Modeling conditional distributions with mixture models: Applications in finance and financial decision-making

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Journal of Applied Econometrics Invited Lecture

Università di Venezia
Italia

June 3, 2005

Compound Markov normal mixture models

$$\mathbf{s}_t = \begin{pmatrix} s_{t1} \\ s_{t2} \end{pmatrix}; \;\; s_{t1} \in \{1, \ldots, m_1\}, \, s_{t2} \in \{1, \ldots, m_2\}$$

$$P\left(s_{t1} = j \mid s_{t-1,1} = i, s_{t-1,2}, \mathbf{s}_{t-2}, \mathbf{s}_{t-3}, \ldots\right) = p_{ij}$$

$$P\left(s_{t2} = j \mid s_{t1} = i, \mathbf{s}_{t-1}, \mathbf{s}_{t-2}, \ldots\right) = r_{ij}$$

$$y_t \mid (\mathbf{x}_t, s_{t1} = i, s_{t2} = j) \sim N\left(\beta + \phi_i + \psi_{ij}, \sigma^2 \cdot \sigma_i^2 \cdot \sigma_{ij}^2\right)$$

Summary of parameterization

$$\boldsymbol{\beta}, \quad \boldsymbol{\phi} = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{m_1} \end{pmatrix}, \quad \boldsymbol{\Psi} = \begin{bmatrix} \psi_{11} & \cdots & \psi_{1m_2} \\ \vdots & & \vdots \\ \psi_{m_1 1} & \cdots & \psi_{m_1 m_2} \end{bmatrix},$$

$$\mathbf{P} = \begin{bmatrix} p_{11} & \cdots & p_{1m_1} \\ \vdots & & \vdots \\ p_{m_1 1} & \cdots & p_{m_1 m_1} \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} r_{11} & \cdots & r_{1m_2} \\ \vdots & & \vdots \\ r_{m_2 1} & \cdots & r_{m_2 m_2} \end{bmatrix},$$

$$\sigma^2, \quad \boldsymbol{\sigma} = \begin{pmatrix} \sigma_1^2 \\ \vdots \\ \sigma_{m_1}^2 \end{pmatrix}, \quad \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11}^2 & \cdots & \sigma_{1m_2}^2 \\ \vdots & & \vdots \\ \sigma_{m_1 1}^2 & \cdots & \sigma_{m_1 m_2}^2 \end{bmatrix};$$

$$\boldsymbol{\theta} = \left\{ \boldsymbol{\beta}, \boldsymbol{\phi}, \boldsymbol{\Psi}, \mathbf{P}, \mathbf{R}, \boldsymbol{\sigma}^2, \boldsymbol{\sigma}, \boldsymbol{\Sigma} \right\}$$

Restrictions on parameters

Let

$$\boldsymbol{\pi} : \boldsymbol{\pi}'\mathbf{P} = \boldsymbol{\pi}'$$

$$E\left(y_t \mid s_{t1} = i\right) = \beta + \phi_i + \sum_{j=1}^{m_2} r_{ij}\psi_{ij} = \beta + \phi_i$$

$$E\left(y_t\right) = \beta + \sum_{i=1}^{m_1} \pi_i\phi_i + \sum_{i=1}^{m_1} \pi_i \sum_{j=1}^{m_2} r_{ij}\psi_{ij} = \beta$$

Number of "identified" parameters in the model is $m_1\left(m_1 + 3m_2 - 3\right)$.

If $\boldsymbol{\phi} = \mathbf{0}$, it is $m_1\left(3m_2 + m_1 - 4\right) + 1$.

Prior distribution

$$\beta \sim N(0,1),$$
$$\phi_i^* \mid \sigma^2 \overset{iid}{\sim} N\left(0, 4\sigma^2\right) \quad (i = 1, \ldots, m_1)$$
$$\text{For } i = 1, \ldots m_1: \quad \psi_{ij}^* \mid \left(\sigma^2\sigma_i^2\right) \overset{iid}{\sim} N\left(0, 4\sigma^2\sigma_i^2\right) \quad (j = 1, \ldots, m_2)$$

$$5/\sigma^2 \sim \chi^2(5)$$
$$2/\sigma_i^2 \overset{iid}{\sim} \chi^2(2) \quad (i = 1, \ldots m_1)$$
$$2/\sigma_{ij}^2 \overset{iid}{\sim} \chi^2(2) \quad (i = 1, \ldots m_1; j = 1, \ldots, m_2)$$

$$\left(p_{i1}, \ldots, p_{im_1}\right) \overset{iid}{\sim} \text{Dirichlet}(1, \ldots, 1)$$
$$\left(r_{i1}, \ldots, p_{im_1}\right) \overset{iid}{\sim} \text{Dirichlet}(1, \ldots, 1)$$

If no serial correlation is permitted then $\boldsymbol{\phi} = \mathbf{0}$.

All results for S&P 500 daily returns, Jan. 2, 1990 - Dec. 31, 1999

1. 12,000 MCMC iterations

2. Every 10'th iteraton recorded

3. Of the 1,200 iterations recorded

   (a) First 200 discarded

   (b) Remaining 1000 used for analysis

Comparison of models by means of average log-likelihood,

$$\int_{\Theta} \log p\left(\mathbf{y} \mid \boldsymbol{\theta}\right) p\left(\boldsymbol{\theta} \mid \mathbf{y}\right) d\boldsymbol{\theta}$$

	$m_1 = 2$	$m_1 = 3$	$m_1 = 4$	$m_1 = 5$
$m_2 = 1$	-3043.9	-3003.8	-2985.3	-2986.1
$m_2 = 2$	-2992.9	**-2979.4**	-2979.7	-2982.1
$m_2 = 3$	-2989.8	-2977.2	-2977.0	-2980.3
$m_2 = 4$	-2989.4	-2976.8	-2978.1	-2980.4
$m_5 = 5$	-2988.7	-2976.5	-2978.0	-2980.4

Serial correlation is prohibited.

There are $2m_1\left(2m_2 - 1\right)$ parameters in the models.

With serial correlaton prohibited:

	$m_1 = 2$	$m_1 = 3$	$m_1 = 4$	$m_1 = 5$
$m_2 = 1$	-3043.9	-3003.8	-2985.3	-2986.1
$m_2 = 2$	-2992.9	**-2979.4**	-2979.7	-2982.1
$m_2 = 3$	-2989.8	-2977.2	-2977.0	-2980.3
$m_2 = 4$	-2989.4	-2976.8	-2978.1	-2980.4
$m_5 = 5$	-2988.7	-2976.5	-2978.0	-2980.4

With serial correlation permitted:

	$m_1 = 2$	$m_1 = 3$	$m_1 = 4$	$m_1 = 5$
$m_2 = 1$	-3044.2	-3004.4	-2985.0	-2976.4
$m_2 = 2$	-2993.4	-2980.3	-2980.4	-2979.9
$m_2 = 3$	-2990.2	-2977.8	-2978.8	-2979.0
$m_2 = 4$	-2985.5	-2977.1	-2978.6	-2979.0
$m_5 = 5$	-2989.2	-2976.5	-2978.5	-2977.4

Characterizing the persistent states

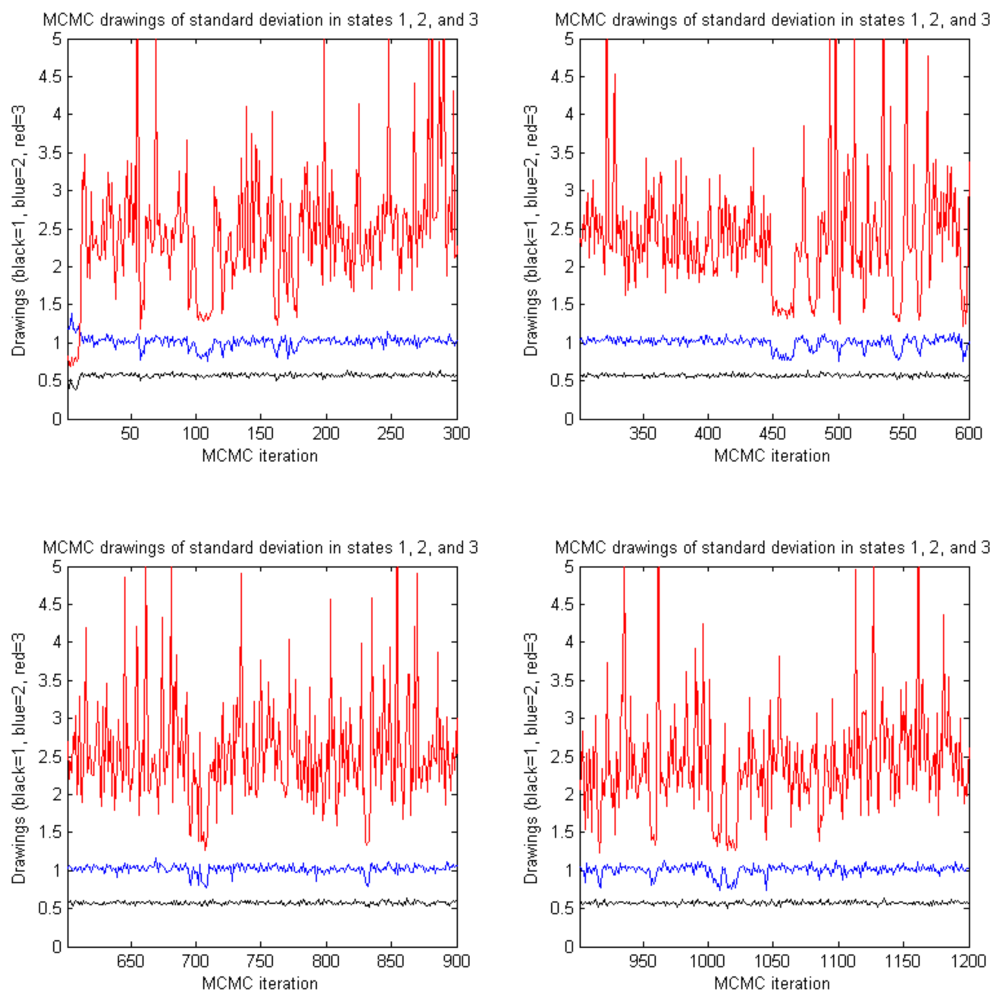
Standard deviation in state i:

$$\left[\sum_{j=1}^{m_2} \left(\psi_{ij}^2 + \sigma^2 \sigma_i^2 \sigma_{ij}^2\right)\right]^{1/2}$$

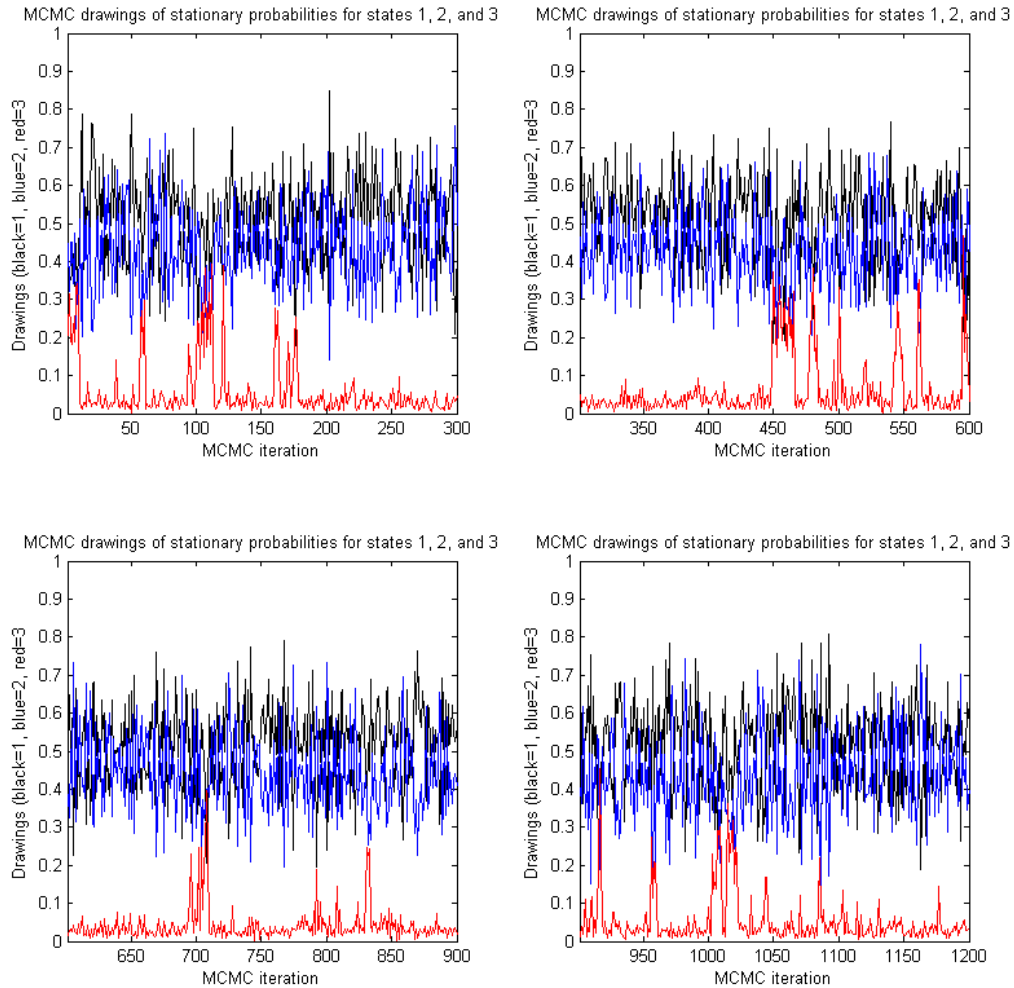
Invariant probabilities:

$$\boldsymbol{\pi} : \boldsymbol{\pi}'\mathbf{P} = \boldsymbol{\pi}'$$

States are not identified with respect to re-labeling.



MCMC simulations of component standard deviations, Markov mixture model



MCMC simulations of component invariant probabilities, Markov mixture model

State dynamics in the model

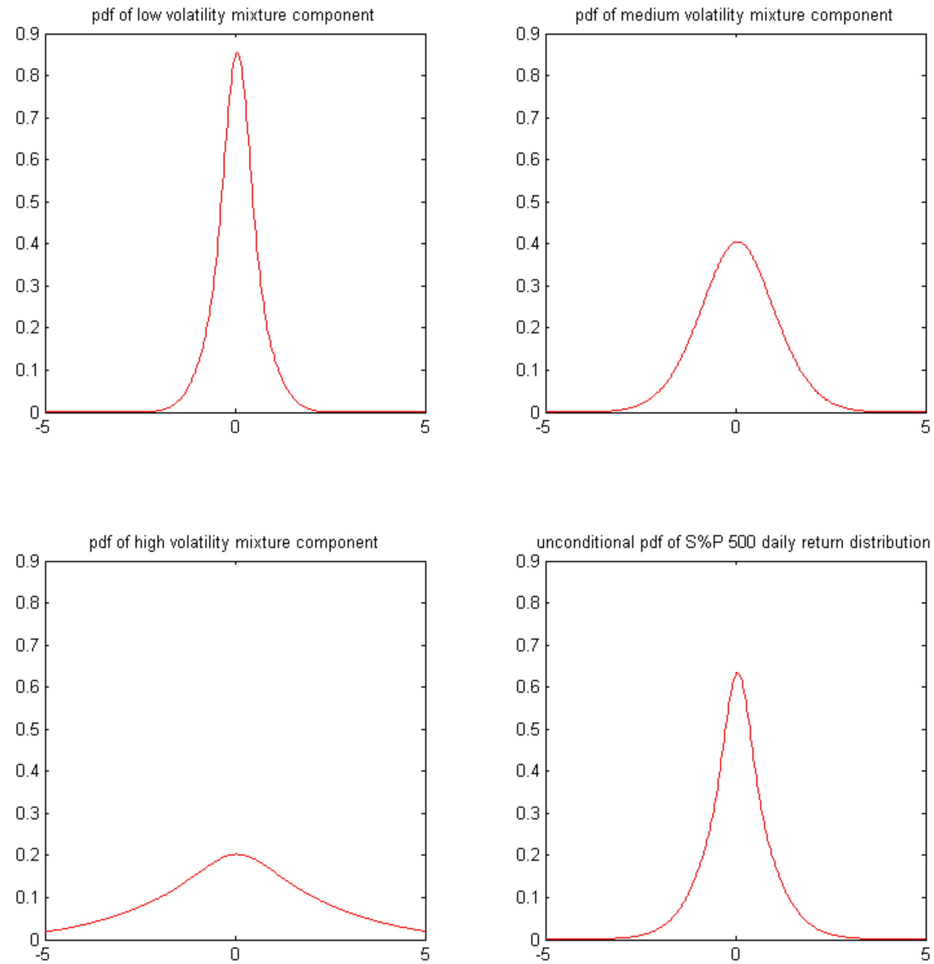
Posterior mean of transition matrix is

$$\mathbf{P} = \begin{bmatrix} 0.9917 & 0.0061 & 0.0022 \\ 0.0072 & 0.9843 & 0.0085 \\ 0.0465 & 0.2011 & 0.7524 \end{bmatrix}$$

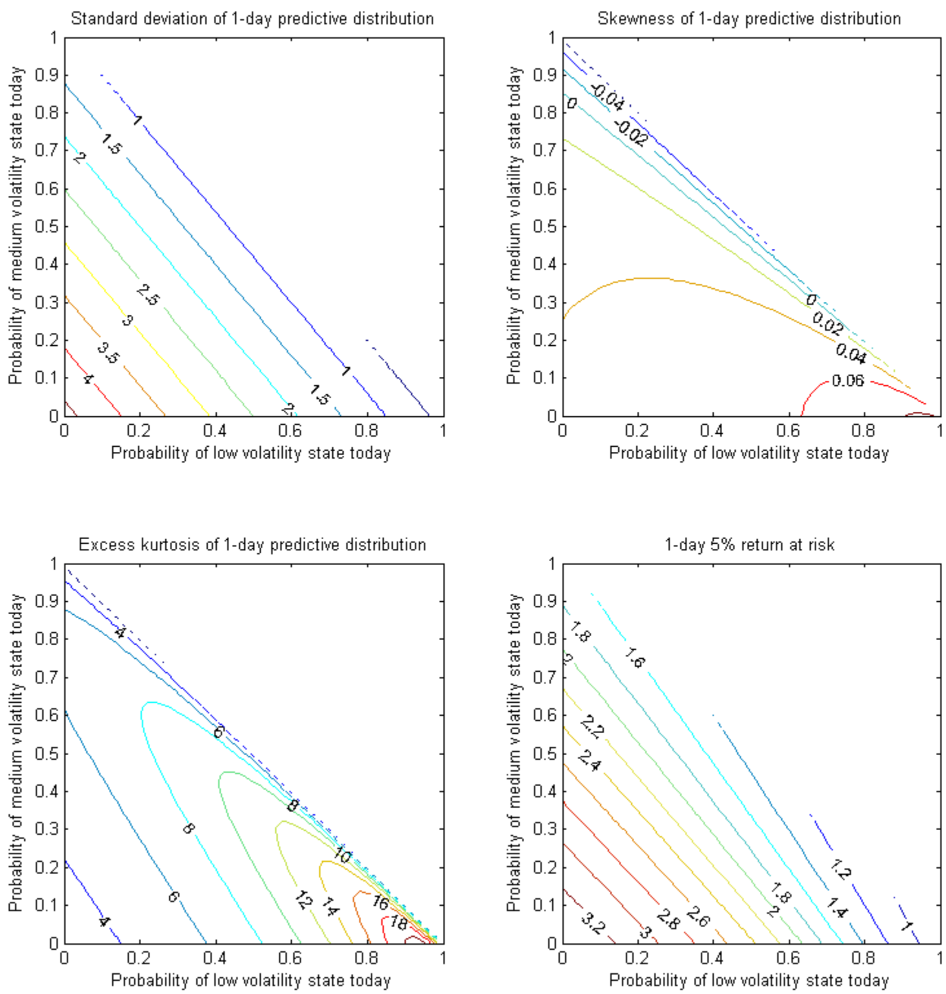
where states are ordered as low, medium and high volatility.

Posterior mean of invariant probability vector is

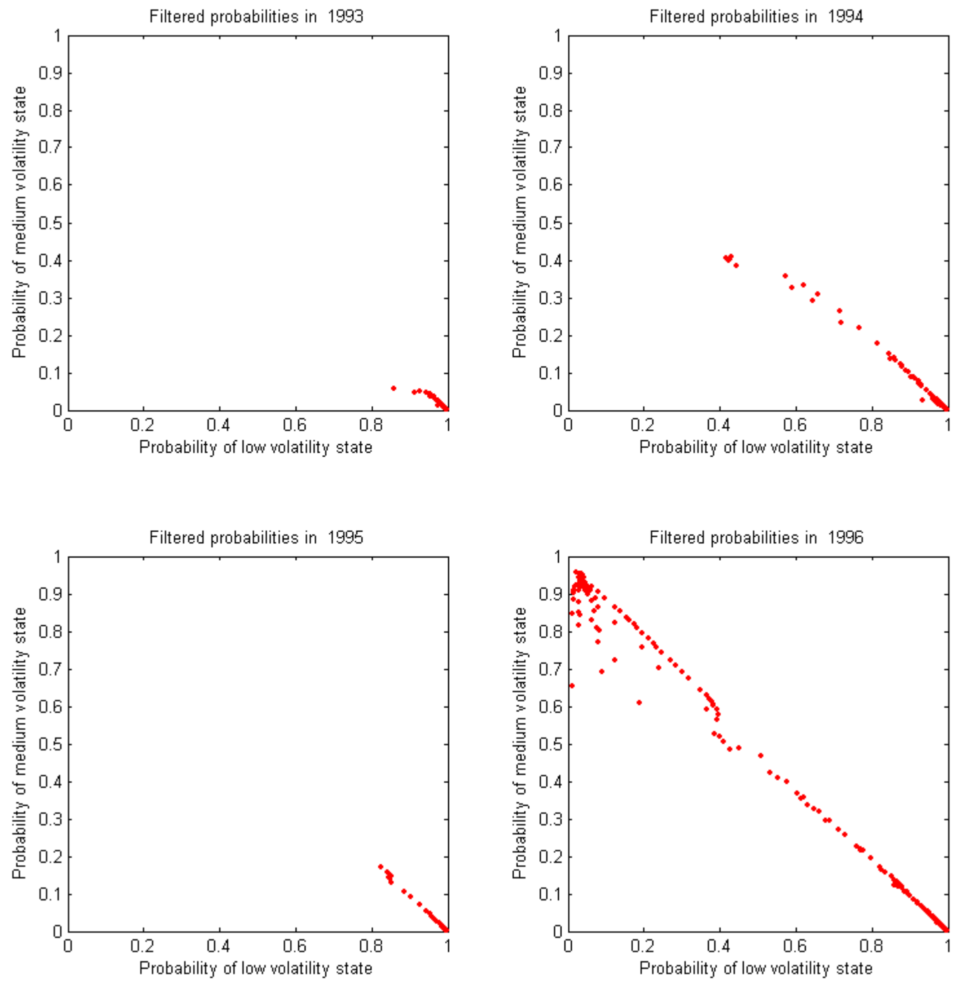
$$\boldsymbol{\pi}' = \begin{pmatrix} 0.5184 & 0.4588 & 0.0228 \end{pmatrix}$$



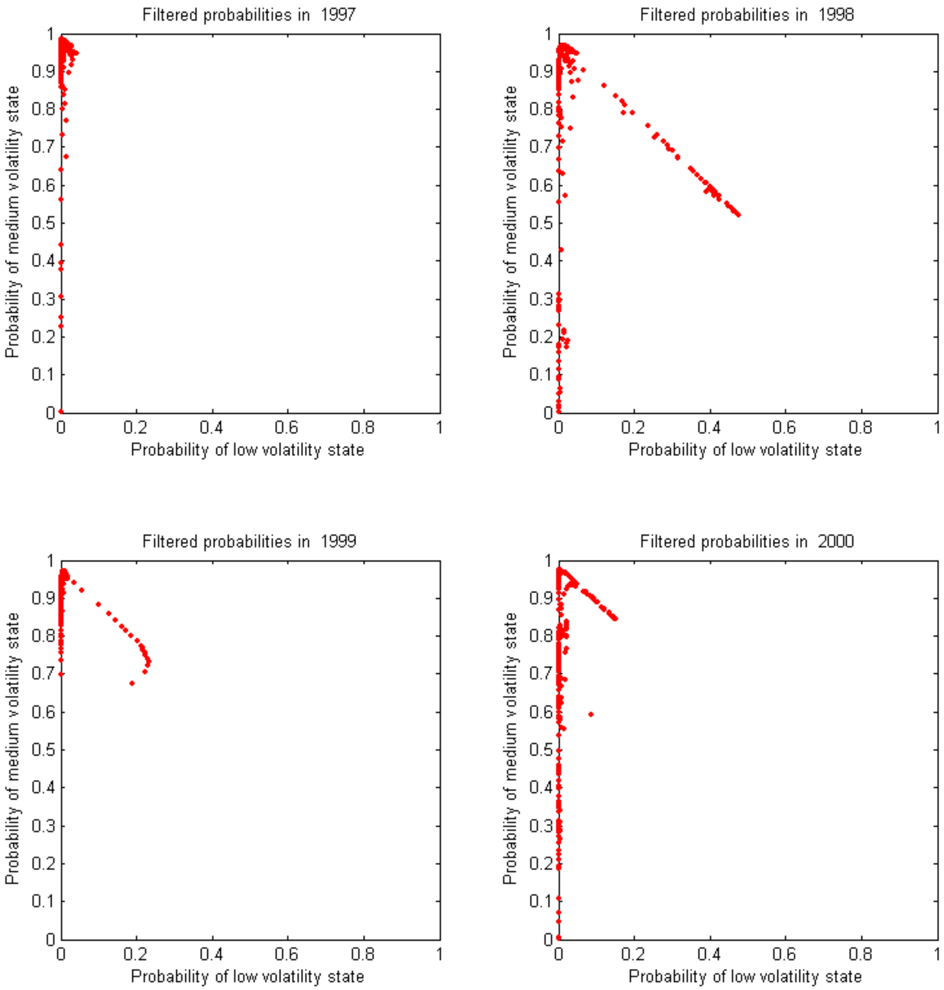
Component p.d.f.s and unconditional p.d.f., compound Markov normal mixture model



One-day predictive distribution given current filtered probabilties



Daily filtered persistent state probabilities, compound Markov normal mixture model



Daily filtered persistent state probabilities, compound Markov normal mixture model

Smoothly mixing regression models

Begin with the normal mixture model

$$y_t \mid (\mathbf{x}_t, \mathbf{v}_t, s_t = j) \sim N\left(\underset{k\times 1}{\boldsymbol{\beta}}{}'\mathbf{x}_t + \underset{p\times 1}{\boldsymbol{\alpha}_j'}\mathbf{v}_t, \sigma_j^2\right),$$

$j = 1, \ldots, m.$

Determination of latent states s_t:

$$\underset{m\times 1}{\widetilde{\mathbf{w}}_t} = \boldsymbol{\Gamma}\underset{q\times 1}{\mathbf{z}_t} + \boldsymbol{\zeta}_t;\ \ \boldsymbol{\zeta}_t \overset{iid}{\sim} N(\mathbf{0}, \mathbf{I}_m)$$

$$\widetilde{s}_t = j \ \text{ iff } \ \widetilde{w}_{tj} \geq \widetilde{w}_{ti}\ \forall\ i = 1, \ldots, m$$

Variable of interest:

y_t: Daily S&P 500 returns, 1990 - 1999

The covariate vectors $\mathbf{x}_t$, $\mathbf{v}_t$ and $\mathbf{z}_t$ are interactive polynomials in the variables

$$\begin{aligned} a_t &= \text{Return in period } t-1,\ a_t = y_{t-1} \\ b_t &= g \cdot b_{t-1} + (1-g)\,|a_{t-1}|^{\kappa} = \sum_{s=0}^{\infty} g^s\,|y_{t-2-s}|^{\kappa} \end{aligned}$$

Gaussian priors:

$$\begin{array}{ll} \boldsymbol{\beta}: & \mu = 0,\ \tau^2 = 1 \\ \boldsymbol{\alpha}: & \mu = 0,\ \tau^2 = 9 \\ \boldsymbol{\Gamma}^* & \mu = 0,\ \tau^2 = 16 \end{array}$$

Inverse gamma priors:

$$\begin{aligned} 2/\sigma^2 &\sim \chi^2(2) \\ 2/\sigma_j^2 &\sim \chi^2(2) \end{aligned}$$

Models considered

$$y_t \sim N\left(\underset{k\times 1}{\boldsymbol{\beta}}{}'\mathbf{x}_t + \underset{p\times 1}{\boldsymbol{\alpha}'_j}\mathbf{v}_t, \sigma_j^2\right), \quad \underset{m\times 1}{\widetilde{\mathbf{w}}_t} = \boldsymbol{\Gamma}\underset{q\times 1}{\mathbf{z}_t} + \boldsymbol{\zeta}_t, \quad \widetilde{s}_t = \text{iff } \widetilde{w}_{tj} \geq \widetilde{w}_{ti}\ \forall\ i$$

(A) $q = 1$, $k > 1$, $p = 1$ Linear regression, normal mixture

(B) $q = 1$, $k = 1$, $p > 1$ Mixture of linear regressions

(C) $q > 1$, $k = 1$, $p = 1$ $\mathbf{w}_t$-weighted mixture of normals

(D) $q > 1$, $k > 1$, $p = 1$ Regression, $\mathbf{w}_t-$dependent normal mixture

(E) $q > 1$, $k = 1$, $p > 1$ $\mathbf{w}_t$-dependent mixture of linear regressions

Some comparisons across models by means of average log-likelihood,

$$\int_{\Theta} \log p(\mathbf{y} \mid \boldsymbol{\theta})\, p(\boldsymbol{\theta} \mid \mathbf{y})\, d\boldsymbol{\theta}$$

Polynomials of order 3 in both a_t and b_t,

Mixture of $m = 3$ normals,

b_t parameters $g = 0.9$, $\kappa = 1$

Model	Description	Average log-likelihood
A	Linear regression, normal mixture	−3124.9
B	Mixture of linear regressions	−3094.9
C	$\mathbf{w}_t$-weighted mixture of normals	−2859.4
D	Regression, $\mathbf{w}_t$−dependent normal mixture	−2875.5
E	$\mathbf{w}_t$-dependent mixture of linear regressions	−2872.1

Some comparisons across models by means of average log-likelihood,

$$\int_{\Theta} \log p\left(\mathbf{y} \mid \boldsymbol{\theta}\right) p\left(\boldsymbol{\theta} \mid \mathbf{y}\right) d\boldsymbol{\theta}$$

Model C, $\mathbf{w}_t$-weighted mixture of normals,

b_t parameters $g = 0.9$, $\kappa = 1$

Mixture of $m = 3$ normals

Polynomial orders		
a_t	b_t	Average log-likelihood
3	3	-2859.4
3	5	-2864.1
5	3	-2875.6

Some comparisons across models by means of average log-likelihood,

$$\int_{\Theta} \log p\left(\mathbf{y} \mid \boldsymbol{\theta}\right) p\left(\boldsymbol{\theta} \mid \mathbf{y}\right) d\boldsymbol{\theta}$$

Model C, $\mathbf{w}_t$-weighted mixture of normals,

Polynomials of order 3 in both a_t and b_t,

b_t parameters $g = 0.9$, $\kappa = 1$

Number of mixture components m	Average log-likelihood
2	-2913.5
3	-2859.4
4	-2848.8
5	-2851.4

Some comparisons across models by means of average log-likelihood,

$$\int_{\Theta} \log p\left(\mathbf{y} \mid \boldsymbol{\theta}\right) p\left(\boldsymbol{\theta} \mid \mathbf{y}\right) d\boldsymbol{\theta}$$

Model C, $\mathbf{w}_t$-weighted mixture of normals,

Polynomials of order 3 in both a_t and b_t,

Mixture of $m = 3$ normals

g	κ	Average log-likelihood
0.90	1.0	-2860.4
0.90	0.7	-2904.9
0.90	1.5	-2930.9
0.70	1.0	-2843.7
0.80	1.0	$\mathbf{-2839.0}$
0.95	1.0	-2895.1

All results for S&P 500 daily returns, Jan. 2, 1990 - Dec. 31, 1999

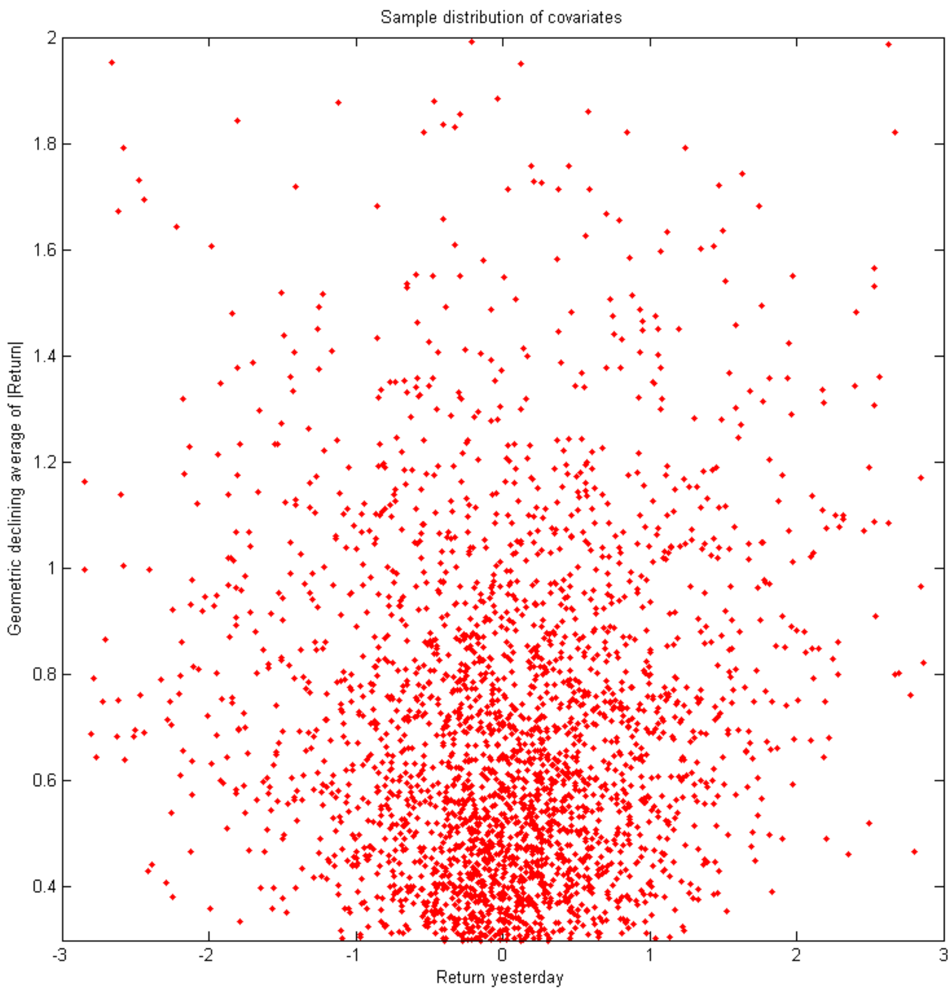
1. 12,000 MCMC iterations

2. Every 100'th iteraton recorded

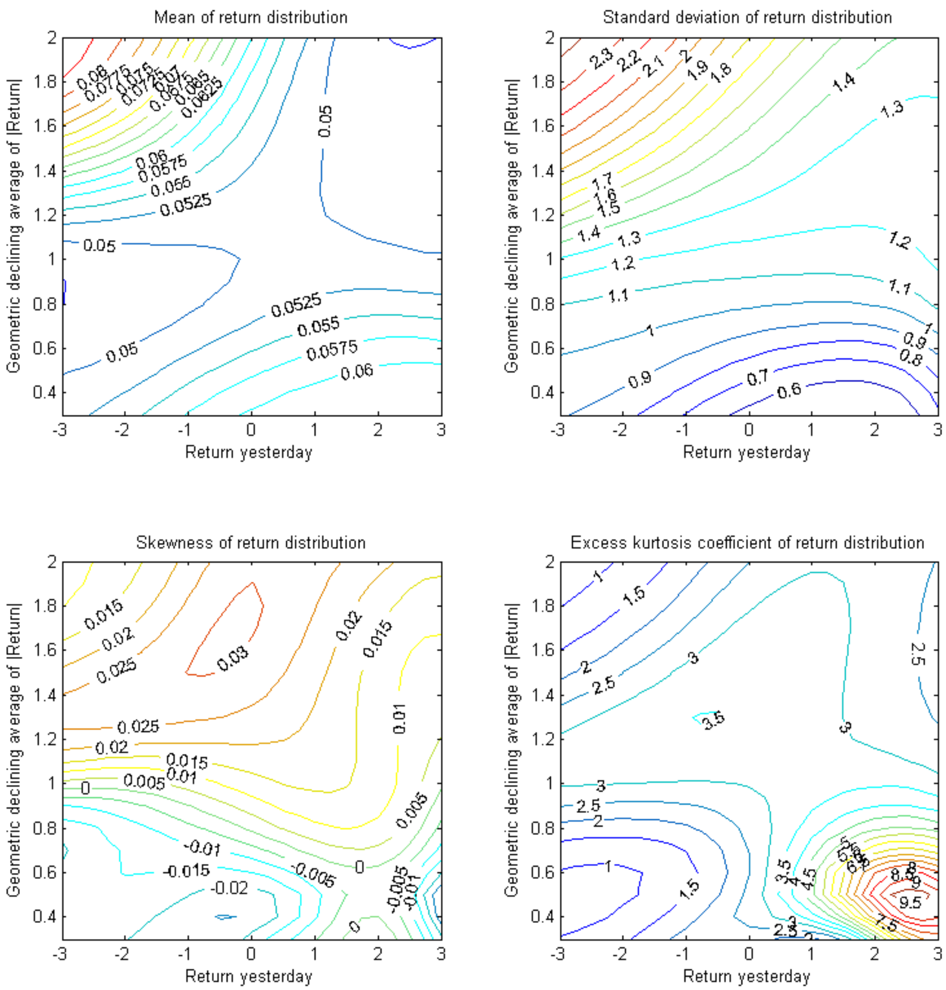
3. Of the 1200 iterations recorded

   (a) First 20 discarded

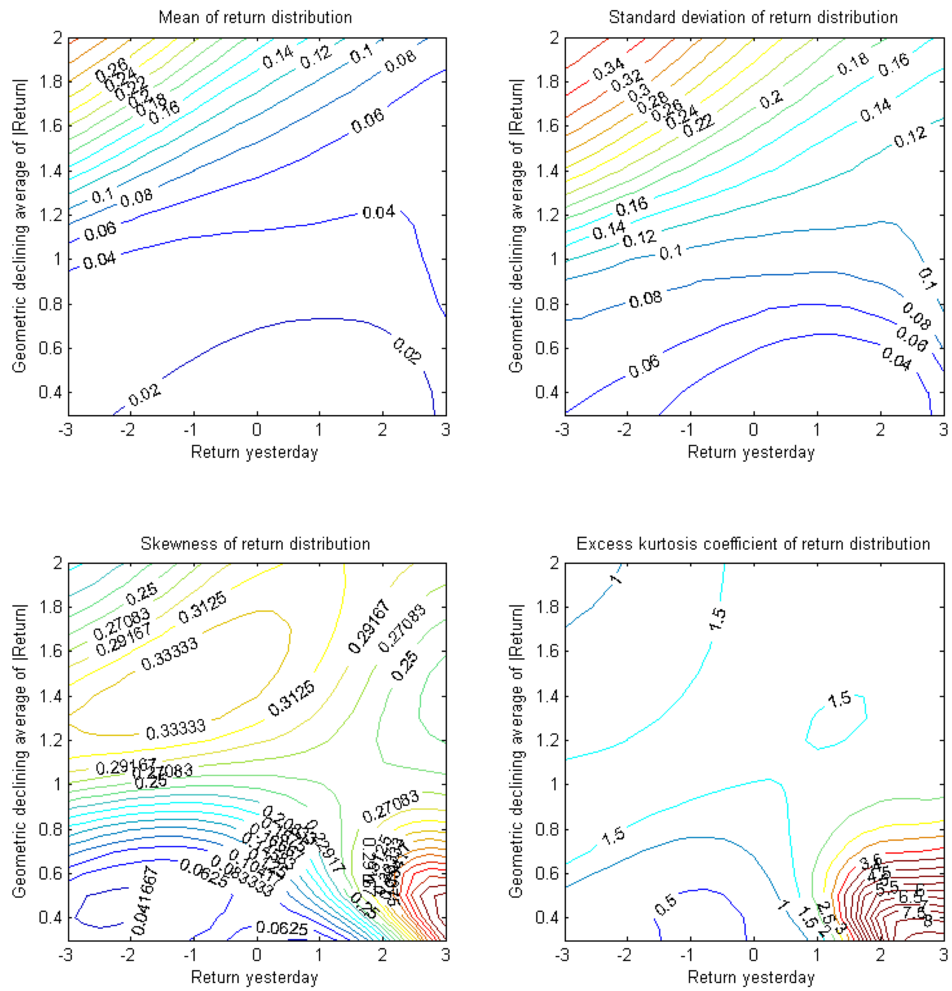
   (b) Remaining 100 used for analysis



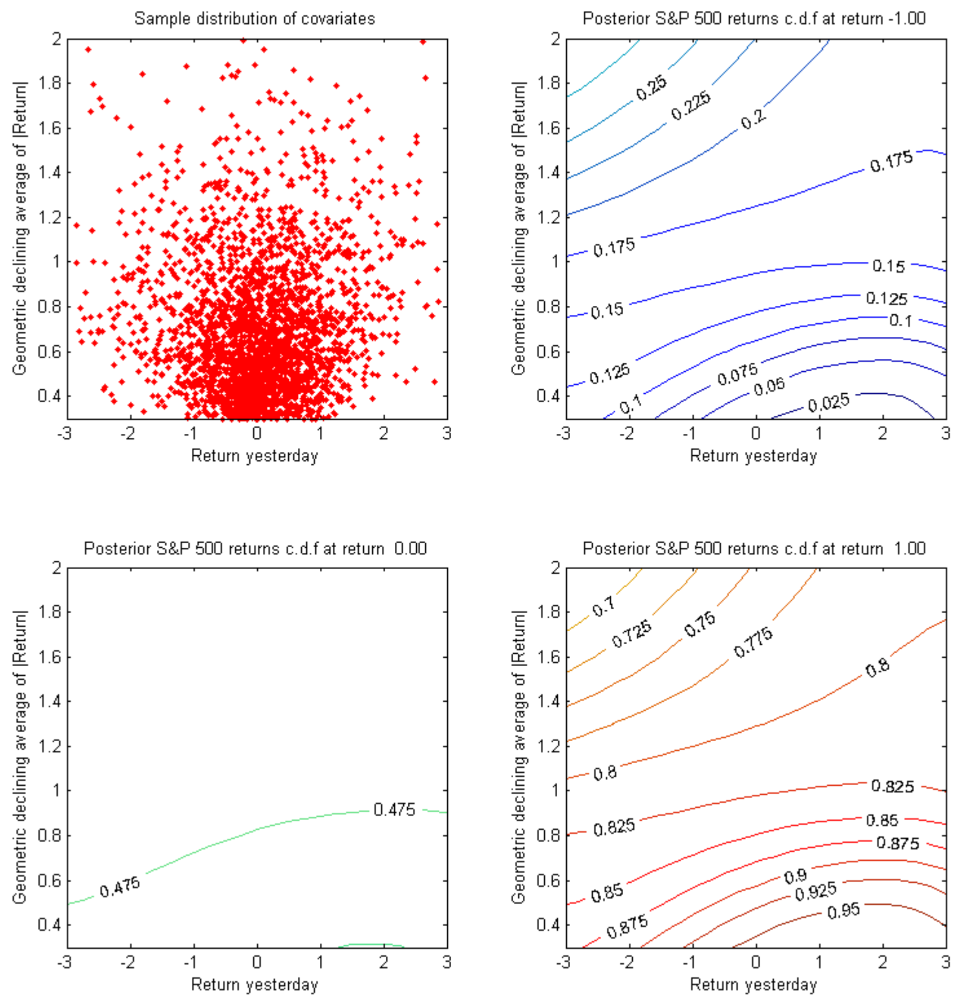
Sample distribution of state variables, S&P 500 return



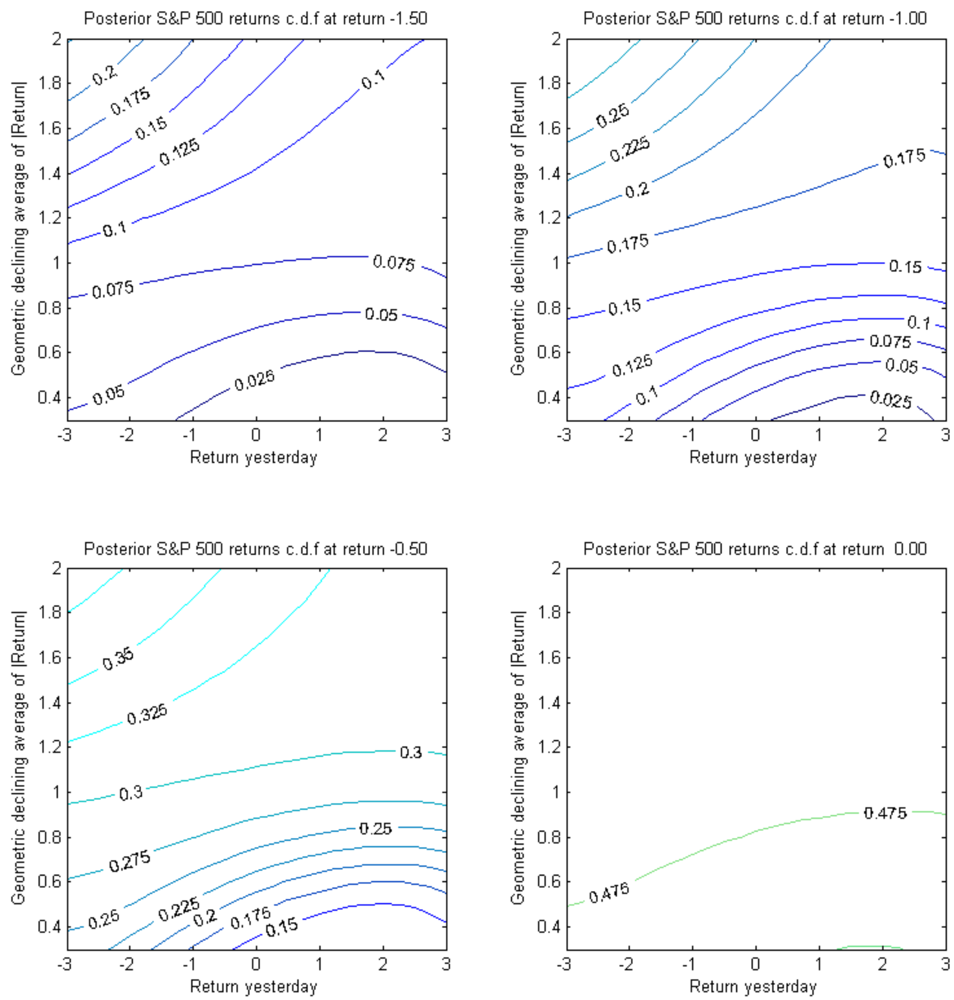
Posterior means of population moments, S&P 500 return, 1990-1999



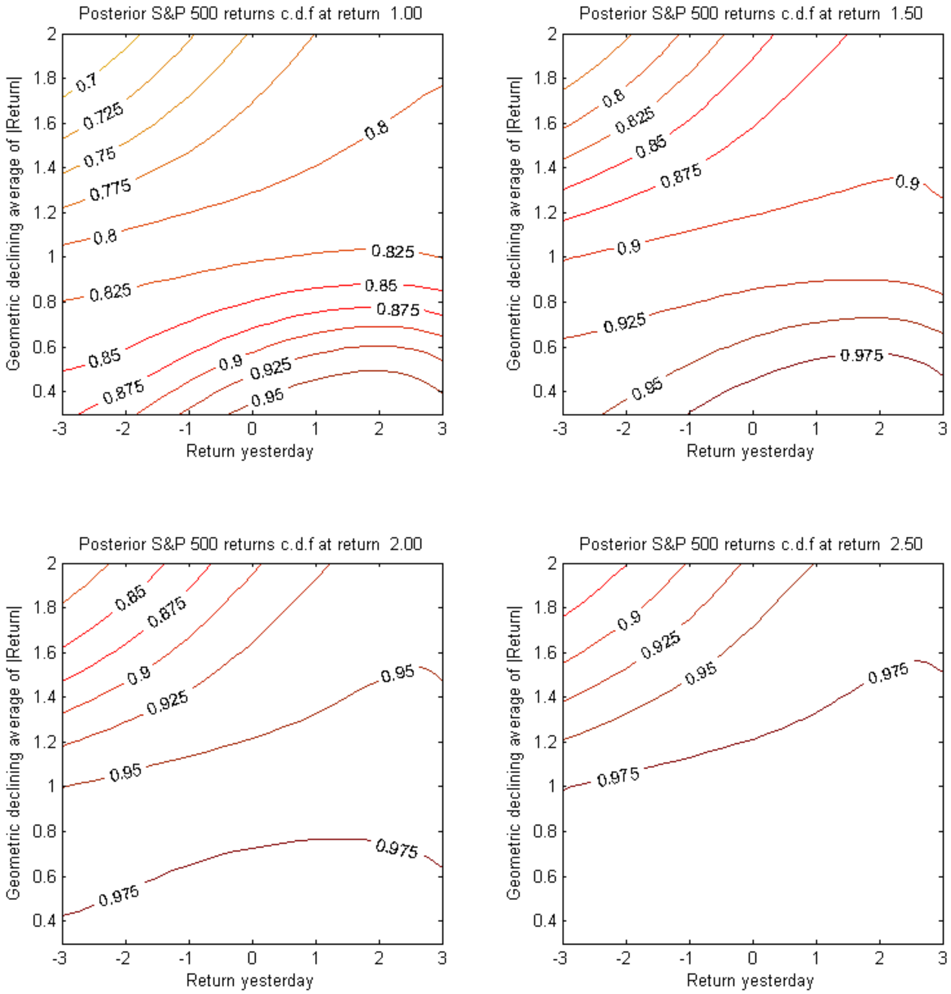
Posterior standard deviations of population moments, S&P 500 return, 1990-1999



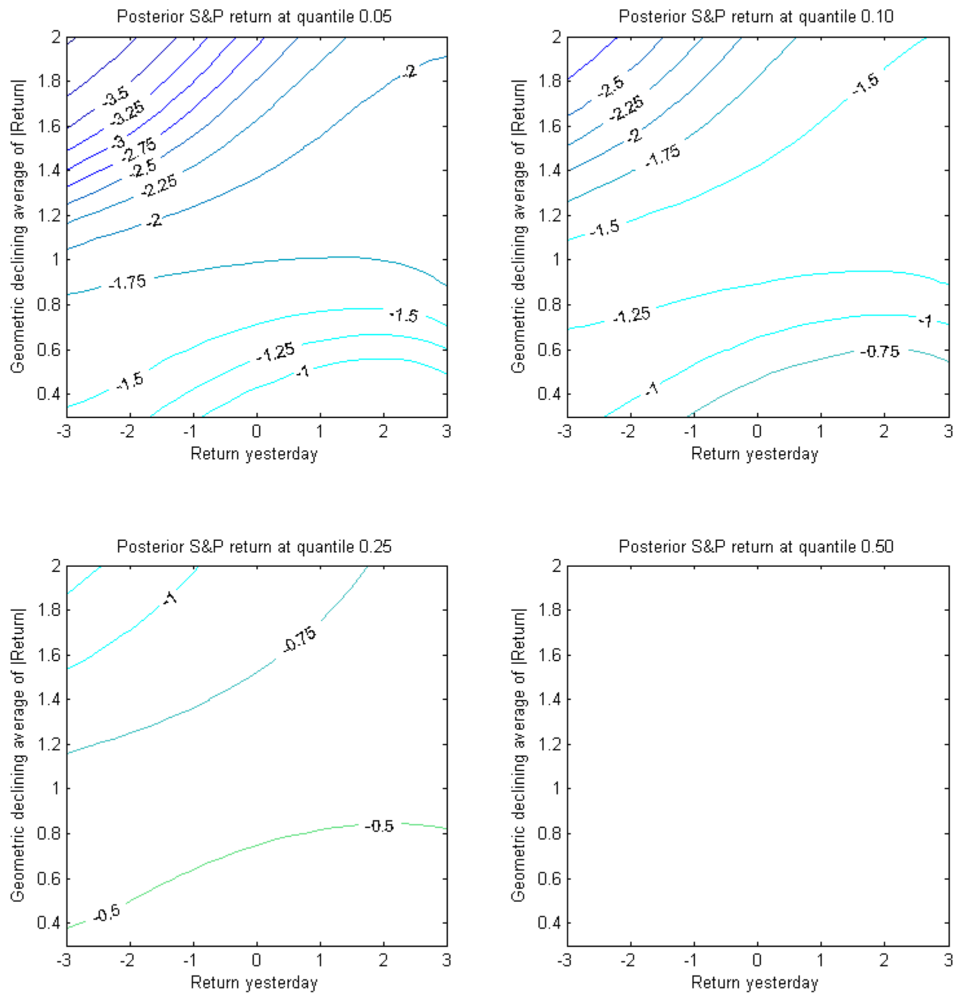
Distribution of state varaibles (1990-1999) and conditional c.d.f. of returns



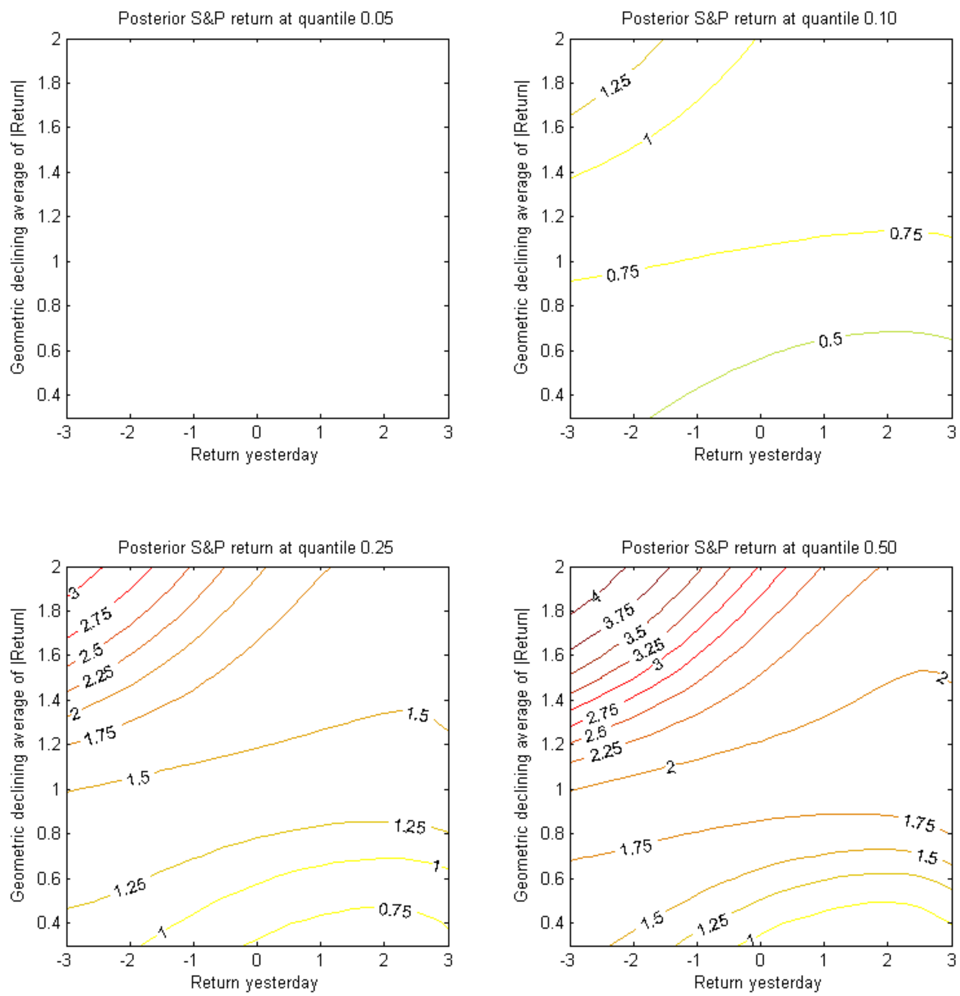
Conditional c.d.f. of returns



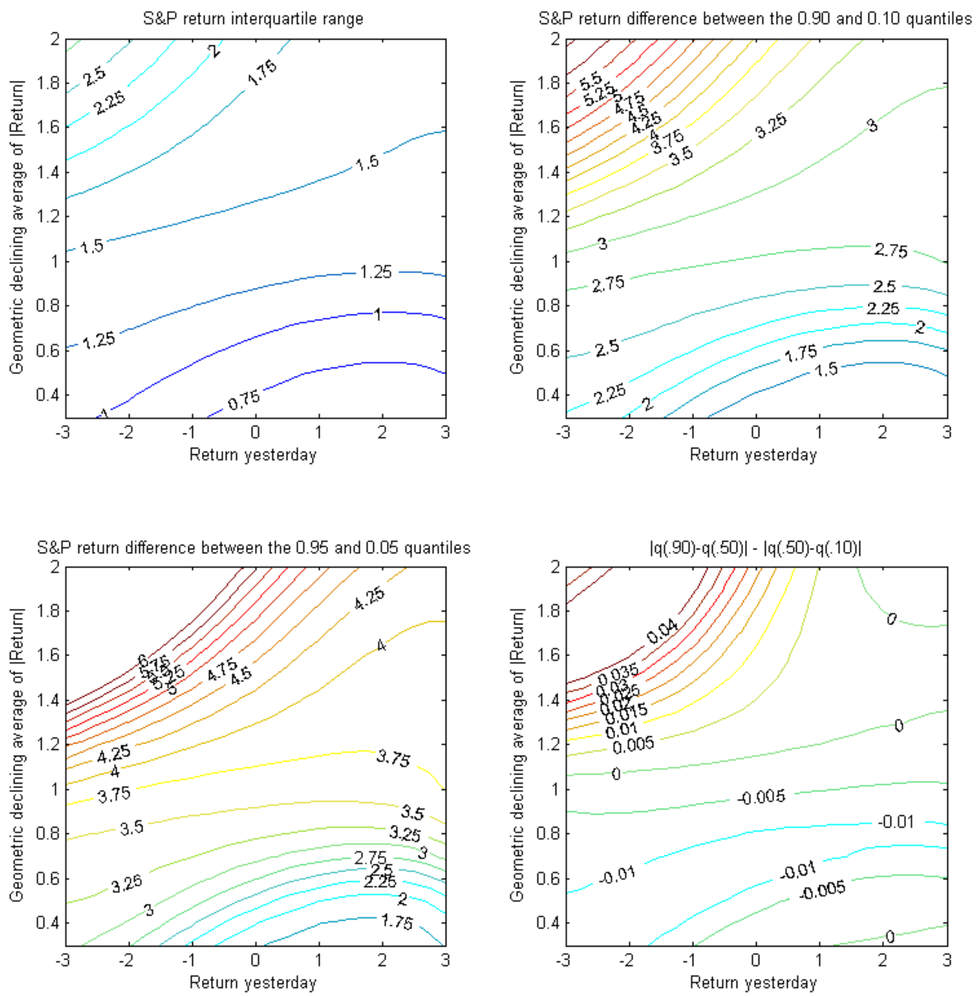
Conditional c.d.f. of returns



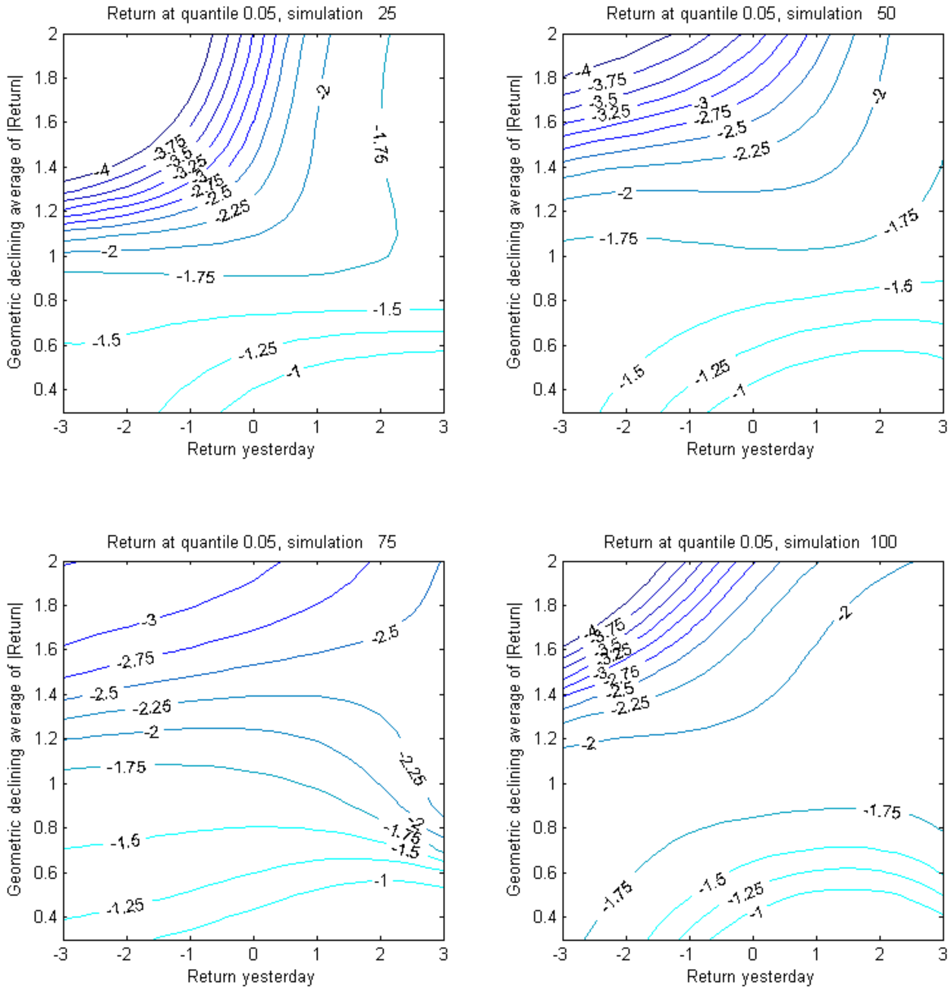
Posterior quantiles of S&P 500 predictive distribution, 1990-1999



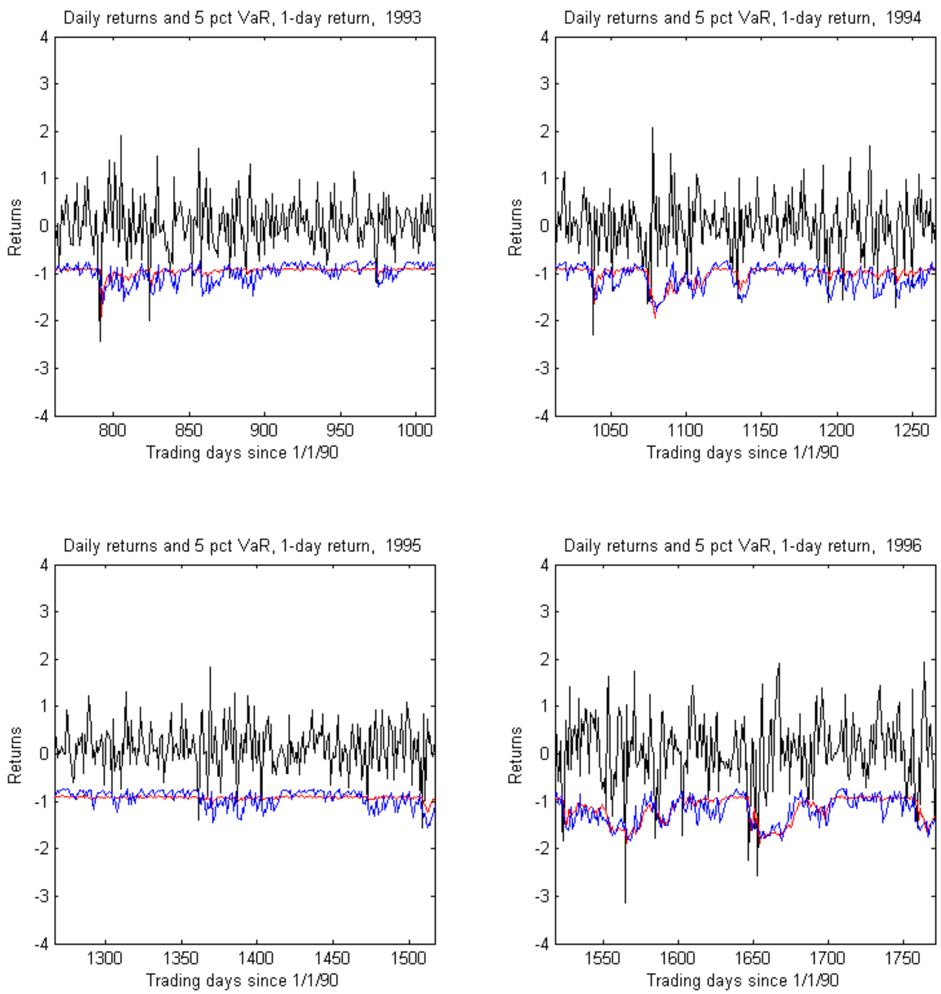
Posterior quantiles of S&P 500 predictive distribution, 1990-1999



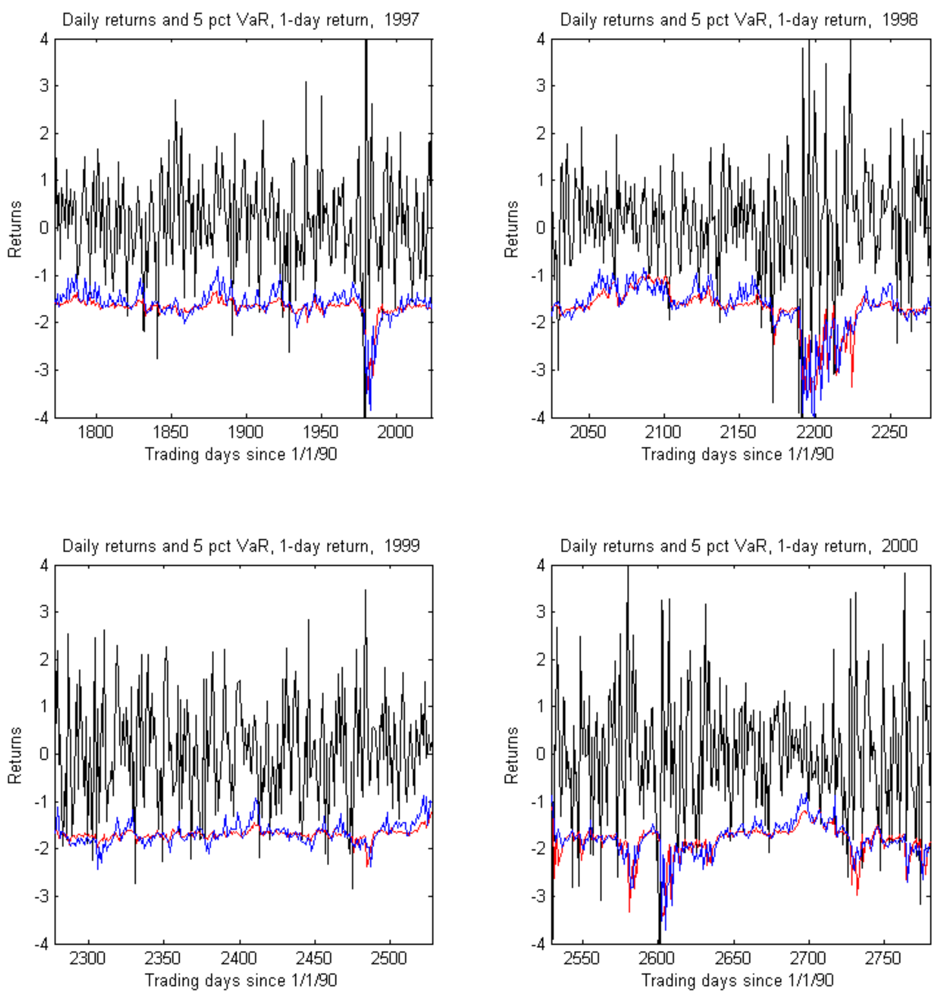
Posterior quantiles of S&P 500 predictive distribution



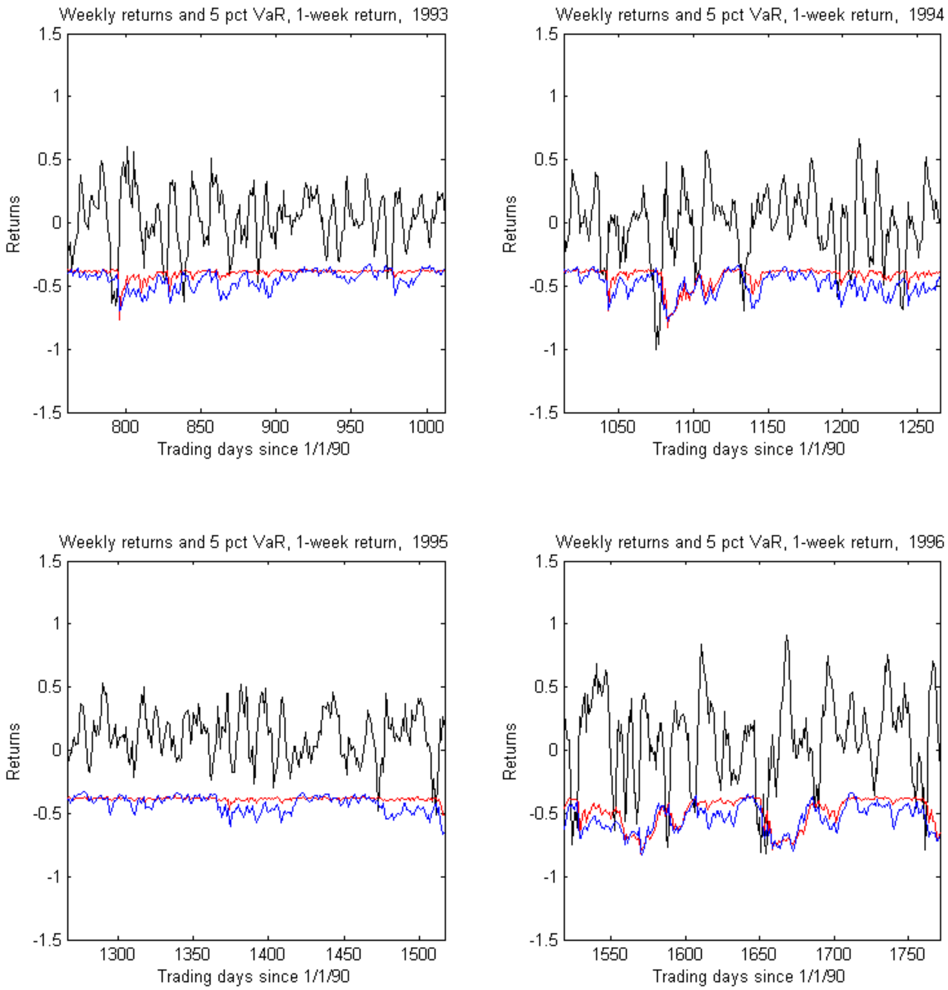
Populaton quantiles of S&P 500 predictive distribution in selected MCMC replications



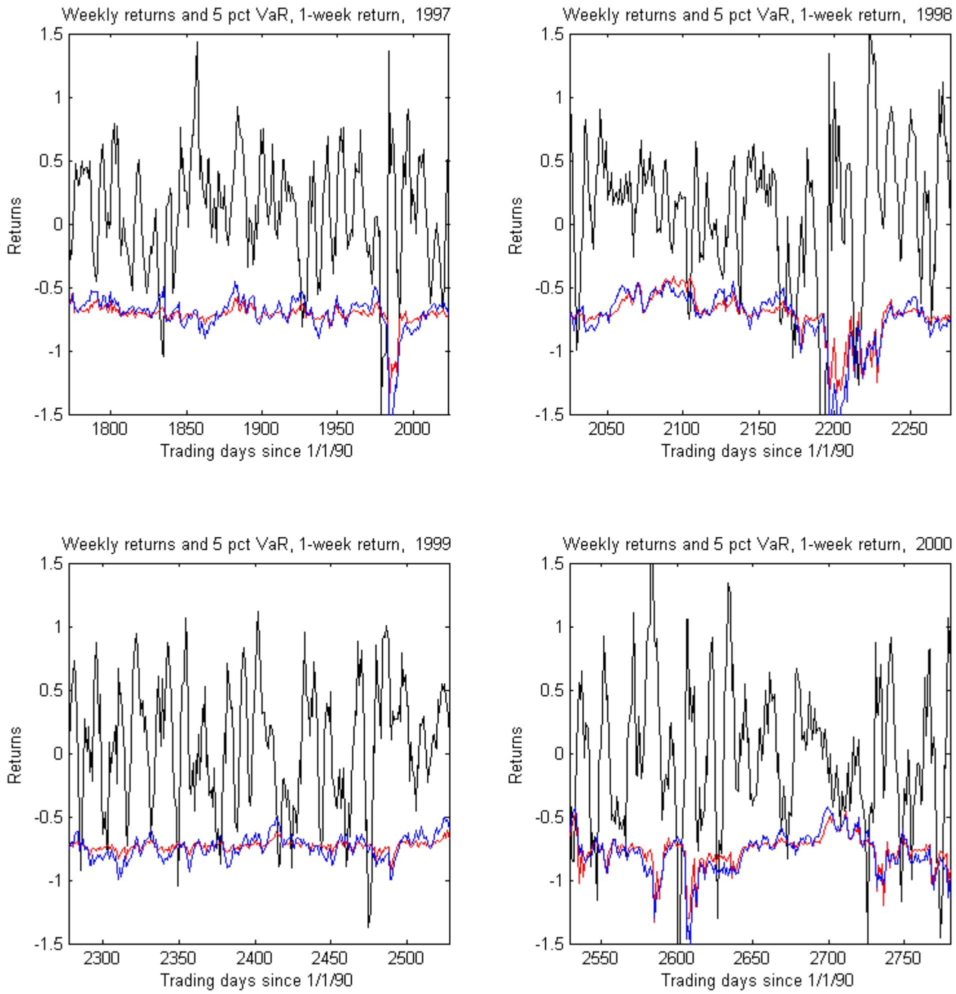
Returns and 1-day 5% return at risk in two models



Returns and 1-day 5% return at risk in two models



Returns and 1-week 5% return at risk in two models



Returns and 1-week 5% return at risk in two models