Data. The estimation data span the period from January 2008 to July 2016 at the monthly frequency ($T = 103$). For each of the four countries, we consider series of CDS premiums with maturities of 1, 2, 5 and 10 years. These observations are obtained from Datastream, which collects CDS market quotation data from industry sources. Our estimation procedure also involves a series of per capita real consumption. Since we consider a representative agent of a reduced four-country euro area economy, the series of real consumption is computed as the sum of the four national series of private consumption expenditures released by Eurostat. In order to get per capita consumption, we divide the previous total consumption by the total population of the four countries.[14]

Calibrated parameters. We set the preference parameters as in the baseline case of Bansal and Yaron (2004): $\delta^* = 0.998$, $\psi^* = 1.5$ and $\gamma^* = 7$. In order to facilitate the estimation, additional constraints are imposed on model parameters. First, we set $\mu^{(\delta)} = 0.6$, which makes our model consistent with the 1983-2015 average of sovereign-default recovery rates (see Appendix A.2). Second, in Equation (57), we assume that θ_c is of the form $[0, u_x\omega', u_\delta\omega']'$, where ω is an E-dimensional vector giving the shares of the countries' GDPs in the total GDP of this reduced euro area (i.e. $\sum_i \omega_i = 1$). Therefore, θ_c depends on two parameters only, that are u_x and u_δ. Empirical observations of the changes in consumption that took place at the time of sovereign default suggest that the slope of an estimated linear relationship between δ_t and Δc_t is of -20% (see Appendix A.2). The fact that this coefficient is negative indicates that the lower the recovery rate, the stronger the associated decrease in consumption. Quantitatively, $u_\delta = -0.2$ means for instance that a French sovereign default characterized by a recovery rate of 37% (given by $\exp(-\delta_t^{(e)})$ with $\delta_t^{(e)} = 1$) would be followed by a

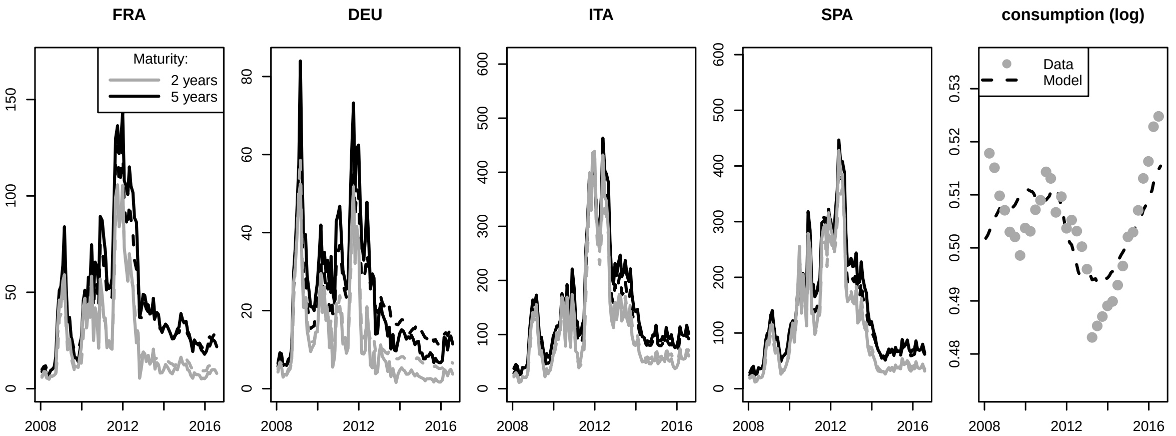
[14]Population figures are available at the annual frequency only. To get a quarterly series of per capita consumption expenditures, we report the same population figure for all four quarters of each year.

5.5% decrease in the consumption of the representative euro-area agent.[15] Third, $\alpha^{(y)}$ and $\alpha^{(1)}$ are taken to be such that the marginal means of all components of w_t^* are equal to one. Fourth, we assume that the specification of the country-specific factors $x_t^{(e)}$ (Equation 54) are the same across member states. Fifth, we assume that, for each member state, the standard deviation of the measurement errors $\eta_{cds,t}$ (Equation 62) are the same across maturities. Sixth, the standard deviation of the measurement error $\eta_{c,t}$ (Equation 62) is set to 0.20%.

Filtering technique. Because the four considered countries have not defaulted over the estimation period, we have $\delta_t = 0$, for $t \in [1, T]$. Hence, the only latent factors are the components of vector w_t^*. In the measurement Equations (62), function f is not affine in w_t. To handle this non-linearity, we use the Extended Kalman Filter.[16]

The filtering approach also has to deal with the fact that the variance of the latent factor w_t^*, conditional on $\underline{w_{t-1}}$, depends on it past values.[17] This situation is common in the estimation of term structure models involving latent affine processes with stochastic volatilities. This is addressed by implementing the standard modifications of Kalman-type filtering algorithms (see e.g. Duan and Simonato (1999) and Monfort, Pegoraro, Renne, and Roussellet (2016)).

Figure 1: Model fit



Note: For the first four charts, the dashed lines correspond to the model-implied CDS spreads, expressed in basis points, and the solid lines correspond to the data. The data span the period from January 2008 to July 2016 at the monthly frequency (end of month). The last plot compares the log of per capita real consumption (available at the quarterly frequency) with its model-implied counterpart (monthly frequency).

Results. Figure 1 illustrates the fit of the model. The first four charts compare observed and model-implied CDS with maturities of 2 and 5 years. The fifth plot shows the fit of per capita real consumption. Figure 2 compares the model-implied risk-neutral probabilities of default to their physical counterparts. It appears that the former are about two times larger than the latter (see also

[15]The share of the French GDP in our reduced euro area is 27.5%, the decrease in consumption of the representative agent is then given by $27.5\% \times \delta_t^{(e)} \times u_\delta = 27.5\% \times 1 \times 0.2 = 5.5\%$.

[16]Non-reported results show that the CDS premiums are relatively well approximated by bond credit spreads, measured as the differences between yields of defaultable bond and those of credit-risk-free bonds with same maturity. Under the RMV assumptions, the latter spreads are linear in w_t^*. This suggests that function f is close enough to a linear relationship for the Extended Kalman Filter —which is based on linear approximations of the measurement equation— to be a relevant filtering method.

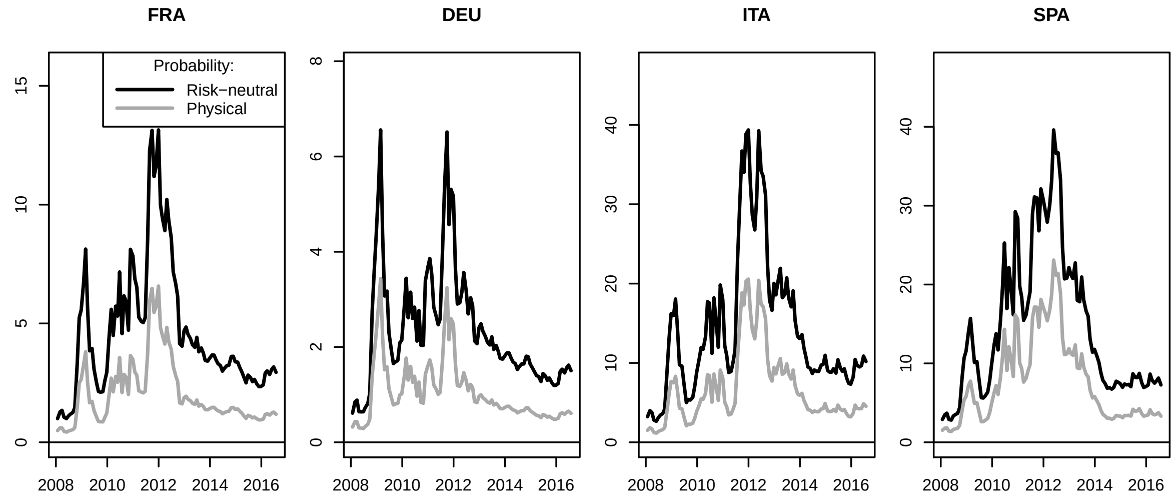
[17]In addition, the conditional distribution of w_t^* is not Gaussian.

Table 1: Parameter estimates

	Adjust.	Value	St.dev.		Adjust.	Value	St.dev.
$\alpha^{(y)}$	$(\times 10^2)$	2.375	*0.007*	σ_1	$(\times 10^2)$	7.833	*0.398*
$\alpha^{(e)}$	$(\times 10^2)$	1.302	*0.057*	σ_2	$(\times 10^2)$	4.095	*0.208*
				σ_3	$(\times 10^2)$	24.410	*1.241*
$\beta_y^{(y)}$		0.976	*0.000*	σ_4	$(\times 10^2)$	24.536	*1.248*
$\beta_x^{(e)}$		0.981	*0.000*				
				c_0		0.501	*0.004*
$\beta_y^{(e)}$	$(\times 10^2)$	0.575	*0.057*	μ_c	$(\times 10^5)$	80.825	*21.209*
$\beta_{1,x}^{(\delta)}$	$(\times 10^4)$	0.433	*0.052*				
$\beta_{2,x}^{(\delta)}$	$(\times 10^4)$	0.219	*0.021*	γ		7.000	–
$\beta_{3,x}^{(\delta)}$	$(\times 10^4)$	0.640	*0.117*	ψ		1.500	–
$\beta_{4,x}^{(\delta)}$	$(\times 10^4)$	0.932	*0.163*	δ		0.998	–
u_x		−1.092	*0.234*				
u_δ		−0.200	–				

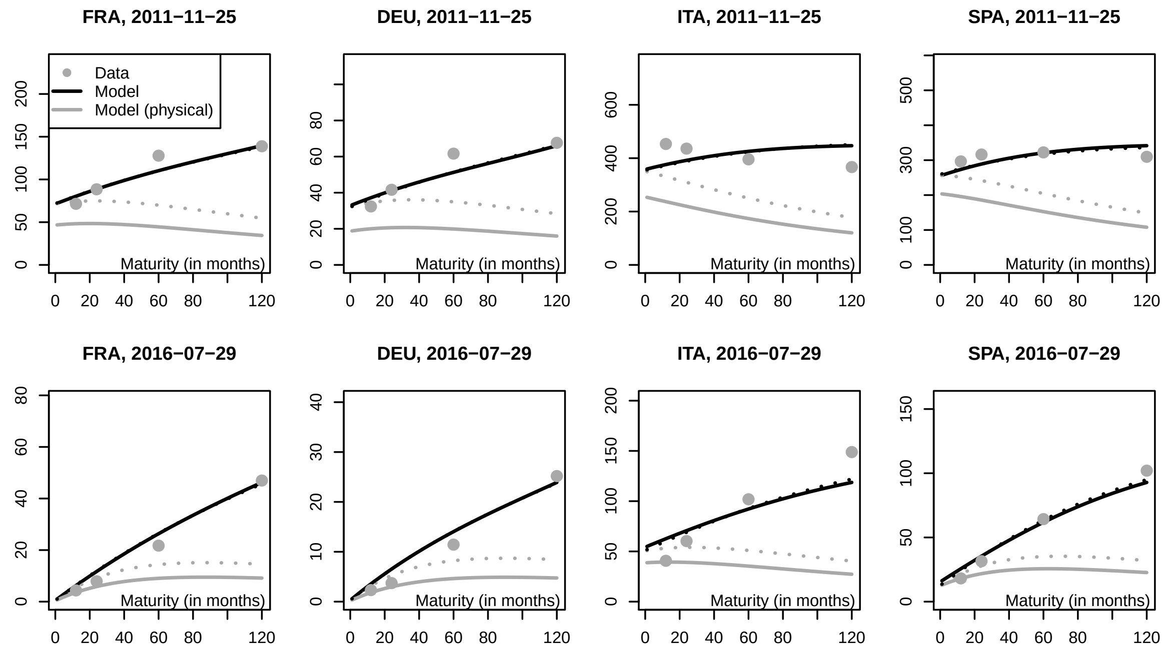
Note: The model is estimated by maximizing an approximate likelihood function derived from a modified Kalman filter. Standard deviations (in italics) are based on an estimate of the asymptotic covariance matrix involving both the Hessian matrix of the log-likelihood function and the outer-product of the log-likelihood gradient, evaluated at the estimated parameter values. Those parameters for which a "–" is reported in the St.dev. column are calibrated (see text). Parameter c_0 is the estimate of the first value of the fitted real consumption. France, Germany, Italy and Spain are respectively indexed by 1, 2, 3 and 4.

Figure 2: Model-implied physical and risk-neutral probabilities of default (5-year horizon)



Note: This figure displays model-implied probabilities of default at the 5-year horizon. Probabilities of default are expressed in percentage points.

Figure 3: Term structures of CDS spreads under the physical and risk-neutral measures



Note: This figure displays term structures of CDS spreads for two dates. The grey circles indicate observed CDS spreads. The black lines are the model-implied CDS spreads (under the risk-neutral measure). The grey lines correspond to the physical CDS spreads, that are the spreads that would be observed if agents were not risk-averse. The dotted lines (black and grey) are based on an alternative model where $u_\delta = 0$, i.e. where there is no direct effect of sovereign defaults on consumption. CDS spreads are expressed in basis points.

Ang (2013)). This is consistent with the fact that a sovereign default is a bad state of the world, characterized by low consumption and therefore a high marginal utility and, hence, a high value of the stochastic discount factor. Figure 3 displays the term structure of model-implied CDS spreads for two dates (black solid lines). The first date, November 2011, corresponds to a peak of the euro-area sovereign debt crisis. The second is the last date in our sample. On each plot, the grey solid line is the term structure of spreads we would observe if agents were not risk-averse, obtained by using the physical dynamics instead of the risk-neutral one in the pricing formula. For the crisis date (first row of charts), we observe sizeable credit-risk premiums for all maturities, including short ones. Such high short-term credit-risk premiums could not be obtained with a model where defaults do not negatively cause consumption. To illustrate this, we have estimated a model where there is no direct influence of sovereign defaults on consumption, i.e. where $u_\delta = 0$ (under this condition, δ_t does not enter the SDF anymore). The associated physical and risk-neutral term structures of CDS spreads are, respectively, the grey and black dotted lines displayed on the charts. In this case, short-term credit-risk premiums are small.

In the large literature on the topic (see e.g. Duffie, Pedersen, and Singleton (2003), Pan and Singleton (2008), Longstaff, Pan, Pedersen, and Singleton (2011) and the recent survey by Augustin, Subrahmanyam, Tang, and Wang (2015)), it is now widely acknowledged that the observed sovereign CDS spreads and bond prices are plagued by other components than pure credit risk, such as counterparty risk of protection buyer and seller (see Pu, Junbo, and Wu (2011)), the capital fluctuation of CDS' protection sellers (see Siriwardane (2016)), and illiquidity (see e.g. Bao, Pan, and Wang (2011), Coro, Dufour, and Varotto (2013) or Oehmke and Zawadowski (2016)). There is a particularly large literature on the latter – notably surveyed by Brigo, Predescu, and Capponi (2009) – where the failure of structural models to replicate the *credit spread puzzle* is largely attributed to the omission of liquidity pricing or too restrictive pricing kernels (see e.g. Huang and Huang (2012), Bao and Pan (2013), Berndt (2015)). Although intensity-based pricing models such as ours can be successfully used to identify liquidity components in asset prices (see Monfort and Renne (2014) or Dubecq, Monfort, Renne, and Roussellet (2016) for instance), departing from independence between credit and liquidity risks is needed (see e.g. He and Xiong (2012)) and finding appropriate proxies for the latter can be extremely difficult (see e.g. Helwege, Huang, and Wang (2014), Dick-Nielsen, Feldhutter, and Lando (2012)). As a result, the size of the illiquidity component in both bonds and CDSs and which leg is mostly affected in the latter are still debated (see e.g. Brigo, Predescu, and Capponi (2009), Chen, Cui, He, and Milbradt (2017), Tang and Yan (2007) and Buhler and Trapp (2008)). We thus leave this avenue for further research, and consider that (at least) part of the credit spread puzzle can be solved by the introduction of credit-event variables in the pricing kernel, as in our current framework.

5.2 Sovereign defaults and exchange rates

In the previous illustration, we were implicitly using euro-denominated CDS. However, CDS protection on many international corporations and on sovereign entities are available in euros and in U.S. dollars. While most of European sovereign bonds are denominated in euros, a large share of European CDS are denominated in dollars. This is because the latter provides a better protection against a potential severe depreciation of the bond's currency in case of credit event (see Fontana and Scheicher (2010)).[18] To understand this, recall that the notional of a euro-denominated CDS is fixed in euros and that of a

[18]The study of the potential liquidity differences between euro-denominated and dollar-denominated bonds, mentioned e.g. in Credit Suisse (2010), is beyond the scope of this paper.

dollar-denominated CDS is fixed in dollars. Therefore, a euro depreciation leads to an increase of the notional of the dollar-denominated CDS expressed in euros. Formally, consider two CDS negotiated at date t: the first is a maturity-h euro-denominated CDS and the second is a dollar-denominated one with the same maturity. At inception, we consider that both CDS have identical face values, say N euros for the former and $N\exp(-s_t)$ dollars for the latter (where s_t is the log of the exchange rate between the domestic and the foreign currency). Assume that entity e defaults before the maturity of the contract and that the default triggers a euro depreciation: $s_{\tau^{(e)}} - s_t > 0$. Then, the payoff of the protection leg is higher for the dollar-denominated CDS than for the euro-denominated one. Indeed, we have $N(1 - RR^{(e)}_{\tau^{(e)}})\exp(s_{\tau^{(e)}} - s_t) > N(1 - RR^{(e)}_{\tau^{(e)}})$. Therefore, if the defaults of euro member states tend to be accompanied by euro depreciations, we expect dollar-denominated CDS to have higher spreads than euro-denominated ones. The data are consistent with this view: the quanto CDS, defined by the deviation between a dollar-denominated CDS spread and a euro-denominated one, are mostly positive (see the grey lines in Figure 4).

In the following, we show that, once the previous model is augmented with the €-$ exchange rate, it can capture the main fluctuations of the term structure of the quanto CDS for the four countries into consideration.

Let us assume that s_t, the logarithm of the exchange rate, is given by:

$$s_t = \chi + y_{2,t} + u'_{s,\delta}\delta_t, \tag{64}$$

where $y_{2,t}$ is an additional component of w^*_t, namely $w^*_t = (y_{1,t}, y_{2,t}, x'_t)'$, where $y_{1,t}$ denotes now the scalar common factor introduced in Section 5.1. As long as the elements of $u_{s,\delta}$ are positive, a sovereign default implies a depreciation of the euro with respect to the U.S. dollar.[19] Since we have observed no default during the considered period, we have $s_t = \chi + y_{2,t}$ for $t \in [1, T]$. We assume that process $y_{2,t}$ is autonomous and the parameterization of this process is defined so as to match the sample mean, variance and autocorrelation of a measure of real exchange rate over our estimation period.[20] We get:

$$\left(y_{2,t}|\underline{w_{t-1}}\right) \sim \gamma_{326.6}\left(2676 \times y_{2,t-1}, 3.5 \times 10^{-4}\right). \tag{65}$$

Estimated values of the elements of $u_{s,\delta}$ are obtained by minimizing the squared deviations between the observed and the model-implied quanto CDS. According to the results, on average, sovereign defaults in France, Germany, Italy and Spain would respectively trigger euro depreciations of 17%, 20%, 9% and 12%. As expected, these results suggest in particular that defaults by France and Germany, the two largest economies of this reduced euro-area, would have stronger impacts on the €-$ exchange rate. Figure 4 compares observed and model-implied quanto CDS. On average across countries and maturities, this simple model accounts for more than 50% of the variances of observed quanto CDS. Let

[19] Because $y_{2,t} \geq 0$, χ has to be negative enough to allow for possible large euro appreciation (assuming the elements of u_δ are nonnegative). We set $\chi = -2$. This implies that the lowest possible exchange rate is 1 \$ = 0.13 €.

[20] We build a measure of real €-$ exchange rate by combining a nominal €-$ exchange rate series extracted from Datastream and consumer price indices coming from Eurostat for the euro area and from the U.S. Bureau of Labor Statistics for the U.S.. The hypothesis according to which process $y_{2,t}$ is autonomous is notably supported by close-to-zero sample correlations between the exchange rate and observed CDS spreads.

us stress that this extended model does not involve an additional latent factor; $y_{2,t}$ is indeed observed for $t \in [1, T]$.

Equation (64) assumes that credit risk affects the exchange rate through the credit event variables only. What if we allow w_t^* to have a direct impact on the exchange rate? To investigate this, we consider this alternative specification for the exchange rate:

$$s_t = \chi + y_{3,t} + u_{s,y}\, y_{1,t} + u_{s,x}'\, x_t\,, \tag{66}$$

where $y_{3,t}$ replaces $y_{2,t}$ in the vector w_t^*. As in the previous case, we estimate $u_{s,y}$ and $u_{s,x}$ by optimizing the fit of observed quanto CDS.[21] The fit of the quanto CDS resulting from this new model is approximately the same as the previous one. However, its implications are unreasonable, in the sense that the estimated effect of $\left(y_{1,t}, x_t^{(1)}, \dots, x_t^{(E)}\right)$ on the exchange rate is too strong. For instance, during the summer of 2011, the euro depreciated by about 10% while, according to the estimated $u_{s,y}$ and $u_{s,x}$, the increase in the components of $\left(y_{1,t}, x_t^{(1)}, \dots, x_t^{(E)}\right)$ alone would have implied a 50% depreciation of the euro. To compensate this effect, $y_{3,t}$ had to experience a sharp decrease. Such a strong adjustment of $y_{3,t}$ is not consistent with the assumption that $y_{3,t}$ is independent from $\left(y_{1,t}, x_t^{(1)}, \dots, x_t^{(E)}\right)$.

Therefore, these results suggest that it is the relationship between the exchange rate and the credit events *per se*, and less between the exchange rate and conditional default probabilities – driven by $\left(y_{1,t}, x_t^{(1)}, \dots, x_t^{(E)}\right)$ – that is key to explain the fluctuation of quanto CDS. These results are in line with those of Ehlers and Schonbucher (2004) and of Brigo, Pede, and Petrelli (2015).

5.3 Financial contagion

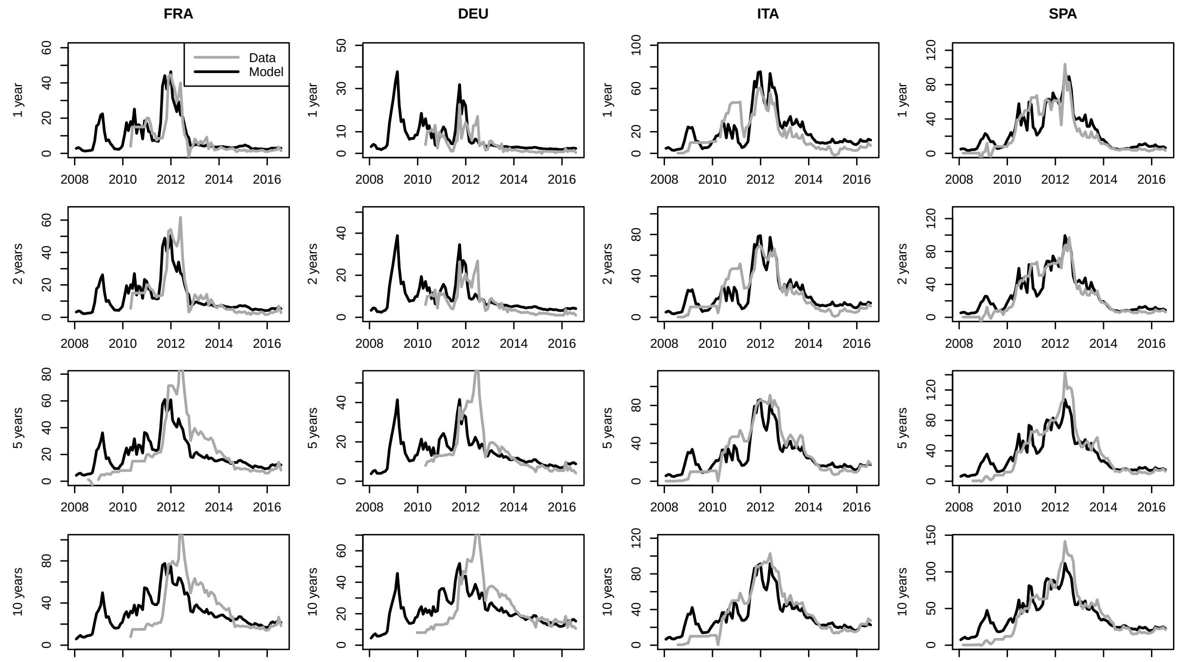
In our framework, contagion can arise from different channels. For instance, we can have direct contagion effects if off-diagonal elements of $\beta_\delta^{(\delta)}$ are different from zero. Indirect contagion effects are obtained if some components of δ_t causes w_{t+1}^* and if the latter, in turn, causes other elements of δ_{t+1}. In this subsection, we focus on the indirect contagion context and propose a specification able to qualitatively match the behavior of banks' CDS spreads that was observed in the aftermath of the Lehman Brothers' bankruptcy.

Figure 5 displays the CDS of four large banks in the months pre- and post-Lehman Brothers' default. All CDS spreads were positively affected by the event. The figure also shows that the effects are not quantitatively the same across entities: for instance, while the 2-year CDS premium of Goldman Sachs increased by 400 basis points from August to September 2008, the increase was twice lower for Citigroup.

Let us show how our model can generate a similar situation. We consider $E = 3$ defaultable entities. For the sake of simplicity, it is assumed here that the risk-free short-term rate is null and that the SDF is equal to one. The calibration of this model is ad hoc, the goal of this exercise being merely to illustrate the potential of our framework in terms of contagion modelling.

[21] During the estimation process, for each trials of parameters $(u_{s,y}, u_{s,x}')'$, $y_{3,t}$ is computed as $s_t - \chi - u_{s,y}\, y_{1,t} - u_{s,x}'\, x_t$ and, as previously, an auto-regressive gamma specification for $y_{3,t}$ (as in Equation (65)) is obtained by matching its sample mean, variance and autocorrelation. Because we expect an increase in credit risk to depreciate the euro, the elements of $u_{s,y}$ and $u_{s,x}$ are imposed to be positive.

Figure 4: Observed and model-implied quanto CDS



Note: This figure compares observed and model-implied quanto CDS (expressed in basis points). Quanto CDS are given by the differences between dollar-denominated CDS premiums and their euro-denominated counterparts. For some countries and maturities, Datastream-extracted CDS premiums are the same for the euro- and dollar-denominated CDS; in these cases, the data are removed from the estimation sample.

For a given entity e, the credit event process $\delta_t^{(e)}$ is instantaneously caused by two specific factors $x_{1,t}^{(e)}$ and $x_{2,t}^{(e)}$. That is, we have:

$$\left(\delta_t^{(e)}|\underline{w_{t-1}}, y_t\right) \sim \gamma_0 \left(4.10^{-5}\, x_{1,t}^{(e)} + 5.10^{-3}\, x_{2,t}^{(e)}, 1\right).$$

An additional factor, denoted by y_t, is aimed at creating commonality among the $x_{1,t}^{(e)}$'s. Factor y_t is characterized by an autonomous dynamics:

$$\left(y_t|\underline{w_{t-1}}\right) \sim \gamma_{0.1} \left(0.97\, y_{t-1}, 1\right).$$

Hence, in our model, the vector w_t^* is of dimension $2E+1 = 7$: $w_t^* = \left(y_t, x_{1,t}^{(1)}, x_{2,t}^{(1)}, x_{1,t}^{(2)}, x_{2,t}^{(2)}, x_{1,t}^{(3)}, x_{2,t}^{(3)}\right)$. The conditional distribution of factors $x_{1,t}^{(e)}$ is:

$$\left(x_{1,t}^{(e)}|\underline{w_{t-1}}\right) \sim \gamma_0 \left(5.10^{-3} + 0.5\, y_{t-1} + 0.9\, x_{1,t-1}^{(e)}, 1\right).$$

Therefore, the credit-event processes are Granger-caused by y_t and the $x_{1,t}^{(e)}$'s, but the reverse is not true. By contrast, factors $x_{2,t}^{(e)}$ are caused by the credit-event variables:

$$\left(x_{2,t}^{(e)}|\underline{w_t}\right) \sim \gamma_0 \left([100, 100, 100]\, \delta_{t-1} + 90\, x_{2,t-1}^{(e)}, 0.01\right).$$

It has to be stressed that, for any entity e, factor $x_{2,t}^{(e)}$ depends on all three credit-event processes. Hence, this model allows for contagion phenomena since, through $x_{2,t}^{(e)}$, the credit-event process of entity e is caused by those of other entities.

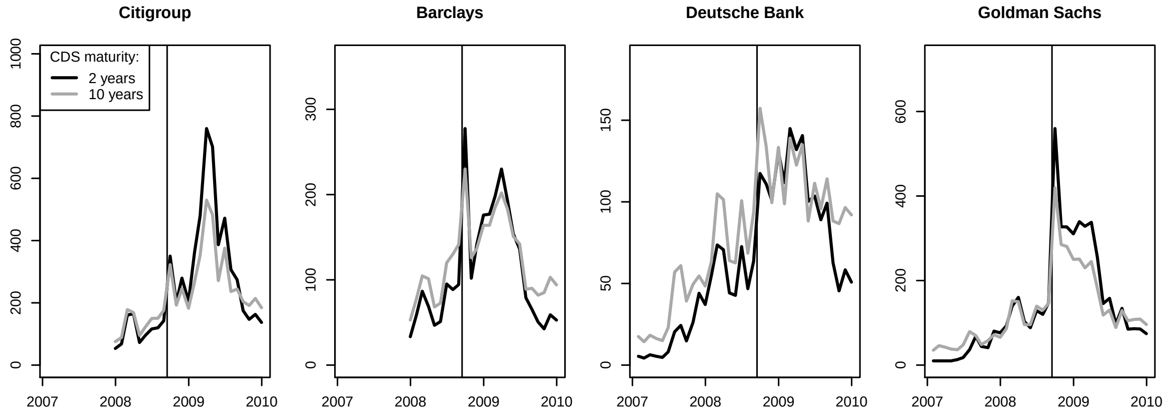
The outputs of a 100-period simulation of this model are displayed on Figure 6. Panel (d) shows that entity 1 defaults on period 50. As can be seen on Panel (c), this default gives rise to sharp and persistent increases in factors $x^{(e)}_{2,t}$. Panels (e) and (f) show how these increases translates into rises in CDS spreads. Besides, even though the specifications of the country-specific processes are identical, the effects of entity 1's default on CDS spreads of different entities are not the same; this is because the country-specific contagion factors react differently to the default of entity 1.

6 Conclusion

We present a general affine credit risk model able to relax the classical restrictive assumptions employed in the reduced-form credit risk literature while preserving tractability in the pricing of default-sensitive securities. Building on the recent non-negative affine Gamma-zero process, we are able to introduce simultaneously the presence of systemic risk, contagion between entities, credit events pricing and stochastic recovery rates. In a recursive affine modeling framework characterized by the presence of common and idiosyncratic factors, and regardless the considered recovery convention, we show that defaultable securities such as defaultable bonds and CDS can be easily priced with explicit or quasi-explicit formulas.

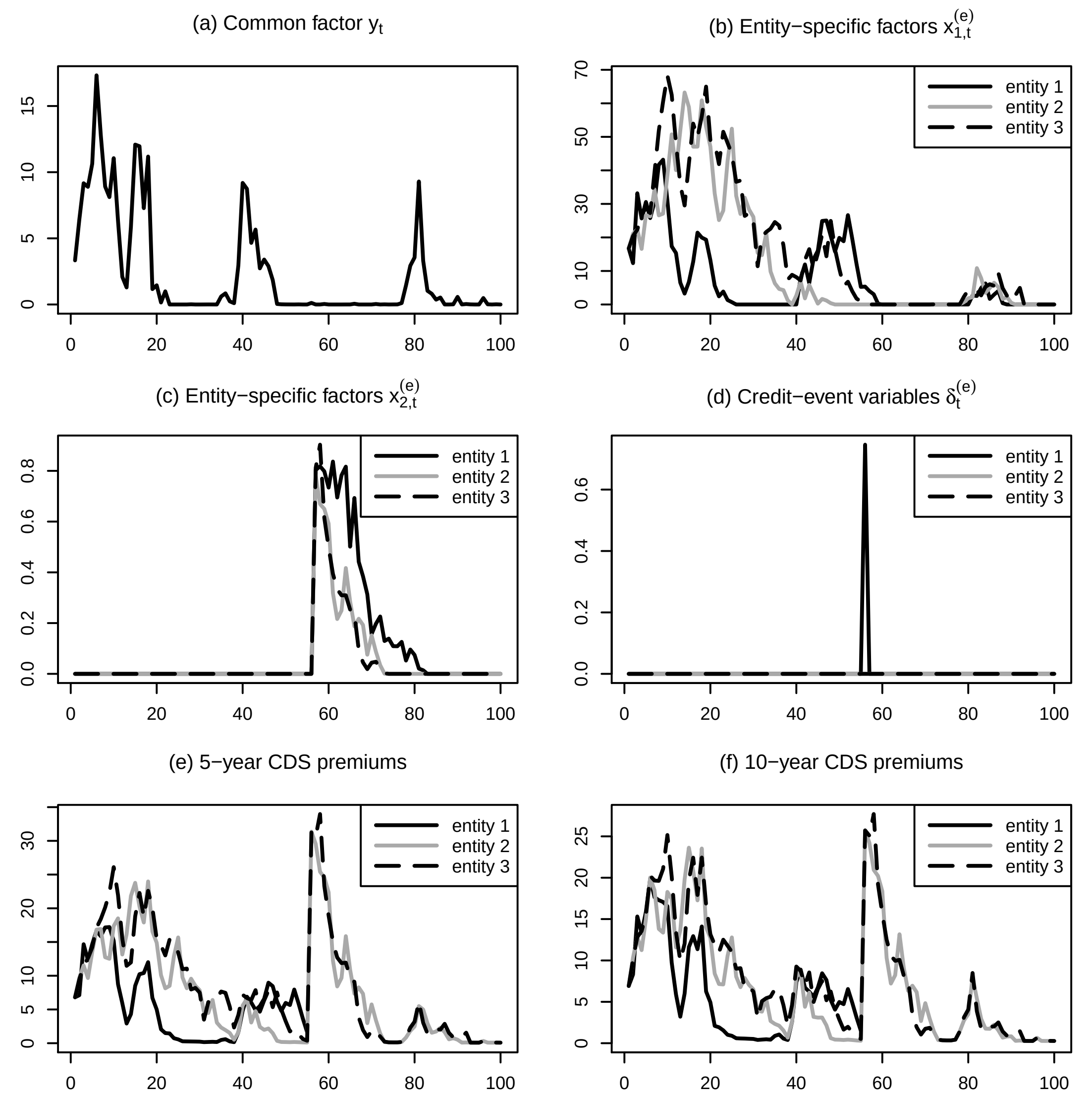
We believe that our framework opens the way to a wide range of financial applications. We provide examples of applications highlighting the potential of our model to investigate the pricing of credit risk. Our results show in particular that the systemic nature of a sovereign default has implications on the shape of the term structure of associated credit-risk premiums. Besides, the flexibility of our framework is exploited in order to model quanto CDSs, which are the differences between CDS premiums on the same entity but denominated in different currencies. The latter model is used to compute market-based expectations of the depreciation that would be triggered by a sovereign default. Eventually, we show the ability of the model to replicate the behavior of banks' CDS spreads that were observed in the aftermath of the Lehman Brothers' bankruptcy.

Figure 5: Lehman Brothers' bankruptcy and banks' CDS



Note: The vertical bar indicates Lehman Brothers' bankruptcy. CDS premiums are expressed in basis points.

Figure 6: Simulation of contagion phenomena



Note: This figure displays the results of a simulation of the model presented in Section 5.3. See text for details.

Appendix

A.1 Sovereign Credit Risk Application: Solving the model

Assume that z_t is of the form $A_0 + A_1' w_t$. Equation (59) becomes:

$$r_{a,t+1} \approx \kappa_0 + \kappa_1 (A_0 + A_1' w_{t+1}) - (A_0 + A_1' w_t) + \mu_c + \theta_c' w_{t+1}, \tag{a.1}$$

Substituting the previous equation in the Euler Equation (58), we get:

$$\begin{aligned} 1 \quad = \quad & \mathbb{E}\Big[\exp \Big(\theta^* \log \delta^* - \frac{\theta^*}{\psi^*} (\mu_c + \theta_c' w_{t+1}) + \\ & \theta^* \left\{ \kappa_0 + \kappa_1 (A_0 + A_1' w_{t+1}) - (A_0 + A_1' w_t) + \mu_c + \theta_c' w_{t+1} \right\} \Big) | \underline{w_t} \Big] \end{aligned}$$

which must be satisfied for any w_t. The previous equation can be written:

$$1 = \mathbb{E}_t \left[\exp(\nu_0 + \nu_1' w_{t+1} - \nu_2' w_t) \right] \tag{a.2}$$

where

$$\begin{aligned} \nu_0 &= \theta^* \log \delta^* + \theta^* (\kappa_0 + A_0 [\kappa_1 - 1]) + \left(\theta^* - \frac{\theta^*}{\psi^*} \right) \mu_c \\ \nu_1 &= \left(\theta^* - \frac{\theta^*}{\psi^*} \right) \theta_c + \theta^* \kappa_1 A_1 \\ \nu_2 &= \theta^* A_1. \end{aligned}$$

Hence we must have:

$$\nu_0 + B^{(w)}(\nu_1) + \left(A^{(w)}(\nu_1) - \nu_2 \right)' w_t = 0, \tag{a.3}$$

where function $A^{(w)}$ and $B^{(w)}$ define the log conditional Laplace transform of w_t (see Appendix 2).

Because the previous equation has to be satisfied for any value of w_t, we get:

$$\left\{ \begin{aligned} \nu_0 + B^{(w)}(\nu_1) &= 0 \\ A^{(w)}(\nu_1) - \nu_2 &= 0. \end{aligned} \right. \tag{a.4}$$

In the previous system, the number of equations $(2E+2)$ is equal to the number of unknowns –that are A_0 and the components of A_1. Hence, solving the model amounts to solving this system of equations. Because these equations are non-linear, they have to be solved numerically. After having computed the analytical first-order derivatives of $(A_0, A_1) \rightarrow B^{(w)}(\nu_1)$ and $(A_0, A_1) \rightarrow A^{(w)}(\nu_1)$, the Gauss-Newton algorithm makes this resolution fast.

At this stage, an inconsistency may hold and has to be dealt with. The A_0 and A_1 resulting from the resolution of System (a.4) depend on $\bar{z}$ (through equations 60). Let's denote by $A_0(\bar{z})$ and $A_1(\bar{z})$ these solutions. Since $z_t = A_0(\bar{z}) + A_1(\bar{z})'w_t$, we should have

$$\bar{z} = A_0(\bar{z}) + A_1(\bar{z})'\bar{w}, \tag{a.5}$$

where $\bar{w}$ is the unconditional mean of w_t. The previous equation, whose unknown is $\bar{z}$, can also be solved by means of the Gauss-Newton algorithm. Note here a difficulty: for each iteration of the Gauss-Newton algorithm employed to solve Equation (a.5), we must use the Gauss-Newton algorithm to solve System (a.4).

A.2 Sovereign Credit Risk Application: Recovery rates and the relationship between δ_t and Δc_t

Data on 1983-2015 sovereign defaults are used to calibrate two parameters of the model presented in Subsection 5.1: $\mu^{(\delta)}$, which defines the scale of credit events $\delta_t^{(e)}$ (Equation 55), and u_δ, which measures the influence of sovereign defaults on consumption growth Δc_t (Equation 57). Default data are from Moody's (2016). They cover 22 sovereign defaults:

Russia (1998), Pakistan (1999), Ecuador (1999), Ukraine (2000), Ivory Coast (2000), Argentina (2001), Moldova (2002), Uruguay (2003), Nicaragua (2003), Grenada (2004), Dominican Republic (2005), Belize (2006), Seychelles (2008), Ecuador (2008), Jamaica (2010), Greece (2012), Greece (2012), Belize (2012), Cyprus (2013), Jamaica (2013), Argentina (2013), Ukraine (2015).

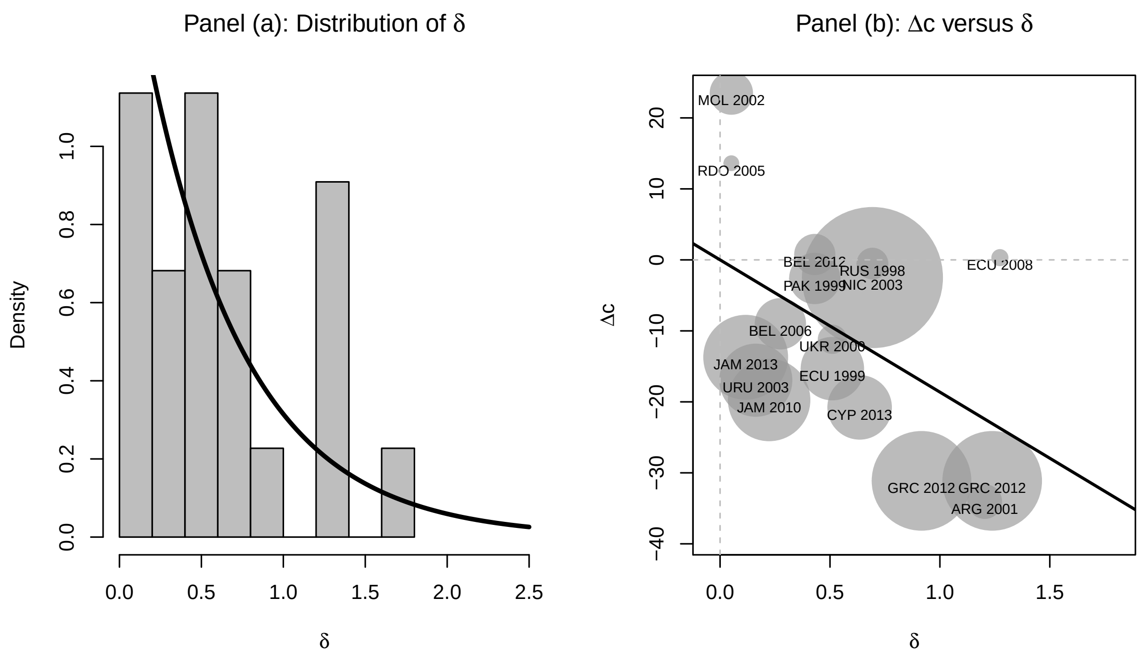
The scale parameter $\mu^{(\delta)}$. Two kinds of recovery rate estimates are considered by Moody's (2016, Exhibit 11). The first one is based on the 30-day post-default price or distressed exchange trading price. The second is the ratio of the present value of cash flows received as a result of the distressed exchange versus those initially promised, discounted using yield to maturity immediately prior to default. For each default, we compute the average of the two ratios when both are available and we take the only one that is available otherwise. Let's denote by $\overline{RR}_i$, $i \in 1, \dots, 22$, the resulting recovery rates. Panel (a) of Figure 7 shows an histogram of $-\log(\overline{RR}_i)$.

Conditional on a default at date t (i.e. $\delta_t^{(e)} > 0$), the distribution of $\delta_t^{(e)}$ is approximately a gamma distribution with a unit shape parameter and a scale parameter of $\mu^{(\delta)}$. (The approximation is accurate if the date-t probabilities of default, conditional on $(\underline{w_{t-1}}, w_t^*)$ are small.) Note further that, in our specification, we have $\delta_t^{(e)} \equiv -\log(RR_t^{(e)})$. Therefore, the sample average of the $-\log(\overline{RR}_i)$, that is 0.6, is used as an estimate of $\mu^{(\delta)}$. The red line in Panel (a) of Figure 7 shows the resulting approximate distribution of $-\log(RR_t^{(e)})$.

The relationship between δ_t and Δc_t. The fact that sovereign defaults are generally accompanied by falls in real consumption is consensual in the literature. In our model, it is assumed, more specifically, that there is a negative linear relationship between consumption growth Δc and the credit event variable δ.

According to Panel (b) of Figure 7, the fact that consumption growth Δc is negatively correlated to $-\log(RR)$ is supported by the data. The per capita consumption data used for this plot come from the World Bank. For each defaulted country, we have computed the change in real consumption over the period $[Y-1, Y+1]$, where Y is the year of default. In order to adjust these measures for trend growth, we subtract the average global consumption growth (3.5% per year) to the previous figure. This provided us with the y-coordinates of the points; the x-coordinates are the $-\log(\overline{RR}_i)$ discussed above. The slope of the fitted linear relationship is -18%. As a result, we set $u_\delta = -20\%$. The sizes of the circles are proportional to the debt-to-GDP ratios prevailing in the defaulted countries on the year of default: It appears that for the two cases where the default was followed by an increase in consumption (Moldova in 2002 and Dominican Republic in 2005), the debt-to-GDP ratios were relatively small.

Figure 7: Sovereign recovery rates and consumption growth



Note: Panel (a) displays an histogram of $-\log(\overline{RR}_i)$, where $\overline{RR}_i$, $i \in 1, \dots, 22$, are estimates of the recovery rates of sovereign defaults that took place over the last thirty years (Moody's, 2016). In our specification, $-\log(\overline{RR})$ is identical to the credit event variable δ. The red line shows the density function of a gamma distribution with a shape parameter of 1 and a scale parameter of 0.6, which is the sample mean of $-\log(\overline{RR})$. In the model, this gamma distribution approximately corresponds to the distribution of $\delta_t^{(e)}$ conditional on default (i.e. on $\delta_t^{(e)} > 0$). Panels (b) is meant to guide the calibration of u_δ, which defines the influence of the credit event variables δ_t on consumption growth Δc_t. It displays changes in real per capita consumer expenditures over the period $[Y-1, Y+1]$, where Y denotes the year of default, against $\delta \equiv -\log(\overline{RR})$. (In order to adjust from trend growth, we subtract the global average consumption growth rate from that of the considered countries.) The sizes of the green circles are proportional to the debt-to-GDP ratio prevailing in the defaulted countries on the year of default.

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Online Appendix

Affine Modelling of Credit Risk, Pricing of Credit Events and Contagion

Alain Monfort, Fulvio Pegoraro, Guillaume Roussellet and Jean-Paul Renne

A.3 Classical Credit Risk Models: Proof of Propositions

Proof of Proposition 2.3

a) is a direct consequence of the definition of $f^{\mathbb{Q}}(w_t \,|\, w_{t-1})$;

b) $p_2^{\mathbb{Q}}(d_{2,t} \,|\, d_{1,t}, y_t, w_{t-1})$ is proportional to $f^{\mathbb{Q}}(w_t \,|\, w_{t-1})$ and therefore equal to $p_2(d_{2,t} \,|\, y_t, y_{t-1}, d_{2,t-1})$;

c) is a direct consequence of *b*);

d) is obtained by summing $f^{\mathbb{Q}}(w_t \,|\, w_{t-1})$ over the values of $d_{2,t}$ (i.e. 0 and 1);

e) is obtained by noting that $p_1^{\mathbb{Q}}(d_{1,t} \,|\, y_t, w_{t-1})$ is proportional to $f^{\mathbb{Q}}(y_t, d_{1,t} \,|\, w_{t-1})$;

f) comes from *e*) by putting $d_{1,t} = 0$, $d_{1,t-1} = 1$ and noting that $p_1(0 \,|\, y_t, y_{t-1}, 1) = 0$;

g) is obtained by noting that $p_1^{\mathbb{Q}}(0 \,|\, y_t, y_{t-1}, 0)$ is proportional to $M_{t-1,t}(y_t, y_{t-1}, 0, 0) \exp\left[-\lambda_{1,t}(y_t, y_{t-1})\right]$, $p_1^{\mathbb{Q}}(1 \,|\, y_t,\, y_{t-1}, 0)$ is proportional to $M_{t-1,t}(y_t, y_{t-1}, 1, 0)\, \{1 - \exp\left[-\lambda_{1,t}(y_t, y_{t-1})\right]\}$, and that the coefficient of proportionality (function of the conditioning information set) is cancelling out in the ratio.

Proof of Proposition 2.4.

Assuming $d_{e,t} = 0$, given a recovery payment $\mathcal{P}^{(e)}_{t+i,h-i}$, under assumptions S.1 to S.4 and using $\mathbb{1}_{\{d'_{e,t:t+i-1}\mathbf{1}=0\}}\, \mathbb{1}_{\{d_{e,t+i}>0\}} = \mathbb{1}_{\{d'_{e,t:t+i-1}\mathbf{1}=0\}} - \mathbb{1}_{\{d'_{e,t:t+i}\mathbf{1}=0\}}$ (with $d_{e,t:t+i} = (d_{e,t}, \dots, d_{e,t+i})'$ and $\mathbf{1} = (1, \dots, 1)'$ of conformable dimension), we have:

$$\begin{aligned}
B_e(t,h) &= \sum_{i=1}^{h} \mathbb{E}_t^{\mathbb{Q}} \Big\{ \exp\Big(- r_t - \ldots - r_{t+i-1} \Big) \mathcal{P}^{(e)}_{t+i,h-i} \Big[\mathbb{1}_{\{d'_{e,t+i-1}\mathbf{1}=0\}} - \mathbb{1}_{\{d'_{e,t+i}\mathbf{1}=0\}} \Big] \Big\} \\
&\qquad + \mathbb{E}_t^{\mathbb{Q}} \Big[\exp\Big(- r_t - \ldots - r_{t+h-1} \Big) \mathbb{1}_{\{d'_{e,t+h}\mathbf{1}=0\}} \Big] \\
&= \sum_{i=1}^{h} \mathbb{E}_t^{\mathbb{Q}} \Big\{ \exp\Big(- r_t - \ldots - r_{t+i-1} \Big) \mathcal{P}^{(e)}_{t+i,h-i} \\
&\qquad \times \Big[\mathbb{Q}\Big(d'_{e,t:t+i-1}\mathbf{1} = 0 \,|\, \underline{y_{t+i}}, d_{e,t} = 0 \Big) - \mathbb{Q}\Big(d'_{e,t:t+i}\mathbf{1} = 0 \,|\, \underline{y_{t+i}}, d_{e,t} = 0 \Big) \Big] \Big\} \\
&\qquad + \mathbb{E}_t^{\mathbb{Q}} \Big[\exp\Big(- r_t - \ldots - r_{t+h-1} \Big) \mathbb{Q}\Big(d'_{e,t:t+h}\mathbf{1} = 0 \,|\, \underline{y_{t+h}}, d_{e,t} = 0 \Big) \Big]
\end{aligned}$$

$$
\begin{aligned}
= & \sum_{i=1}^{h} \mathbb{E}_t^{\mathbb{Q}} \Big\{ \exp \Big(- r_t - \ldots - r_{t+i-1} \Big) \mathcal{P}_{t+i,h-i}^{(e)} \\
& \times \Big[\prod_{j=1}^{i-1} \mathbb{Q} \Big(d_{e,t+j} = 0 \,|\, d_{e,t+j-1} = 0, \underline{y_{t+i}} \Big) - \prod_{j=1}^{i} \mathbb{Q} \Big(d_{e,t+j} = 0 \,|\, d_{e,t+j-1} = 0, \underline{y_{t+i}} \Big) \Big] \Big\} \\
& + \mathbb{E}_t^{\mathbb{Q}} \Big[\exp \Big(- r_t - \ldots - r_{t+h-1} \Big) \prod_{j=1}^{h} \mathbb{Q} \Big(d_{e,t+j} = 0 \,|\, d_{e,t+j-1} = 0, \underline{y_{t+h}} \Big) \Big] .
\end{aligned}
$$

Since $d_{e,t}$ does not Granger cause y_t under $\mathbb{Q}$, and using the equivalence between Granger non-causality and Sims non-causality, y_{t+i} can be replaced by y_{t+j} in the conditional probabilities:

$$
\mathbb{Q} \Big(d_{e,t+j} = 0 \,|\, d_{e,t+j-1} = 0, \underline{y_{t+i}} \Big) = \mathbb{Q} \Big(d_{e,t+j} = 0 \,|\, d_{e,t+j-1} = 0, \underline{y_{t+j}} \Big) = \exp \left[-\lambda_e \left(y_{t+j}, y_{t+j-1} \right) \right] ,
$$

and, therefore, the zero-coupon bond price can be written as:

$$
\begin{aligned}
B_e(t,h) = & \sum_{i=1}^{h} \mathbb{E}_t^{\mathbb{Q}} \Big\{ \exp \Big(- r_t - \ldots - r_{t+i-1} \Big) \mathcal{P}_{t+i,h-i}^{(e)} \\
& \times \Big[\prod_{j=1}^{i-1} \mathbb{Q} \Big(d_{e,t+j} = 0 \,|\, d_{e,t+j-1} = 0, \underline{y_{t+j}} \Big) - \prod_{j=1}^{i} \mathbb{Q} \Big(d_{e,t+j} = 0 \,|\, d_{e,t+j-1} = 0, \underline{y_{t+j}} \Big) \Big] \Big\} \\
& + \mathbb{E}_t^{\mathbb{Q}} \Big\{ \exp \Big[- \sum_{j=1}^{h} \Big(r_{t+j-1} + \lambda_{e,t+j} \Big) \Big] \Big\} \\
= & \sum_{i=1}^{h} \mathbb{E}_t^{\mathbb{Q}} \Big\{ \exp \Big(- r_t - \ldots - r_{t+i-1} \Big) \mathcal{P}_{t+i,h-i}^{(e)} \Big[\exp \Big(- \sum_{j=1}^{i-1} \lambda_{e,t+j} \Big) \Big(1 - \exp(-\lambda_{e,t+i}) \Big) \Big] \Big\} \\
& + \mathbb{E}_t^{\mathbb{Q}} \Big\{ \exp \Big[- \sum_{j=1}^{h} \Big(r_{t+j-1} + \lambda_{e,t+j} \Big) \Big] \Big\} . \qquad \blacksquare
\end{aligned}
$$

A.4 Defaultable Securities Pricing Formulas

Proof of Proposition 3.1: We first check that, for $i = 1$, Equation (19) provides the right expression $\varphi_{w,t,1}^{\mathbb{Q}} \left(u_1 \right) = \exp \Big[\widetilde{A}_w \left(u_1 \right)' w_t + \widetilde{B}_w \left(u_1 \right) \Big]$ in line with relation (50). Let us now assume that (19) is satisfied for $i-1$ and any t. For a given date $t \in \{1, \ldots, T\}$ and horizon $i \in \{1, \ldots, I\}$, applying the property of iterated expectations, the multi-horizon conditional risk-neutral Laplace transform can be written as:

$$
\begin{aligned}
\varphi_{w,t,i}^{\mathbb{Q}} \left(u_i, \ldots, u_1 \right) & = \mathbb{E}^{\mathbb{Q}} \left[\exp \left(u_i' \, w_{t+1} + \ldots + u_1' \, w_{t+i} \right) \,|\, \underline{w_t} \right] \\
& = \mathbb{E}^{\mathbb{Q}} \Big\{ \exp \left(u_i' \, w_{t+1} \right) \mathbb{E}^{\mathbb{Q}} \Big[\exp \left(u_{i-1}' \, w_{t+2} + \ldots + u_1' \, w_{t+i} \right) \,|\, \underline{w_{t+1}} \Big] \,|\, \underline{w_t} \Big\} \qquad \text{(a.1)} \\
& = \mathbb{E}^{\mathbb{Q}} \Big\{ \exp \left(u_i' \, w_{t+1} \right) \varphi_{w,t+1,i-1}^{\mathbb{Q}} \left(u_{i-1}, \ldots, u_1 \right) \,|\, \underline{w_t} \Big\} \; ;
\end{aligned}
$$

replacing $\varphi^{\mathbb{Q}}_{w,t+1,i-1}(u_{i-1},\ldots,u_1)$ by $\exp\left[\mathcal{A}'_{i-1}w_{t+1}+\mathcal{B}_{i-1}\right]$ into relation (a.1) we find:

$$\begin{aligned}\varphi^{\mathbb{Q}}_{w,t,i}(u_i,\ldots,u_1) &= \exp\left[\widetilde{A}_w(u_i+\mathcal{A}_{i-i})'\,w_t+\widetilde{B}_w(u_i+\mathcal{A}_{i-i})+\mathcal{B}_{i-1}\right] \\ &= \exp\left[\mathcal{A}'_i\,w_t+\mathcal{B}_i\right]\end{aligned} \tag{a.2}$$

and therefore, by identification we have the recursion:

$$\begin{cases}\mathcal{A}_i &= \widetilde{A}_w(u_i+\mathcal{A}_{i-1})\,, \\ \mathcal{B}_i &= \widetilde{B}_w(u_i+\mathcal{A}_{i-1})+\mathcal{B}_{i-1}\,,\end{cases} \tag{a.3}$$

that has to be run only once regardless the date t and the horizon of interest i.

If the reverse order structure (18) is not satisfied, then the date-t conditional multi-horizon Laplace transform (17) is still exponential-affine in w_t but now the $\mathcal{A}_i$ and $\mathcal{B}_i$ loadings are obtained at the i^{th} step of the recursive system $\mathcal{A}_i=\mathcal{A}^{(i)}_i$, $\mathcal{B}_i=\mathcal{B}^{(i)}_i$ (see Proposition 3 in Gourieroux, Monfort, Pegoraro, and Renne (2014) and Proposition (a.1) of this appendix):

$$\begin{cases}\mathcal{A}^{(i)}_0 &= 0\,,\;\; \mathcal{B}^{(i)}_0=0\,, \\ \mathcal{A}^{(i)}_j &= \widetilde{A}_w\left(u^{(i)}_{i+1-j}+\mathcal{A}^{(i)}_{j-1}\right)\,, \\ \mathcal{B}^{(i)}_j &= \widetilde{B}_w\left(u^{(i)}_{i+1-j}+\mathcal{A}^{(i)}_{j-1}\right)+\mathcal{B}^{(i)}_{j-1}\,.\end{cases} \tag{a.4}$$

From the computational point of view, this means that for any pair of horizons $i\neq\kappa$, the associated two pairs of loadings $\left(\mathcal{A}^{(i)}_i,\mathcal{B}^{(i)}_i\right)$ and $\left(\mathcal{A}^{(\kappa)}_\kappa,\mathcal{B}^{(\kappa)}_\kappa\right)$ have two be calculated separately running for each of them a i-step and a κ-step recursion (a.4), respectively. ■

Proof of Proposition 3.2: Given relation (13), Assumption H.4 and assuming that $\mathcal{V}^{(e)}_{t+i,h-i}\left(\underline{\widetilde{w}^{(e)}_{t+i}},\right.$

$$\underline{\delta_{t+i-1}^{(e)}}, \delta_{t+i}^{(e)}\Big) = \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=i}^{h-1}\left(r_{t+j} + \delta_{t+j+1}^{(e)}\right)\right] \mid \underline{w_{t+i}}\right\} \text{ for } i = 1, \ldots, h-1, \text{ then we get:}$$

$$\begin{aligned}
& B_e(t,h) \\
= & \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{i} r_{t+j-1}\right) \exp\left(-\delta_{t+i}^{(e)}\right) \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=i}^{h-1} r_{t+j} - \sum_{j=i+1}^{h} \delta_{t+j}^{(e)}\right] \mid \underline{w_{t+i}}\right\} \mathbb{1}_{\{\delta_{t:t+i-1}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_t}\right\} \\
& -\sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{i} r_{t+j-1}\right) \exp\left(-\delta_{t+i}^{(e)}\right) \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=i}^{h-1} r_{t+j} - \sum_{j=i+1}^{h} \delta_{t+j}^{(e)}\right] \mid \underline{w_{t+i}}\right\} \mathbb{1}_{\{\delta_{t:t+i}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_t}\right\} \\
& +\mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{h} r_{t+j-1}\right) \mathbb{1}_{\{\delta_{t:t+h}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_t}\right\} \\
= & \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{\mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=1}^{h} r_{t+j-1} - \sum_{j=i}^{h} \delta_{t+j}^{(e)}\right] \mathbb{1}_{\{\delta_{t:t+i-1}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_{t+i}}\right\} \mid \underline{w_t}\right\} \\
& -\sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{\mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=1}^{h} r_{t+j-1} - \sum_{j=i}^{h} \delta_{t+j}^{(e)}\right] \mathbb{1}_{\{\delta_{t:t+i}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_{t+i}}\right\} \mid \underline{w_t}\right\} \\
& +\mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{h} r_{t+j-1}\right) \mathbb{1}_{\{\delta_{t:t+h}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_t}\right\} \\
= & \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{\mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=1}^{h} r_{t+j-1} - \sum_{j=1}^{h} \delta_{t+j}^{(e)}\right] \mathbb{1}_{\{\delta_{t:t+i-1}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_{t+i}}\right\} \mid \underline{w_t}\right\} \\
& -\sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{\mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=1}^{h} r_{t+j-1} - \sum_{j=1}^{h} \delta_{t+j}^{(e)}\right] \mathbb{1}_{\{\delta_{t:t+i}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_{t+i}}\right\} \mid \underline{w_t}\right\} \\
& +\mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{h} r_{t+j-1}\right) \mathbb{1}_{\{\delta_{t:t+h}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_t}\right\};
\end{aligned}$$

Now, by the law of iterated expectations, we can write:

$$\begin{aligned}
B_e(t,h) \quad = \quad & \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=1}^{h} r_{t+j-1} - \sum_{j=1}^{h} \delta_{t+j}^{(e)}\right] \sum_{i=1}^{h} \mathbb{1}_{\{\delta_{t:t+i-1}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_t}\right\} \\
& -\mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=1}^{h} r_{t+j-1} - \sum_{j=1}^{h} \delta_{t+j}^{(e)}\right] \sum_{i=1}^{h} \mathbb{1}_{\{\delta_{t:t+i}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_t}\right\} \\
& +\mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{h} r_{t+j-1}\right) \mathbb{1}_{\{\delta_{t:t+h}^{(e)\prime}\mathbf{1}=0\}} \mid \underline{w_t}\right\}.
\end{aligned}$$

Given that $\delta_t^{(e)} = 0$:

$$\sum_{i=1}^{h} \mathbb{1}_{\{\delta_{t:t+i-1}^{(e)\prime}\mathbf{1}=0\}} - \sum_{i=1}^{h} \mathbb{1}_{\{\delta_{t:t+i}^{(e)\prime}\mathbf{1}=0\}} + \mathbb{1}_{\{\delta_{t:t+h}^{(e)\prime}\mathbf{1}=0\}} = \mathbb{1}_{\{\delta_t^{(e)}=0\}} = 1\,,$$

and we have:

$$B_e(t,h) = \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=1}^{h}\left(r_{t+j-1} + \delta_{t+j}^{(e)}\right)\right] \mid \underline{w_t}\right\} = \mathcal{V}_{t,h}^{(e)}\left(\underline{\widetilde{w}_t^{(e)}}, 0, 0\right) .$$

Let us prove now relations (23). Given $r_t = \xi_0 + \xi_1' w_t$ and $\delta_t^{(e)} = e_\delta' \delta_t = \widetilde{e}_\delta' w_t$, $\mathcal{V}_{t,h}^{(e)}\left(\underline{w_t}\right)$ can be written as:

$$\begin{aligned}
\mathcal{V}_{t,h}^{(e)}\left(\underline{w_t}\right) &= \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=0}^{h-1}\left(r_{t+j} + \delta_{t+j+1}^{(e)}\right)\right] \mid \underline{w_t}\right\} \\
&= \exp\left[-\left(\xi_0 + \xi_1' w_t\right)\right] \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\sum_{j=1}^{h-1}\left(\xi_0 + \xi_1' w_{t+j}\right) - \sum_{j=1}^{h} \widetilde{e}_\delta' w_{t+j}\right] \mid \underline{w_t}\right\} \\
&= \exp\left[-\xi_0\, h - \xi_1' w_t\right] \mathbb{E}^{\mathbb{Q}}\left\{\exp\left[-\left(\xi_1 + \widetilde{e}_\delta\right)' w_{t+1} - \ldots - \left(\xi_1 + \widetilde{e}_\delta\right)' w_{t+h-1} - \widetilde{e}_\delta' w_{t+h}\right] \mid \underline{w_t}\right\} \\
&= \exp\left[-\xi_0\, h - \xi_1' w_t\right] \varphi_{w,t,h}^{\mathbb{Q}}\left[-\left(\xi_1 + \widetilde{e}_\delta\right), \ldots, -\left(\xi_1 + \widetilde{e}_\delta\right), -\widetilde{e}_\delta\right] \\
&= \exp\left[-\xi_0\, h - \xi_1' w_t\right] \varphi_{w,t,h}^{\mathbb{Q}}\left(u_2, u_1\right)
\end{aligned}$$

where $u_1 = -\widetilde{e}_\delta$ and $u_2 = -(\xi_1 + \widetilde{e}_\delta)$. Then, from relation (19) we have $\varphi_{w,t,h}^{\mathbb{Q}}\left(u_2, u_1\right) = \exp\left[\mathcal{A}_h' w_t + \mathcal{B}_h\right]$, with the recursions $\mathcal{A}_h$ and $\mathcal{B}_h$ immediately obtained from (20). ■

Proof of Lemma 3.1 : we have that

$$\lim_{u_2 \to -\infty} \mathbb{E}\left[\exp(u_1' Z_1 + u_2\, Z_2)\right] = \mathbb{E}\left[\exp(u_1' Z_1)\, \mathbb{1}_{\{Z_2=0\}}\right] + \lim_{u_2 \to -\infty} \mathbb{E}\left[\exp(u_1' Z_1 + u_2\, Z_2)\, \mathbb{1}_{\{Z_2>0\}}\right] ,$$

and since in the second term on the right-hand side $\exp(u_2\, Z_2)\, \mathbb{1}_{\{Z_2>0\}} \to 0$ when $u_2 \to -\infty$, relation (25) is a consequence of the Lebesgue theorem. ■

Proof of Proposition 3.3

Given relation (13), Assumption H.4 and $\mathcal{V}_{t+i}^{(e)}\left(\underline{w_{t+i}}\right) = 1$ for any $i \in \{1, \ldots, h\}$, we know that the no-arbitrage price at date $t < \tau^{(e)}$ of the defaultable zero-coupon bond of interest can be written as:

$$
\begin{aligned}
B_e(t,h) \;=\; & \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left(-\sum_{j=1}^{i} r_{t+j-1} \right) \exp\left(-a_e - a'_{w,e} w_{t+i} \right) \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i-1}\mathbf{1}=0\}} \,|\, \underline{w_t} \right\} \\
& - \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left(-\sum_{j=1}^{i} r_{t+j-1} \right) \exp\left(-a_e - a'_{w,e} w_{t+i} \right) \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i}\mathbf{1}=0\}} \,|\, \underline{w_t} \right\} \\
& + \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left(-\sum_{j=1}^{h} r_{t+j-1} \right) \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+h}\mathbf{1}=0\}} \,|\, \underline{w_t} \right\} .
\end{aligned} \tag{a.5}
$$

If we consider Lemma 3.1 and Proposition 3.1, relation (a.5) can be written as:

$$
\begin{aligned}
B_e(t,h) \;=\; & \sum_{i=1}^{h} \lim_{u\to-\infty} \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left[\left(-\sum_{j=1}^{i} r_{t+j-1} - a_e - a'_{w,e} w_{t+i} \right) + u\, \delta^{(e)\prime}_{t:t+i-1}\mathbf{1} \right] |\, \underline{w_t} \right\} \\
& - \sum_{i=1}^{h} \lim_{u\to-\infty} \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left[\left(-\sum_{j=1}^{i} r_{t+j-1} - a_e - a'_{w,e} w_{t+i} \right) + u\, \delta^{(e)\prime}_{t:t+i}\mathbf{1} \right] |\, \underline{w_t} \right\} \\
& + \lim_{u\to-\infty} \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left[\left(-\sum_{j=1}^{h} r_{t+j-1} \right) + u\, \delta^{(e)\prime}_{t:t+h}\mathbf{1} \right] |\, \underline{w_t} \right\}
\end{aligned} \tag{a.6}
$$

$$
\begin{aligned}
= & \sum_{i=1}^{h} \lim_{u\to-\infty} \exp\left[-i\xi_0 - a_e + (u\widetilde{e}_\delta - \xi_1)' w_t \right] \varphi^{\mathbb{Q}}_{w,t,i}(u\widetilde{e}_\delta - \xi_1, -a_{w,e}) \\
& - \sum_{i=1}^{h} \lim_{u\to-\infty} \exp\left[-i\xi_0 - a_e + (u\widetilde{e}_\delta - \xi_1)' w_t \right] \varphi^{\mathbb{Q}}_{w,t,i}(u\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta - a_{w,e}) \\
& + \lim_{u\to-\infty} \exp\left[-h\xi_0 + (u\widetilde{e}_\delta - \xi_1)' w_t \right] \varphi^{\mathbb{Q}}_{w,t,h}(u\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta) \,,
\end{aligned}
$$

and relations from (26) to (28) are thus proved. ■

Proof of Proposition 3.4

Given Lemma 3.1, relation (29) can be written as:

$$PB_{t,t+h}^{(e)f} = \mathcal{S}_{t,t+h}^{(e)\$} \sum_{i=1}^{h} \lim_{u \to -\infty} \exp\left[-i\xi_0 + \chi + (u\widetilde{e}_\delta - \xi_1)' w_t\right] \varphi_{w,t,i}^{\mathbb{Q}} \left(u\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta + u_s\right) , \tag{a.7}$$

while, relation (30) can be written as follows:

$$\begin{aligned} & PS_{t,t+h}^{(e)f} \\ = & \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}} \left[\exp\left(\chi + u_s' w_{t+i} - \sum_{j=1}^{i} r_{t+j-1}\right) \left(\mathbb{1}_{\{\delta_{t:t+i-1}^{(e)\prime}\mathbf{1}=0\}} - \mathbb{1}_{\{\delta_{t:t+i}^{(e)\prime}\mathbf{1}=0\}}\right) \mid \underline{w_t}\right] \\ & - \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}} \left[\exp\left((\chi - a_e) + (u_s - a_{w,e})' w_{t+i} - \sum_{j=1}^{i} r_{t+j-1}\right) \left(\mathbb{1}_{\{\delta_{t:t+i-1}^{(e)\prime}\mathbf{1}=0\}} - \mathbb{1}_{\{\delta_{t:t+i}^{(e)\prime}\mathbf{1}=0\}}\right) \mid \underline{w_t}\right] , \\ = & \sum_{i=1}^{h} \lim_{u \to -\infty} \Psi_{(t,i)}^{\mathbb{Q}} \left(\chi, u\,\widetilde{e}_\delta - \xi_1, u_s\right) - \sum_{i=1}^{h} \lim_{u \to -\infty} \Psi_{(t,i)}^{\mathbb{Q}} \left(\chi, u\,\widetilde{e}_\delta - \xi_1, u\widetilde{e}_\delta + u_s\right) \\ & - \sum_{i=1}^{h} \lim_{u \to -\infty} \Psi_{(t,i)}^{\mathbb{Q}} \left(\chi - a_e, u\,\widetilde{e}_\delta - \xi_1, u_s - a_w\right) + \sum_{i=1}^{h} \lim_{u \to -\infty} \Psi_{(t,i)}^{\mathbb{Q}} \left(\chi - a_e, u\,\widetilde{e}_\delta - \xi_1, u\,\widetilde{e}_\delta + u_s - a_w\right) . \end{aligned} \tag{a.8}$$

The price of default protection (32) is easily obtained by imposing (a.7) = (a.8), thus proving Proposition 3.4. ■

A.5 Defaultable Bonds Pricing under Recovery of Treasury (RT) Convention

In order to deal with the Recovery of Treasury (RT) convention, the following result turns out to be useful:

Proposition a.1 *The exponential-affine nature of the conditional risk-neutral Laplace transform of (w_t) (see Relation (50)) implies the following exponential-affine multi-horizon Laplace transform:*

$$\begin{aligned} \varphi_{w,t,i}^{\mathbb{Q}} \left(u_{1:i}^{(i)}\right) &= \mathbb{E}^{\mathbb{Q}} \left[\exp\left(u_1^{(i)\prime}\, w_{t+1} + \ldots + u_i^{(i)\prime}\, w_{t+i}\right) \mid \underline{w_t}\right] \\ &= \exp\left(\mathcal{A}_i' w_t + \mathcal{B}_i\right) , \end{aligned} \tag{a.9}$$

where $u_{1:i}^{(i)} = \left(u_1^{(i)}, \ldots, u_i^{(i)}\right)$ and where, for any $j \in \{1, \ldots, i\}$, $u_j^{(i)}$ is an N-dimensional vector. The $\mathcal{A}_i$ and $\mathcal{B}_i$ loadings are obtained at the i^{th} step of the recursive system $\mathcal{A}_i = \mathcal{A}_i^{(i)}$, $\mathcal{B}_i = \mathcal{B}_i^{(i)}$:

$$\begin{cases} \mathcal{A}_0^{(i)} = 0\,, \quad \mathcal{B}_0^{(i)} = 0\,, \\ \mathcal{A}_j^{(i)} = \widetilde{A}_w \left(u_{i+1-j}^{(i)} + \mathcal{A}_{j-1}^{(i)}\right) , \\ \mathcal{B}_j^{(i)} = \widetilde{B}_w \left(u_{i+1-j}^{(i)} + \mathcal{A}_{j-1}^{(i)}\right) + \mathcal{B}_{j-1}^{(i)} \,. \end{cases} \tag{a.10}$$

Proof *See Proposition 3 in Gourieroux, Monfort, Pegoraro, and Renne (2014).*

The recovery of Treasury (RT), introduced by Jarrow and Turnbull (1995) and Longstaff and Schwartz (1995), states that the creditor receives a fraction (corresponding to the recovery rate) of the present value of the principal. This means that, in case of default at date $\tau^{(e)} = t+i$, the payoff is $\mathcal{P}^{(e)}_{t+i,h-i} = \exp(-\delta^{(e)}_{t+i})\, B(t+i, h-i)$, where $B(t,h) = \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{h} r_{t+j-1}\right) \mid \underline{w_t}\right\}$ is the date-t market price of an otherwise equivalent default-free zero-coupon bond maturing at $t+h$. Given $r_t = \xi_0 + \xi_1' w_t$ and relation (50), it is easily seen that $B(t,h) = \exp\left[\mathcal{C}_h' w_t + \mathcal{D}_h\right]$, where $\mathcal{C}_h = \widetilde{A}_w\left(\mathcal{C}_{h-1}\right) - \xi_1$, $\mathcal{D}_h = \widetilde{B}_w\left(\mathcal{C}_{h-1}\right) + \mathcal{D}_{h-1} - \xi_0$, with $\mathcal{C}_0 = 0$ and $\mathcal{D}_0 = 0$. In this case, and assuming (for ease of presentation) the recovery rate specification (16), we have:

Proposition a.2 *Under the RT convention, the no-arbitrage price at date $t < \tau^{(e)}$ of a defaultable zero-coupon bond issued by an entity $e \in \{1, \ldots, E\}$ and maturing in h periods is given by:*

$$B_e(t,h) \quad = \quad \sum_{i=1}^{h} \left(\vartheta^{\mathbb{Q}}_{1,t,i} - \vartheta^{\mathbb{Q}}_{2,t,i}\right) + \vartheta^{\mathbb{Q}}_{2,t,h}\,, \tag{a.11}$$

where:

$$\begin{aligned} \vartheta^{\mathbb{Q}}_{1,t,i} \quad &:= \quad \lim_{u \to -\infty} \Psi^{\mathbb{Q}}_{(t,i,c)}\left(u\,\widetilde{e}_\delta - \xi_1, \mathcal{C}_{h-i} - \widetilde{e}_\delta\right) \\ \vartheta^{\mathbb{Q}}_{2,t,i} \quad &:= \quad \lim_{u \to -\infty} \Psi^{\mathbb{Q}}_{(t,i,c)}\left(u\,\widetilde{e}_\delta - \xi_1, \mathcal{C}_{h-i} + u\widetilde{e}_\delta\right)\,, \end{aligned} \tag{a.12}$$

with $u \in \mathbb{R}$ and where:

$$\begin{aligned} \Psi^{\mathbb{Q}}_{(t,i,c)}\left(u_2, u_i^{(i)}\right) \quad &:= \quad \exp\left(\mathcal{D}_{h-i} - i\xi_0 + u_2' w_t\right)\, \varphi^{\mathbb{Q}}_{w,t,i}\left(u_2, \ldots, u_2, u_i^{(i)}\right) \\ &= \quad \exp\left[\left(u_2 + \mathcal{A}_i\right)' w_t + \left(\mathcal{B}_i + \mathcal{D}_{h-i} - i\xi_0\right)\right]\,; \end{aligned} \tag{a.13}$$

the $\mathcal{A}_i$ and $\mathcal{B}_i$ loadings are obtained as the final values $\mathcal{A}_i = \mathcal{A}_i^{(i)}$, $\mathcal{B}_i = \mathcal{B}_i^{(i)}$ of the i-step recursion (a.4) with $u_1^{(i)} = \ldots = u_{i-1}^{(i)} = u_2$.

Proof of Proposition a.2

Given relation (13), Assumption H.4 and $\mathcal{V}^{(e)}_{t+i}\left(\underline{w_{t+i}}\right) = B_{t+i}(h-i)$ for any $i \in \{1, \ldots, h\}$, we have that:

$$\begin{aligned} B_e(t,h) \quad = \quad & \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{i} r_{t+j-1}\right)\, \exp\left(-\delta^{(e)}_{t+i}\right)\, B_{t+i}(h-i)\, \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i-1}\mathbf{1}=0\}} \mid \underline{w_t}\right\} \\ & - \sum_{i=1}^{h} \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{i} r_{t+j-1}\right)\, \exp\left(-\delta^{(e)}_{t+i}\right)\, B_{t+i}(h-i)\, \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+i}\mathbf{1}=0\}} \mid \underline{w_t}\right\} \\ & + \mathbb{E}^{\mathbb{Q}}\left\{\exp\left(-\sum_{j=1}^{h} r_{t+j-1}\right)\, \mathbb{1}_{\{\delta^{(e)\prime}_{t:t+h}\mathbf{1}=0\}} \mid \underline{w_t}\right\}\,. \end{aligned} \tag{a.14}$$

Now, if we consider Lemma 3.1 and Proposition a.1, and given that $B(t+i, h-i) = \exp\left[\mathcal{C}_{h-i}' w_{t+i} + \mathcal{D}_{h-i}\right]$, where $\mathcal{C}_{h-i} = \widetilde{A}_w\left(\mathcal{C}_{h-i-1}\right) - \xi_1$, $\mathcal{D}_{h-i} = \widetilde{B}_w\left(\mathcal{C}_{h-i-1}\right) + \mathcal{D}_{h-i-1} - \xi_0$ (with $\mathcal{C}_0 = 0$ and $\mathcal{D}_0 = 0$), then relation (a.14) can be written as:

$$
\begin{aligned}
B_e(t,h) \quad = \quad & \sum_{i=1}^{h} \lim_{u\to-\infty} \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left(-\sum_{j=1}^{i} r_{t+j-1} - \delta_{t+i}^{(e)} + \mathcal{C}_{h-i}' w_{t+i} + \mathcal{D}_{h-i} + u\left(\delta_{t:t+i-1}^{(e)'}\, \mathbf{1}\right)\right) \mid \underline{w_t} \right\} \\
& - \sum_{i=1}^{h} \lim_{u\to-\infty} \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left(-\sum_{j=1}^{i} r_{t+j-1} + \mathcal{C}_{h-i}' w_{t+i} + \mathcal{D}_{h-i} + u\left(\delta_{t:t+i}^{(e)'}\, \mathbf{1}\right)\right) \mid \underline{w_t} \right\} \\
& + \lim_{u\to-\infty} \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left(-\sum_{j=1}^{h} r_{t+j-1} + u\left(\delta_{t:t+h}^{(e)'}\, \mathbf{1}\right)\right) \mid \underline{w_t} \right\} \\
= \quad & \sum_{i=1}^{h} \lim_{u\to-\infty} \exp\left[-i\,\xi_0 + (u\widetilde{e}_\delta - \xi_1)' w_t + \mathcal{D}_{h-i}\right] \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left[\sum_{j=1}^{i-1} (u\,\widetilde{e}_\delta - \xi_1)' w_{t+j} + (\mathcal{C}_{h-i} - \widetilde{e}_\delta)' w_{t+i}\right] \mid \underline{w_t} \right\} \\
& - \sum_{i=1}^{h} \lim_{u\to-\infty} \exp\left[-i\,\xi_0 + (u\widetilde{e}_\delta - \xi_1)' w_t + \mathcal{D}_{h-i}\right] \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left[\sum_{j=1}^{i-1} (u\,\widetilde{e}_\delta - \xi_1)' w_{t+j} + (\mathcal{C}_{h-i} + u\,\widetilde{e}_\delta)' w_{t+i}\right] \mid \underline{w_t} \right\} \\
& + \lim_{u\to-\infty} \exp\left[-h\,\xi_0 + (u\widetilde{e}_\delta - \xi_1)' w_t\right] \mathbb{E}^{\mathbb{Q}} \left\{ \exp\left[\sum_{j=1}^{h-1} (u\,\widetilde{e}_\delta - \xi_1)' w_{t+j} + u\,\widetilde{e}_\delta' w_{t+h}\right] \mid \underline{w_t} \right\} \\
= \quad & \sum_{i=1}^{h} \lim_{u\to-\infty} \Psi^{\mathbb{Q}}_{(t,i,c)}\left(u\,\widetilde{e}_\delta - \xi_1, \mathcal{C}_{h-i} - \widetilde{e}_\delta\right) \\
& - \sum_{i=1}^{h} \lim_{u\to-\infty} \Psi^{\mathbb{Q}}_{(t,i,c)}\left(u\,\widetilde{e}_\delta - \xi_1, \mathcal{C}_{h-i} + u\,\widetilde{e}_\delta\right) \\
& + \lim_{u\to-\infty} \Psi^{\mathbb{Q}}_{(t,h,c)}\left(u\,\widetilde{e}_\delta - \xi_1, u\,\widetilde{e}_\delta\right) ,
\end{aligned} \tag{a.15}
$$

with:

$$
\begin{aligned}
\Psi^{\mathbb{Q}}_{(t,i,c)}\left(u_2, u_i^{(i)}\right) \quad & := \quad \exp\left(\mathcal{D}_{h-i} - i\xi_0 + u_2' w_t\right) \varphi^{\mathbb{Q}}_{w,t,i}\left(u_2, \ldots, u_2, u_i^{(i)}\right) \\
& = \quad \exp\left[(u_2 + \mathcal{A}_i)' w_t + (\mathcal{B}_i + \mathcal{D}_{h-i} - i\xi_0)\right]
\end{aligned} \tag{a.16}
$$

and relations from (a.11) to (a.13) are thus proved. ■

A.6 The Class of Recursive Affine Processes

A.6.1 Definition of Recursive Affine Process

Let us consider a multivariate discrete-time stochastic process $\{w_t\,, t \in \mathbb{N}\}$. The vector w_t is partitioned into n subvectors $w_{i,t}$, $i \in \{1,\ldots,n\}$, of size n_i. The size of $w_t = (w_{1,t}^{'},\ldots,w_{n,t}^{'})^{'}$ is thus $N = \sum_{i=1}^{n} n_i$ and $\underline{w}_t = (w_t^{'},\ldots,w_1^{'})^{'}$ denotes the overall information at date t.

Definition a.1 *The process $\{w_t\}$ is recursive discrete-time affine if the conditional Laplace transforms:*

$$\mathbb{E}\left[\exp\left(u_i^{'}\,w_{i,t}\right) \mid w_{i-1,t}^{'},\ldots,w_{1,t}^{'},\underline{w}_{t-1}\right]\,,\ \ i \in \{1,\ldots,n\}\,, \tag{a.17}$$

are of the form:

$$\begin{aligned} &\exp\left[\sum_{j=1}^{i-1} c_{i,j}^{'}(u_i)w_{j,t} + \sum_{j=1}^{n} a_{i,j}^{'}(u_i)w_{j,t-1} + b_i(u_i)\right]\,,\ \ \textit{for } i \in \{2,\ldots,n\}\,, \\ &\exp\left[\sum_{j=1}^{n} a_{1,j}^{'}(u_1)w_{j,t-1} + b_1(u_1)\right]\,,\ \ \textit{if } i = 1\,. \end{aligned} \tag{a.18}$$

This definition thus implies that, for $i \in \{1,\ldots,n\}$, the conditional Laplace transform of $w_{i,t}$ given the present values $w_{i-1,t}^{'},\ldots,w_{1,t}^{'}$ and all the past values $\underline{w}_{t-1}$, are exponential-affine in $w_{i-1,t}^{'},\ldots,w_{1,t}^{'},w_{t-1}$.

A first important result is that, for any $i \in \{1,\ldots,n\}$, the joint conditional Laplace transform of $(w_{1,t}^{'},\ldots,w_{i,t}^{'})^{'}$, given $\underline{w}_{t-1}$, is exponential-affine in w_{t-1}. In particular, this is true for $i = n$ and, therefore, the process $\{w_t\}$ is affine. More precisely, we have the following result.

Proposition a.3 *For any $i \in \{1,\ldots,n\}$, the joint conditional Laplace transform of $(w_{1,t}^{'},\ \ldots,w_{i,t}^{'})^{'}$, given $\underline{w}_{t-1}$, is given by:*

$$\mathbb{E}\left[\exp(u_1' w_{1,t} + \ldots + u_i' w_{i,t}) \mid \underline{w_{t-1}}\right] = \exp\left[\sum_{j=1}^{n} \widetilde{a}_{i,j}(u_1,\ldots,u_i)' w_{j,t-1} + \widetilde{b}_i(u_1,\ldots,u_i)\right] \tag{a.19}$$

where the functions $\widetilde{a}_{i,j}$ and $\widetilde{b}_i$, for any $j \in \{1,\ldots,n\}$, i varying, are obtained recursively from:

$$\left\{\begin{array}{lcl} \widetilde{a}_{1,j}(u_1) & = & a_{1,j}(u_1)\,,\ \widetilde{b}_1(u_1) = b_1(u_1) \\ \widetilde{a}_{i,j}(u_1,\ldots,u_i) & = & \widetilde{a}_{i-1,j}\left[u_1 + c_{i,1}(u_i),\ldots,u_{i-1} + c_{i,i-1}(u_i)\right] + a_{i,j}(u_i) \\ \widetilde{b}_i(u_1,\ldots,u_i) & = & \widetilde{b}_{i-1}\left[u_1 + c_{i,1}(u_i),\ldots,u_{i-1} + c_{i,i-1}(u_i)\right] + b_i(u_i)\,, \end{array}\right. \tag{a.20}$$

Proof The result is obvious for $i = 1$. Let us assume that it is true $i-1$, and let us show that is also true for i. We have:

$$
\begin{aligned}
& \mathbb{E}\left[\exp(u_1^{'}w_{1,t}+\ldots+u_i^{'}w_{i,t}) \,|\, \underline{w}_{t-1}\right] \\
= \; & \mathbb{E}\left\{\exp\left(\sum_{k=1}^{i-1} u_k^{'}w_{k,t}\right)\mathbb{E}\left[\exp(u_i w_{i,t}) \,|\, w_{1,t},\ldots,w_{i-1,t},\underline{w}_{t-1}\right] \,|\, \underline{w}_{t-1}\right\} \\
= \; & \mathbb{E}\left\{\exp\left(\sum_{k=1}^{i-1} u_k^{'}w_{k,t}\right)\exp\left(\sum_{k=1}^{i-1} c_{i,k}^{'}(u_i)w_{k,t}+\sum_{k=1}^{n} a_{i,k}^{'}(u_i)w_{k,t-1}+b_i(u_i)\right) \,|\, \underline{w}_{t-1}\right\} \\
= \; & \exp\left(\sum_{k=1}^{n} a_{i,k}^{'}(u_i)w_{k,t-1}+b_i(u_i)\right)\mathbb{E}\left\{\exp\left(\sum_{k=1}^{i-1}(u_k+c_{i,k}(u_i))^{'}w_{k,t}\right) \,|\, \underline{w}_{t-1}\right\} \\
= \; & \exp\left(\sum_{k=1}^{n} a_{i,k}^{'}(u_i)w_{k,t-1}+b_i(u_i)\right) \\
& \quad \times \exp\left[\sum_{k=1}^{n}\widetilde{a}_{i-1,k}(u_1+c_{i,1}(u_i),\ldots,u_{i-1}+c_{i,i-1}(u_i))^{'}w_{k,t-1}\right. \\
& \qquad \left. +\widetilde{b}_{i-1}(u_1+c_{i,1}(u_i),\ldots,u_{i-1}+c_{i,i-1}(u_i))\right] \\
= \; & \exp\left\{\sum_{k=1}^{n}\left[\widetilde{a}_{i-1,k}(u_1+c_{i,1}(u_i),\ldots,u_{i-1}+c_{i,i-1}(u_i))+a_{i,k}(u_i)\right]^{'}w_{k,t-1}\right. \\
& \qquad \left. +\left[\widetilde{b}_{i-1}(u_1+c_{i,1}(u_i),\ldots,u_{i-1}+c_{i,i-1}(u_i))+b_i(u_i)\right]\right\}
\end{aligned}
$$

and the result follows by identification.

Corollary a.3.1 *The n-dimensional stochastic process $\{w_t\}$ is affine and the conditional Laplace transform of w_t, given $\underline{w}_{t-1}$, is:*

$$\varphi_{t-1}(u)=\mathbb{E}\left[\exp(u^{'}w_t) \,|\, \underline{w_{t-1}}\right]=\exp\left[\sum_{j=1}^{n}\widetilde{a}_{n,j}(u)'w_{j,t-1}+\widetilde{b}_n(u)\right]$$

where $u=(u_1,\ldots,u_n)^{'}$.

Proof Straightforward consequence of Proposition a.3.

The previous results provide a convenient way to specify a general multivariate affine process $\{w_t\}$ of dimension $N=\sum_{i=1}^{n} n_i$. Indeed, it is possible to decompose this specification into n specifications of conditional distributions of the sub-vectors $w_{i,t}$. In particular, if $n_i=1$ for any i, we see that the specification of a n-variate affine process can be decomposed into the specification of n univariate conditional distributions. The multivariate processes thus obtained possess all the important properties characterizing the affine class.

A.6.2 Moments, Semi-strong VAR Representations, Stationarity Conditions and Prediction

The purpose of this section is to provide moments, VAR representation, stationarity conditions and prediction formulas of Recursive Affine processes exploiting the exponential-affine form of their conditional Laplace transform. For sake of notational simplicity, we will first consider the case where the components $w_{i,t}$, for any $i \in \{1, \ldots, n\}$, are univariate. The general case will be considered at the end of this section.

CONDITIONAL MOMENTS AND VAR REPRESENTATIONS

According to Definition a.1, the log-Laplace transform of $w_{i,t}$, given $w_{i-1,t}, \ldots, w_{1,t}, w_{t-1}$, namely:

$$\psi_t(u_i) = \sum_{j=1}^{i-1} c_{i,j}(u_i) w_{j,t} + \sum_{j=1}^{n} a_{i,j}(u_i) w_{j,t-1} + b_i(u_i) . \tag{a.21}$$

The recursive conditional mean and conditional variance of $w_{i,t}$, given $w_{i-1,t}, \ldots, w_{1,t}, w_{t-1}$, are respectively given by:

$$\begin{aligned} \overline{m}_{i,t} &= \sum_{j=1}^{i-1} c_{i,j}^{(1)}(0) w_{j,t} + \sum_{j=1}^{n} a_{i,j}^{(1)}(0) w_{j,t-1} + b_i^{(1)}(0) , \\ \overline{\sigma}_{i,t}^2 &= \sum_{j=1}^{i-1} c_{i,j}^{(2)}(0) w_{j,t} + \sum_{j=1}^{n} a_{i,j}^{(2)}(0) w_{j,t-1} + b_i^{(2)}(0) , \end{aligned} \tag{a.22}$$

where the exponents (1) and (2) indicate, respectively, the first-order and the second-order derivatives. Denoting by $\overline{m}_t$ and $\overline{\sigma}_t^2$ the vectors whose components are, respectively, $\overline{m}_{i,t}$ and $\overline{\sigma}_{i,t}^2$, $i \in \{1, \ldots, n\}$, we have, with obvious notations:

$$\begin{aligned} \overline{m}_t &= C_m \, w_t + A_m \, w_{t-1} + b_m , \\ \overline{\sigma}_t^2 &= C_\sigma \, w_t + A_\sigma \, w_{t-1} + b_\sigma , \end{aligned} \tag{a.23}$$

where C_m and C_σ are lower triangular matrices with 0 on the main diagonal. Defining $\varepsilon_t = w_t - \overline{m}_t$, we have:

$$\mathbb{E} \left(\varepsilon_{i,t} \,|\, w_{i-1,t}, \ldots, w_{1,t}, w_{t-1} \right) = 0$$

$$\mathbb{V} \left(\varepsilon_{i,t} \,|\, w_{i-1,t}, \ldots, w_{1,t}, w_{t-1} \right) = \overline{\sigma}_{i,t}^2 ,$$

and therefore $\mathbb{E}(\varepsilon_{i,t} \,|\, w_{t-1}) = 0$, $\mathbb{V}(\varepsilon_{i,t} \,|\, w_{t-1}) = \mathbb{E}(\overline{\sigma}_{i,t}^2 \,|\, w_{t-1})$ and $Cov(\varepsilon_{i,t}, \varepsilon_{j,t} \,|\, w_{t-1}) = 0$, since $\mathbb{E}(\varepsilon_{i,t}\varepsilon_{j,t} \,|\, w_{t-1}) = \mathbb{E} \left[\varepsilon_{i,t} \mathbb{E}(\varepsilon_{j,t} \,|\, w_{j-1,t}, \ldots, w_{1,t}, w_{t-1}) \,|\, w_{t-1} \right] = 0$, if $i < j$. We thus have the recursive semi-strong VAR representation:

$$w_t = C_m \, w_t + A_m \, w_{t-1} + b_m + \varepsilon_t , \tag{a.24}$$

ε_t being a martingale difference whose conditional variance-covariance matrix, given the past, is $diag[\mathbb{E}(\overline{\sigma}_{i,t}^2 \,|\, w_{t-1})]$.

From relation (a.24) we get:

$$w_t = \widetilde{A}_m \, w_{t-1} + \widetilde{b}_m + (I - C_m)^{-1} \, \varepsilon_t , \tag{a.25}$$

where $\widetilde{A}_m = (I - C_m)^{-1} A_m$ and $\widetilde{b}_m = (I - C_m)^{-1} b_m$. In particular, we get the conditional moments:

$$\begin{aligned} m_{t-1} &= \mathbb{E}(w_t \,|\, w_{t-1}) = \widetilde{A}_m \, w_{t-1} + \widetilde{b}_m \\ \Sigma_{t-1} &= \mathbb{V}(w_t \,|\, w_{t-1}) = (I - C_m)^{-1} \, diag[\mathbb{E}(\overline{\sigma}^2_{i,t} \,|\, w_{t-1})] \, (I - C_m)^{-1'} . \end{aligned} \tag{a.26}$$

In addition, from relations (a.23) and (a.26) we have:

$$\begin{aligned} \mathbb{E}(\overline{\sigma}^2_t \,|\, w_{t-1}) &= C_\sigma \, m_t + A_\sigma \, w_{t-1} + b_\sigma \\ &= (C_\sigma \, \widetilde{A}_m + A_\sigma) \, w_{t-1} + C_\sigma \, \widetilde{b}_m + b_\sigma \\ &= A \, w_{t-1} + b \;\; \text{(say)} \end{aligned}$$

and therefore:

$$\Sigma_{t-1} = (I - C_m)^{-1} \, diag[A \, w_{t-1} + b] \, (I - C_m)^{-1'} .$$

So, if we denote $\Omega_{t-1} = (I - C_m)^{-1} \, diag[A \, w_{t-1} + b]^{1/2}$, we have the semi-strong VAR representation:

$$w_t = \widetilde{A}_m \, w_{t-1} + \widetilde{b}_m + \Omega_{t-1} \, \eta_t \, , \tag{a.27}$$

where η_t is a martingale difference with a conditional variance-covariance matrix equal to the identity matrix. Note that, since C_m is lower triangular with diagonal terms equal to zero, it is nihilpotent, i.e. $C^h_m = 0$ for any $h > n$ and, therefore, $(I - C_m)^{-1} = I + C_m + \ldots + C^{n-1}_m$.

Unconditional Moments and Stationarity Conditions

Let us denote by $\widetilde{m}_t$ and $\widetilde{\Sigma}_t$ the unconditional mean and variance-covariance matrix of w_t. From (a.27) we have:

$$\begin{aligned} \widetilde{m}_t &= \widetilde{A}_m \, \widetilde{m}_{t-1} + \widetilde{b}_m \\ \widetilde{\Sigma}_t &= \mathbb{V}(m_t) + \mathbb{E}(\Sigma_t) = \widetilde{A}_m \, \widetilde{\Sigma}_{t-1} \, \widetilde{A}'_m + (I - C_m)^{-1} \, diag[A \, \widetilde{m}_{t-1} + b] \, (I - C_m)^{-1'} , \end{aligned} \tag{a.28}$$

and

$$vec(\widetilde{\Sigma}_t) = \left(\widetilde{A}_m \otimes \widetilde{A}_m\right) \, vec(\widetilde{\Sigma}_{t-1}) + \left[(I - C_m)^{-1} \otimes (I - C_m)^{-1}\right] \, (D \, \widetilde{m}_{t-1} + d) \, , \tag{a.29}$$

where $D\,\widetilde{m}_{t-1} + d = vec\left[diag(A\,\widetilde{m}_{t-1} + b)\right]$. Finally, we get the linear recursive system:

$$\begin{pmatrix} \widetilde{m}_t \\ \\ vec(\widetilde{\Sigma}_t) \end{pmatrix} = \begin{pmatrix} \widetilde{A}_m & 0 \\ \\ \left[(I - C_m)^{-1} \otimes (I - C_m)^{-1}\right] D & \widetilde{A}_m \otimes \widetilde{A}_m \end{pmatrix} \begin{pmatrix} \widetilde{m}_{t-1} \\ \\ vec(\widetilde{\Sigma}_{t-1}) \end{pmatrix} + \begin{pmatrix} \widetilde{b}_m \\ \\ \left[(I - C_m)^{-1} \otimes (I - C_m)^{-1}\right] d \end{pmatrix}. \tag{a.30}$$

Since the eigenvalues of $\widetilde{A}_m \otimes \widetilde{A}_m$ are all the products of the eigenvalues of $\widetilde{A}_m$, we get the following result:

Proposition a.4 *System (a.30) is convergent if the moduli of the eigenvalues of $\widetilde{A}_m$ are all strictly smaller than one.*

In this case we get:

Corollary a.4.1 *The stationary unconditional moments $\widetilde{m}$ and $\widetilde{\Sigma}$ of the process (w_t), obtained as the limits of the system (a.30), are:*

$$\begin{aligned} \widetilde{m} &= (I - \widetilde{A}_m)^{-1}\,\widetilde{b}_m \\ vec(\widetilde{\Sigma}) &= (I - \widetilde{A}_m \otimes \widetilde{A}_m)^{-1} \left[(I - C_m)^{-1} \otimes (I - C_m)^{-1}\right] (D\,\widetilde{m} + d)\,. \end{aligned} \tag{a.31}$$

In addition, if we iterate relation (a.25) we get:

$$w_{t+h} = \widetilde{A}_m^h\, w_t + \sum_{i=0}^{h-1} \widetilde{A}_m^i \left[\widetilde{b}_m + (I - C_m)^{-1}\,\varepsilon_{t+h-i}\right], \tag{a.32}$$

and thus $Cov(w_{t+h}, w_t) = \widetilde{A}_m^h\,\widetilde{\Sigma}$. Therefore, we obtain the following result:

Proposition a.5 *If all the eigenvalues of $\widetilde{A}_m$ have a modulus strictly smaller than one, the process (w_t) is (asymptotically) second order stationary, with mean $\widetilde{m}$ and auto-covariance function $\widetilde{A}_m^h\,\widetilde{\Sigma}$.*

An important particular case occurs if $a_{i,k}(u_i) = 0$ for $k > i$. In this case, the matrix A_m is lower triangular and, since $(I - C_m)$ is lower triangular with diagonal terms equal to 1, the same is true for $(I - C_m)^{-1}$ and, finally, $\widetilde{A}_m = (I - C_m)^{-1} A_m$ is lower triangular with the same main diagonal as A_m. This result implies that the eigenvalues of $\widetilde{A}_m$ are the diagonal terms of A_m and thus we have the following:

Corollary a.5.1 *If $a_{i,k}(u_i) = 0$ for $k > i$, the second-order stationarity conditions are $a_{ii}^{(1)}(0) < 1$ for any $i \in \{1, \ldots, n\}$.*

Note that, if $a_{i,k}(u_i) = 0$ for $k > i$, that is $\widetilde{A}_m$ is lower triangular, this means that $w_{k,t}$ does not Granger cause $w_{i,t}$ (for any $k > i$).

Forecasting

From relation (a.32) we see that the optimal prediction of w_{t+h} at t is:

$$\widehat{w}_{t,t+h} \quad := \quad \mathbb{E}_t(w_{t+h}) = \widetilde{A}_m^h \, w_t + \sum_{i=0}^{h-1} \widetilde{A}_m^i \, \widetilde{b}_m \,, \tag{a.33}$$

and the conditional variance-covariance matrix of the prediction error is:

$$\mathbb{V}_t(w_{t+h}) = \sum_{i=0}^{h-1} \widetilde{A}_m^i \, (I - C_m)^{-1} \, \mathbb{V}_t(\varepsilon_{t+h-i}) \, (I - C_m)^{-1'} \, \widetilde{A}_m^{i'} \,, \tag{a.34}$$

with:

$$\mathbb{V}_t(\varepsilon_{t+h-i}) = \mathbb{E}_t \left[diag \left(A \, w_{t+h-i-1} + b \right) \right] = diag \left[A \, \mathbb{E}_t(w_{t+h-i-1}) + b \right] \,. \tag{a.35}$$

Finally:

$$\mathbb{V}_t(w_{t+h}) = \sum_{i=0}^{h-1} \widetilde{A}_m^i \, (I - C_m)^{-1} \, diag \left[A \, \widehat{w}_{t,t+h-i-1} + b \right] (I - C_m)^{-1'} \, \widetilde{A}_m^{i'} \,, \tag{a.36}$$

where $\widehat{w}_{t,t+h-i-1}$ is given in (a.33). In summary, we have explicit forms of both the optimal predictions and the variance-covariance matrices of the prediction errors.

Extension to the General Case

The purpose of this section is to deal with the general case where $w_t = (w_{1,t}^{'}, \ldots, w_{n,t}^{'})^{'}$, each $w_{i,t}$ is of size n_i and, therefore, w_t is of size $N = \sum_{i=1}^{n} n_i$. The conditional log-Laplace transform of $w_{i,t}$, given $w_{i-1,t}, \ldots, w_{1,t}, \, w_{t-1}$ is:

$$\psi_t(u_i) = \sum_{j=1}^{i-1} c_{i,j}^{'}(u_i) w_{j,t} + \sum_{j=1}^{n} a_{i,j}^{'}(u_i) w_{j,t-1} + b_i(u_i) \,, \tag{a.37}$$

where u_i is of size n_i, whereas $c_{i,j}(\cdot)$ and $a_{i,j}(\cdot)$ are of size n_j. If we denote by $\overline{m}_{i,t} = \mathbb{E}(w_{i,t} \,|\, w_{i-1,t}, \ldots, w_{1,t}, w_{t-1})$, we have again:

$$\overline{m}_{i,t} \quad = \quad \sum_{j=1}^{i-1} c_{i,j}^{(1)}(0) w_{j,t} + \sum_{j=1}^{n} a_{i,j}^{(1)}(0) w_{j,t-1} + b_i^{(1)}(0) \,, \tag{a.38}$$

where $c_{i,j}^{(1)}(0)$ is now a (n_i, n_j) matrix whose k^{th} column is the gradient of $c_{i,j,k}(u_i)$ evaluated at zero, $a_{i,j}^{(1)}(0)$ having a similar definition and $b_i^{(1)}(0)$ being the gradient of $b_i(u_i)$ evaluated at zero. Stacking the $\overline{m}_{i,t}$, for any $i \in \{1, \ldots, n\}$ we get the same Equation as in (a.23):

$$\overline{m}_t \quad = \quad C_m \, w_t + A_m \, w_{t-1} + b_m \,, \tag{a.39}$$

where now the C_m and A_m are (N, N) matrices being defined by the blocks $c_{i,j}^{(1)}(0)$ and $a_{i,j}^{(1)}(0)$, respectively. Note that C_m is block-lower triangular, the square diagonal blocks being equal to zero. We still have the block-recursive VAR representation (a.24):

$$w_t \quad = \quad C_m\, w_t + A_m\, w_{t-1} + b_m + \varepsilon_t\,, \tag{a.40}$$

the error term ε_t being a martingale difference whose conditional variance-covariance matrix, given w_{t-1}, is block-diagonal, with the i^{th} block being $\mathbb{E}(\overline{\Sigma}_{i,t}\,|\,w_{t-1})$ and where $\overline{\Sigma}_{i,t}$ is defined by:

$$\overline{\Sigma}_{i,t} \quad = \quad \sum_{j=1}^{i-1}\sum_{k=1}^{n_j} c_{i,j,k}^{(2)}(0) w_{j,k,t} + \sum_{j=1}^{n}\sum_{k=1}^{n_j} a_{i,j,k}^{(2)}(0) w_{j,k,t-1} + b_i^{(2)}(0)\,, \tag{a.41}$$

the exponent (2) indicating a second-order derivative matrix (of size (n_i, n_i)). Note that $\overline{\Sigma}_{i,t}$ is an affine function of the $w_{j,k,t}$'s and the $w_{j,k,t-1}$'s.

The VAR representation (a.25) remains valid:

$$w_t \quad = \quad \widetilde{A}_m\, w_{t-1} + \widetilde{b}_m + (I - C_m)^{-1}\,\varepsilon_t\,, \tag{a.42}$$

and we still have:

$$\begin{aligned} m_t \quad &= \quad \mathbb{E}(w_t\,|\,w_{t-1}) = \widetilde{A}_m\, w_{t-1} + \widetilde{b}_m \\ \Sigma_t \quad &= \quad \mathbb{V}(w_t\,|\,w_{t-1}) = (I - C_m)^{-1}\, bdiag[\mathbb{E}(\overline{\Sigma}_{i,t}\,|\,w_{t-1})]\,(I - C_m)^{-1'}\,, \end{aligned} \tag{a.43}$$

the matrix $bdiag[\mathbb{E}(\overline{\Sigma}_{i,t}\,|\,w_{t-1})]$ being the block-diagonal matrix whose diagonal blocks are $\mathbb{E}(\overline{\Sigma}_{i,t}\,|\,w_{t-1})$, which are affine functions of the $w_{j,k,t-1}$'s (see (a.41)). The stationarity conditions remain the same and in the particular case where $w_{k,t}$ does not cause $w_{j,t}$ for $k > i$, these conditions are such that, for any $i \in \{1, \ldots, n\}$, the eigenvalues of the (n_i, n_i) matrix $a_{ii}^{(1)}(0)$ have a modulus strictly smaller than one.

A.6.3 Recursive Risk-Neutral dynamics and Esscher Transforms

Let us assume the following specification of our one-period discrete-time stochastic discount factor:

$$M_{t-1,t} = \exp\left[-r_{t-1} + \theta'_{t-1}\, w_t - \psi^{\mathbb{P}}_{w,t-1}(\theta_{t-1})\right] \tag{a.44}$$

where r_{t-1} is the predetermined short rate between $t-1$ and t, $\theta_{t-1} = (\theta'_{1,t-1}, \theta'_{2,t-1}, \ldots, \theta'_{n,t-1})'$ is the vector of stochastic risk-correction parameters function of $\underline{w_{t-1}}$ and $\psi^{\mathbb{P}}_{w,t-1}(u_w) = \log \varphi^{\mathbb{P}}_{w,t-1}(u_w)$ is the log-Laplace transform of w_t, given $\underline{w_{t-1}}$.

Proposition a.6 *Let us suppose that the historical dynamics of the N-dimensional stochastic process $w_t = (w'_{1,t}, \ldots, w'_{n,t})'$ is recursive affine and that the one-period stochastic discount factor is given by* *(a.44), where* $\theta_{t-1} = (\theta_{1,t-1}^{'}, \ldots, \theta_{n,t-1}^{'})^{'}$. *Then, the risk-neutral conditional distribution of* $w_{n-i,t}$*, given* $(w_{n-i-1,t}, \ldots, w_{1,t}, \underline{w}_{t-1})$ *is the conditional Esscher transform of the corresponding historical distributions associated with the parameters* $\widetilde{\theta}_{n-i,t-1}$*, obtained from* $\theta_{i,t-1}$*'s by the backward recursion:*

$$\begin{cases} \widetilde{\theta}_{n,t-1} & = \quad \theta_{n,t-1} \\ \widetilde{\theta}_{n-i,t-1} & = \quad \theta_{n-i,t-1} + \sum_{k=1}^{i} c_{n-k+1,n-i}(\widetilde{\theta}_{n-k+1,t-1}), \quad i \in \{1, \ldots, n-1\}. \end{cases} \tag{a.45}$$

The proof of this proposition is based on the two following lemmas.

Lemma a.1 *Let us consider a random vector* $(w_1^{'}, w_2^{'})^{'}$ *with p.d.f.* $f(w_1, w_2)$ *and the Esscher transform of* $f(w_1, w_2)$ *associated with parameters* $\theta = (\theta_1^{'}, \theta_2^{'})$ *denoted by* $f^{\theta}(w_1, w_2)$*. Then, the conditional distribution* $f^{\theta}(w_2 \,|\, w_1)$ *is the Esscher transform of* $f(w_2 \,|\, w_1)$ *associated with parameter* θ_2*.*

Proof We have by definition, using the notation $\propto$ for indicating proportionality:

$$\begin{aligned} & f^{\theta}(w_1, w_2) \propto f(w_1, w_2) \, \exp\left(\theta_1^{'} w_1 + \theta_2^{'} w_2\right) \\ \text{or} \;\; & f^{\theta}(w_1, w_2) \propto f(w_1) \, \exp(\theta_1^{'} w_1) f(w_2 | w_1) \exp(\theta_2^{'} w_2) \\ \text{and} \;\; & f^{\theta}(w_2 | w_1) \propto f(w_2 | w_1) \exp(\theta_2^{'} w_2)\,, \end{aligned} \tag{a.46}$$

which gives the result.

Lemma a.2 *Let us consider a n-dimensional random vector* $w = (w_1^{'}, \ldots, w_n^{'})^{'}$*, with p.d.f.* $f(w_1, \ldots, w_n)$*, and the Esscher transform of* $f(w_1, \ldots, w_n)$ *associated with parameters* $\theta = (\theta_1^{'}, \ldots, \theta_n^{'})^{'}$ *and denoted* $f^{\theta}(w_1, \ldots, w_n)$*. Let us assume that the Laplace transform associated to the conditional p.d.f.* $f(w_i \,|\, w_{i-1}, \ldots, w_1)$ *is of the form:*

$$\varphi_i(u) \;\; = \;\; \mathbb{E}\left[\exp(u^{'} w_i) \,|\, w_{i-1}, \ldots, w_1\right] = \exp\left[\sum_{j=1}^{i-1} c_{i,j}^{'}(u) w_j + b_i(u)\right], \quad i \in \{2, \ldots, n\}, \tag{a.47}$$

with $\varphi_1(u) = \mathbb{E}\left[\exp(u^{'} w_1)\right] = \exp\left[b_1(u)\right]$*. Then, the following results hold:*

a) *The marginal p.d.f.* $f^{\theta}(w_1, \ldots, w_{n-i})$ *is the Esscher transform of* $f(w_1, \ldots, w_{n-i})$ *associated with the parameters*

$$\theta_j + \sum_{k=1}^{i} c_{n-k+1,j}(\widetilde{\theta}_{n-k+1}), \quad j \in \{1, \ldots, n-i\}, \tag{a.48}$$

(with the convention $\sum_{k=1}^{i} c_{n-k+1,j}(\widetilde{\theta}_{n-k+1}) = 0$*, for* $i = 0$*) where the* $\widetilde{\theta}_j$ *are obtained recursively from:*

$$\begin{cases} \widetilde{\theta}_{n} & = \quad \theta_{n} \\ \widetilde{\theta}_{n-i} & = \quad \theta_{n-i} + \sum_{k=1}^{i} c_{n-k+1,n-i}(\widetilde{\theta}_{n-k+1}), \quad i \in \{1, \ldots, n-1\}. \end{cases} \tag{a.49}$$

b) The conditional distribution $f^\theta(w_{n-i}\,|\,w_{n-i-1},\dots,w_1)$ is the Esscher transform of $f(w_{n-i}\,|\ w_{n-i-1},\dots,w_1)$ associated with the parameter $\widetilde{\theta}_{n-i}$ obtained from (a.49).

Proof Let us consider, first, the computation of $f^\theta(w_1,\dots,w_{n-i})$, and then the computation of the associated $f^\theta(w_{n-i}\,|\,w_{n-i-1},\,\dots,w_1)$.

- if $i=0$, $a)$ is true.
- Let us assume now that $a)$ is true for i:

$$f^\theta(w_1,\dots,w_{n-i}) \propto f(w_1,\dots,w_{n-i})\,\exp\left[\sum_{j=1}^{n-i}\left(\theta_j+\sum_{k=1}^{i}c_{n-k+1,j}(\widetilde{\theta}_{n-k+1})\right)'w_j\right]. \tag{a.50}$$

Replacing $f(w_1,\dots,w_{n-i})$ by $f(w_1,\dots,w_{n-i-1})\,f(w_{n-i}\,|\,w_1,\dots,w_{n-i-1})$ on the rhs of the previous relation and integrating with respect to w_{n-i} gives:

$$\begin{aligned}
&f^\theta(w_1,\dots,w_{n-i-1})\\
&\propto f(w_1,\dots,w_{n-i-1})\,\exp\left[\sum_{j=1}^{n-i-1}\left(\theta_j+\sum_{k=1}^{i}c_{n-k+1,j}(\widetilde{\theta}_{n-k+1})\right)'w_j\right]\\
&\quad\times\int f(w_{n-i}\,|\,w_1,\dots,w_{n-i-1})\,\exp\left[\left(\theta_{n-i}+\sum_{k=1}^{i}c_{n-k+1,n-i}(\widetilde{\theta}_{n-k+1})\right)'w_{n-i}\right]dw_{n-i}
\end{aligned}$$

$$\begin{aligned}
&\propto f(w_1,\dots,w_{n-i-1})\,\exp\left[\sum_{j=1}^{n-i-1}\left(\theta_j+\sum_{k=1}^{i}c_{n-k+1,j}(\widetilde{\theta}_{n-k+1})\right)'w_j\right]\\
&\quad\times E\left\{\exp\left[\left(\theta_{n-i}+\sum_{k=1}^{i}c_{n-k+1,n-i}(\widetilde{\theta}_{n-k+1})\right)'w_{n-i}\right]\,|\,w_1,\dots,w_{n-i-1}\right\}
\end{aligned}$$

$$\begin{aligned}
&\propto f(w_1,\dots,w_{n-i-1})\,\exp\left[\sum_{j=1}^{n-i-1}\left(\theta_j+\sum_{k=1}^{i}c_{n-k+1,j}(\widetilde{\theta}_{n-k+1})\right)w_j\right]\\
&\quad\times\exp\left[\sum_{j=1}^{n-i-1}c_{n-i,j}\left(\theta_{n-i}+\sum_{k=1}^{i}c_{n-k+1,n-i}(\widetilde{\theta}_{n-k+1})\right)'w_j\right]
\end{aligned}$$

$$\propto f(w_1,\dots,w_{n-i-1})\,\exp\left[\sum_{j=1}^{n-i-1}\left(\theta_j+\sum_{k=1}^{i+1}c_{n-k+1,j}(\widetilde{\theta}_{n-k+1})\right)'w_j\right]$$

which is the result for $i+1$. Finally, Lemma 1 and the formula for $f^\theta(w_1,\dots,w_{n-i})$ show that $f^\theta(w_{n-i}\,|\,w_1,\dots,w_{n-i-1})$ is the Esscher transform of $f(w_{n-i}\,|\,w_1,\,\dots,w_{n-i-1})$ associated with the coefficient:

$$\theta_{n-i}+\sum_{k=1}^{i}c_{n-k+1,n-i}(\widetilde{\theta}_{n-k+1})=\widetilde{\theta}_{n-i}\,. \tag{a.51}$$

Now, the proof of Proposition a.6 is a direct consequence of part $b)$ of Lemma a.2 and of formula (a.49), if we consider the conditional distribution given $\underline{w}_{t-1}$.

From proposition a.6 it is clear that if the components $w'_{1,t}, \ldots, w'_{n,t}$ are conditionally independent in the historical world, given w_{t-1}, then the same is true under the risk-neutral one and $\widetilde{\theta}_{n-i,t-1} = \theta_{n-i,t-1}$ for any $i \in \{0, \ldots, n-1\}$. Moreover, if $\theta_{n-i,t-1} = 0$ for $i \in \{0, \ldots, p\}$, we have $\widetilde{\theta}_{n-i,t-1} = 0$ for $i \in \{0, \ldots, p\}$ and, thus, the conditional distributions of $w_{n-i,t}$, given $w_{n-i-1,t}, \ldots, w_{1,t}, \underline{w}_{t-1}$ are the same under both measures, for $i \in \{0, \ldots, p\}$.

Corollary a.6.1 *If we denote by $\psi^{\mathbb{P}}_{i,t}(u_i)$ the historical log-Laplace transform of $w_{i,t}$ given $w_{i-1,t}, \ldots,$ $w_{1,t}$, $\underline{w}_{t-1}$, the corresponding risk-neutral log-Laplace transform is:*

$$\psi^{\mathbb{Q}}_{i,t}(u_i) = \psi^{\mathbb{P}}_{i,t}(u_i + \widetilde{\theta}_{i,t-1}) - \psi^{\mathbb{P}}_{i,t}(\widetilde{\theta}_{i,t-1}) \,. \tag{a.52}$$

Proof Straightforward.

Corollary a.6.2 *If $\widetilde{\theta}_{t-1}$ does not depend on $\underline{w}_{t-1}$, then the risk-neutral dynamics of (w_t) is recursive affine.*