Switching VARMA Term Structure Models

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[Preliminary and incomplete version]

Abstract
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The purpose of the paper is to propose a global discrete-time modeling of the term structure of interest rates able to capture simultaneously the following important features : (i) interest rates with an historical dynamics involving several lagged values, and switching regimes; (ii) a specification of the stochastic discount factor (SDF) with time-varying and regime-dependent risk-premia; (iii) the possibility to derive explicit or quasi explicit formulas for zero-coupon bond and interest rate derivative prices; (iv) the positiveness of the yields at each maturity. We develop the Switching Autoregressive Normal (SAN) Term Structure model of order p and the Switching Autoregressive Gamma (SAG) Term Structure model of order p, and regime shifts are described by a Markov chain with state-dependent transition probabilities. In both cases multifactor generalizations are proposed.

Keywords : Affine Term Structure Models, Stochastic Discount Factor, Car processes, Switching Regimes, VARMA processes, Lags, Positiveness, Derivative Pricing.

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1 INTRODUCTION

In this paper we propose a global discrete-time modeling of the term structure of interest rates, which captures simultaneously the following important features :

- interest rates with an historical dynamics involving several lagged values, and switching regimes;
- a specification of the stochastic discount factor (SDF) with time-varying and regime-dependent risk-premia;
- the possibility to derive explicit or quasi explicit formulas for zero-coupon bond and interest rate derivative prices;
- the positiveness of the yields at each maturity.

It is well known in the literature that interest rates show an historical dynamics characterized by a strong dependence from several of their lagged values, and by switching in the regimes [see, among the others, Hamilton (1988), Cai (1994), Garcia and Perron (1996), Gray (1996), Cochrane and Piazzesi (2005)]; indeed, changes in the business cycle conditions or monetary policy may affect real rates and expected inflation and cause interest rates to behave quite differently in different time periods. In addition, there is a large (discrete-time and continuous-time) empirical literature on bond yields (in particular, short-term rates), based, in general, on the class of Affine Term Structure Models (ATSMs)[3], suggesting that regime switching models describe the historical interest rates data better than single-regime models [see, for example, Driffill and Sola (1994), Ang and Bekaert (2002), Bansal and Zhou (2002), Dai, Singleton and Yang (2003), Driffill, Kenc and Sola (2003), Evans (2003)]. This aspect lead us to propose dynamic term structure models (DTSMs) where the yield curve is driven by an univariate or multivariate factor (x_t) which is function of its p most recent lagged values and for which all the coefficients depend on a latent J-states non homogeneous Markov Chain (s_t) describing different regimes in the economy. The

[3]The Affine family of DTSMs is characterized by the fact that the zero-coupon bond yields are affine functions of Markovian state variables, and it gives closed-form expressions for zero-coupon bond prices which greatly facilitates pricing and econometric implementation [see Duffie and Kan (1996), Duffie, Filipovic and Schachermayer (2003)]. Observe that the Affine Term Structure family is much larger that it has been considered in the literature : indeed, it has been observed recently that the family of Quadratic Term Structure Models (QTSMs) is a special case of the Affine class obtained by stacking the factor values and their squares [see Gourieroux and Sufana (2003), Cheng and Scaillet (2004)].

factor (x_t) is considered as an exogenous variable or an endogenous variable: in the second case the factor is a vector of several yields.

We consider an exponential-affine SDF with time-varying and regime-dependent risk correction coefficients; consequently, in our models, both factor risk and regime-shift risk are priced and investors are assumed to observe the current and past values of the factor and of the regime-indicator variable.

At the same time, we want to maintain the tractability of affine models, that is, we want to obtain explicit or quasi explicit formula for zero-coupon bond and interest rate derivative prices. This result is achieved by matching the historical distribution and the SDF in order to get a Car risk-neutral (pricing) dynamics[4]. We develop the Switching Autoregressive Normal (SAN) Term Structure model of order p and the Switching Autoregressive Gamma[5] (SAG) Term Structure model of order p, and in both cases we propose multifactor generalizations.

Even if the Gaussian family of models does not guarantee the positiveness of the yields for every time to maturity [see, among the others, Vasicek (1977), Dai and Singleton (2000), Bekaert and Grenadier (2001), Ang and Bekaert (2002), Ang and Piazzesi (2003)], we study the Switching Autoregressive Normal model, in our SDF framework, because it extends many standard models, like the ones just mentioned above and the more recent ones like Dai, Singleton and Yang (2003). Indeed, the historical and risk-neutral dynamics of (x_t) depends from several of their lagged values and of the regime-indicator variable; in addition, we are able to derive formulas, for the yield curve and for the price of derivatives, with simple analytical or quasi explicit representations.

The second kind of models we propose in the paper, based on the Switching Autoregressive Gamma process of order p, [that is, a Regime-Switching positive AR(p) process with a martingale difference error], implies the positiveness of the yields for each time to maturity, and regardless of an exogenous or endogenous specification for the factor (x_t). Moreover, the ARG(p) assumption gives the possibility to replicate complex nonlinear dynamics and

[4]A Car process is a Markovian process with an exponential-affine conditional Laplace transform. An important difference between the discrete-time and continuous-time affine framework is that all continuous-time affine processes sampled at discrete points are Car, while there exists a large number of Car processes without a continuous time counterpart [see Darolles, Gourieroux, Jasiak (2002) and Duffie, Filipovic and Schachermayer (2003) for details]

[5]The Autoregressive Gamma (ARG) process is a Car process, and the ARG(1) specification is the discrete-time counterpart of the Cox-Ingersoll-Ross process [Cox-Ingersoll-Ross (1985)].

provides explicit or tractable formulas for zero-coupon bond and derivative prices. In a related study, Bansal and Zhou (2002) propose an (approximate) discrete-time Cox-Ingersoll-Ross DTSM with regime shifts. We extend their framework, by means of the exact discrete-time equivalent of the CIR process generalized to an autoregressive order p larger than one [the ARG(p) process], allowing for a non homogeneous historical transition matrix for (s_t) [in Bansal and Zhou (2002) (s_t) is an homogeneous Markov chain], and pricing the regime-shift risk [in Bansal and Zhou (2002), the risk correction coefficient for regime-switching is assumed equal to zero].

2 LAPLACE TRANSFORMS, CAR(p) PROCESSES AND SWITCHING REGIMES

It is now well documented [see e.g. Darolles, Gourieroux and Jasiak (2003), Gourieroux and Monfort (2002), Gourieroux, Monfort and Polimenis (2002, 2003), Polimenis (2001)] that the Laplace transform (or moment generating function) is a very convenient mathematical tool in many financial domains. It is, in particular, a crucial notion in the theory of Car(p) processes [see Darolles, Gourieroux and Jasiak (2003) for details].

2.1 Definition of a Car(p) process

Definition 1 [Car(p) process]: A n-dimensional process $\tilde{x} = (\tilde{x}_t,\, t \geq 0)$ is a compound autoregressive process of order p [Car(p)] if the distribution of $\tilde{x}_{t+1}$ given the past values $\underline{\tilde{x}_t} = (\tilde{x}_t, \tilde{x}_{t-1}, \ldots)$ admits a real Laplace transform of the following type:

$$\begin{aligned} & E\left[\exp(u'\tilde{x}_{t+1}) \,|\, \underline{\tilde{x}_t}\right] \\ & = E_t[\exp(u'\tilde{x}_{t+1})] \\ & = \exp\left[\tilde{a}_1(u)'\tilde{x}_t + \ldots + \tilde{a}_p(u)'\tilde{x}_{t+1-p} + \tilde{b}(u)\right], \quad u \in \mathbb{R}^n, \end{aligned} \tag{1}$$

where $a_i(u)$, $i \in \{1, \ldots, p\}$, and $b(u)$ are nonlinear functions, and where $a_p(u) \neq 0$, $\forall\, u \in \mathbb{R}^n$. The existence of this Laplace transform in a neighborhood of $u = 0$, implies that all the conditional moments exist, and that the conditional expectations and variance-covariance matrices (and all conditional cumulants) are affine functions of $(\tilde{x}'_t, \tilde{x}'_{t-1}, \ldots, \tilde{x}'_{t+1-p})$.

2.2 Univariate Index-Car(p) process

An important class of Car(p) processes are the Index-Car(p) processes, which are built from a Car(1) process. In this section we consider a univariate process x_t and the multivariate case will be considered in sections 2.6 and 2.7.

Definition 2 [Univariate Index-Car(p) process]: Let $\exp[a(u)y_t + b(u)]$ be the conditional Laplace transform of a univariate Car(1) process, x_{t+1} admitting a conditional Laplace transform defined by:

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[a(u)(\beta_1 x_t + \ldots + \beta_p x_{t+1-p}) + b(u)\right], \quad u \in \mathbb{R}, \tag{2}$$

is called an Univariate Index-Car(p) process.

Note that, if y_t is a positive process and if the parameters $\beta_1, \ldots, \beta_p$ are positive, the process x_t will be positive.

Using the notation $\beta = (\beta_1, \ldots, \beta_p)'$ and $X_t = (x_t, x_{t-1}, \ldots, x_{t+1-p})'$, the Laplace transform (2) can be written as:

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[a(u)\beta' X_t + b(u)\right] . \tag{3}$$

2.3 Examples of Univariate Index-Car(p) processes

a. Gaussian model

If y_t is a Gaussian AR(1) process defined by:

$$y_{t+1} = \nu + \rho y_t + \varepsilon_{t+1}$$

where ε_{t+1} is a gaussian white noise distributed as $\mathcal{N}(0, \sigma^2)$, the conditional Laplace transform of y_{t+1} given $\underline{y_t}$ is:

$$E\left[\exp(uy_{t+1}) \,|\, \underline{y_t}\right] = \exp\left[u\rho y_t + u\nu + \tfrac{\sigma^2}{2}u^2\right] .$$

The process is Car(1) with $a(u) = u\rho$ and $b(u) = u\nu + \frac{\sigma^2}{2}u^2$. The associated Index-Car(p) process has a conditional Laplace transform defined by:

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[u\rho(\beta_1 x_t + \ldots + \beta_p x_{t+1-p}) + u\nu + \tfrac{\sigma^2}{2}u^2\right] ;$$

so, using the notation $\varphi_i = \rho\beta_i$, we have that x_{t+1} is the Gaussian AR(p) process defined by:

$$x_{t+1} = \nu \, + \, \varphi_1 \, x_t \, + \, \ldots \, + \, \varphi_p \, x_{t+1-p} \, + \, \varepsilon_{t+1} \tag{4}$$

and its conditional Laplace transform becomes:

$$E\left[\exp(ux_{t+1}) \,|\, \underline{x_t}\right] = \exp\left[u\varphi' X_t + u\nu + \tfrac{\sigma^2}{2}u^2\right] , \tag{5}$$

where $\varphi = (\varphi_1, \ldots, \varphi_p)'$.

b. Gamma model

Let us now consider an autoregressive gamma of order one [ARG(1)] process y_t. The conditional Laplace transform is [see Gourieroux and Jasiak (2005) for details]:

$$E\left[\exp(uy_{t+1}) \,|\, \underline{y_t}\right] = \exp\left[\tfrac{\rho u}{1-u\mu}\, y_t - \nu \log(1 - u\mu)\right] , \quad \rho > 0 \, , \mu > 0 \, , \nu > 0 \, ,$$

and it is well known that, given y_t, y_{t+1} can be obtained by first drawing a latent variable U_{t+1} in the Poisson distribution $\mathcal{P}(\frac{\rho y_t}{\mu})$ and, then, drawing $\frac{y_{t+1}}{\mu}$ in the gamma distribution $\gamma(\nu+U_{t+1})$. The process y_{t+1} is positive and the associated Index-Car(p) process x_{t+1} is also positive. The conditional Laplace transform of this process is:

$$E\left[\exp(ux_{t+1})\,|\,\underline{x_t}\right] = \exp\left[\tfrac{\rho u}{1-u\mu}(\beta_1 x_t + \ldots + \beta_p x_{t+1-p}) - \nu\log(1-u\mu)\right],$$

with $\beta_i \geq 0$, for $i \in \{1,\ldots,p\}$, or using the same notation as above:

$$E\left[\exp(ux_{t+1})\,|\,\underline{x_t}\right] = \exp\left[\tfrac{u}{1-u\mu}\varphi' X_t - \nu\log(1-u\mu)\right]. \tag{6}$$

Similarly, given X_t, x_{t+1} can be obtained by drawing U_{t+1} in $\mathcal{P}(\frac{\varphi' X_t}{\mu})$ and $\frac{x_{t+1}}{\mu}$ in $\gamma(\nu+U_{t+1})$. It easily seen that the conditional mean and variance of x_{t+1}, given $\underline{x_t}$, are respectively given by $\nu\mu + \varphi' X_t$ and $\nu\mu^2 + 2\mu\varphi' X_t$; so, the process x_{t+1} has the weak AR(p) representation:

$$x_{t+1} = \nu\mu + \varphi' X_t + \varepsilon_{t+1}, \tag{7}$$

where ε_{t+1} is a conditionally heteroscedastic martingale difference, whose conditional variance is $\nu\mu^2 + 2\mu\varphi' X_t$; the process is stationary if and only if $\varphi' e < 1$ [where $e = (1,\ldots,1) \in \mathbb{R}^p$] and, in this case, the processes x_{t+1} and ε_{t+1} have finite unconditional variance given by $\nu\mu^2 + 2\nu\mu^2\frac{\varphi' e}{1-\varphi' e}$; the unconditional mean of x_{t+1} is given by $\frac{\nu\mu}{1-\varphi' e}$.

2.4 Univariate Switching regimes Car(p) process

Let us first consider a J-states homogeneous Markov Chain z_{t+1}, which can take the values $e_j \in \mathbb{R}^J$, $j \in \{1,\ldots,J\}$, where e_j is the j^{th} element of the $(J \times J)$ identity matrix. The transition probability, from state e_i to state e_j is $\pi(e_i, e_j) = Pr(z_{t+1} = e_j \,|\, z_t = e_i)$. It is first worth noting that z_{t+1} is a Car(1) process.

Proposition 1 : The Markov chain process z_{t+1} is a Car(1) process with a conditional Laplace transform given by:

$$E[\exp(v' z_{t+1})|\,\underline{z_t}] = \exp(a_z(v,\pi)' z_t), \tag{8}$$

where

$$a_z(v,\pi) = \left[\log\left(\sum_{j=1}^{J}\exp(v' e_j)\pi(e_1, e_j)\right), \ldots, \log\left(\sum_{j=1}^{J}\exp(v' e_j)\pi(e_J, e_j)\right)\right]'.$$

[Proof : straightforward.]

Let us now consider a univariate Index-Car(p) process with a conditional Laplace transform given by $\exp\left[a(u)\beta' X_t + b(u)\right]$, and let us assume the $b(u)$ can be written:

$$\begin{aligned} b(u) &= \tilde{b}(u)'\lambda \quad \text{where} \\ \tilde{b}(u) &= (b_1(u), \dots, b_m(u))' \text{ and } \lambda = (\lambda_1, \dots, \lambda_m)' \,. \end{aligned} \tag{9}$$

We are now going to generalize this model by assuming that the parameters λ_i are stochastic and linear function of $Z_t = (z_t', \dots, z_{t-p}')'$. More precisely, we assume that the conditional distribution of x_{t+1} given $\underline{x_t}$ and $\underline{z_{t+1}}$ has a Laplace transform given by:

$$E[\exp(u x_{t+1}) | \, \underline{x_t} \,, \underline{z_{t+1}}] = \exp\left[a(u)\beta' X_t + \tilde{b}(u)'\Lambda Z_t\right] \,, \tag{10}$$

where Λ is a $[m, (p+1)J]$ matrix. Note that we assume no instantaneous causality between x_{t+1} and z_{t+1} and we admit one more lag in Z_t that in X_t; if the process z_t is not observed the no instantaneous causality assumption is not really important at this stage since we could rename z_t as z_{t+1}, however it will be useful at the pricing level in order to obtain simple pricing procedures [Dai, Singleton and Yang (2003) also make this kind of assumption]. The joint process $(x_{t+1}, z_{t+1}')'$ is easily seen to be a Car($p+1$) process.

Proposition 2 : The conditional Laplace transform of $(x_{t+1}, z_{t+1}')'$ given $\underline{x_t} \,, \underline{z_t}$ has the following form:

$$\begin{aligned} & E\left[\exp(u x_{t+1} + v' z_{t+1}) \,|\, \underline{z_t}, \, \underline{x_t}\,\right] \\ = \quad & \exp\left\{a(u)\beta' X_t + \left[e_1' \otimes a_z(v, \pi)' + \tilde{b}(u)'\Lambda\right] Z_t\right\} \,, \end{aligned} \tag{11}$$

where e_1 is the first component of the canonical basis in $\mathbb{R}^{p+1}$, and where $\otimes$ denotes the Kronecker product.

[Proof : straightforward.]

2.5 Examples of Univariate Switching regimes Car(p) processes

a. Gaussian case

Let us start from the AR(p) model (4). Its conditional Laplace transform is given by (5):

$$E\left[\exp(ux_{t+1})\,|\,\underline{x_t}\right] = \exp\left[u\varphi' X_t + u\nu + \frac{\sigma^2}{2}u^2\right] ,$$

and the function $b(u)$ has the form (9) with $\tilde{b}(u)' = \left(u, \frac{u^2}{2}\right)$ and $\lambda' = (\nu, \sigma^2)$.

If λ is replaced by ΛZ_t, the joint process $(x_{t+1}, z'_{t+1})'$ is Car$(p+1)$ with a conditional Laplace transform given by:

$$\begin{aligned} & E\left[\exp(ux_{t+1} + v' z_{t+1})\,|\,\underline{z_t},\, \underline{x_t}\right] \\ = \;& \exp\left[u\varphi' X_t + \left(u, \frac{u^2}{2}\right)\Lambda Z_t + a_z(v,\pi) z_t\right] . \end{aligned} \tag{12}$$

More precisely, the dynamics is given by [using the notation $\Lambda = \begin{pmatrix}\lambda'_1 \\ \lambda'_2\end{pmatrix}$]:

$$x_{t+1} = \lambda'_1 Z_t + \varphi' X_t + (\lambda'_2 Z_t)^{1/2}\varepsilon_{t+1} , \tag{13}$$

where ε_{t+1} is a gaussian white noise distributed as $\mathcal{N}(0,\sigma^2)$, $Z_t = (z'_t, \ldots, z'_{t-p})'$ and z_t is a Markov chain such that $Pr(z_{t+1} = e_j\,|\,z_t = e_i) = \pi(e_i,\, e_j)$.

In particular, let us consider the case:

$$\Lambda = \begin{bmatrix} (1, -\varphi_1, \ldots, -\varphi_p) \otimes \nu^{*'} \\ e'_1 \otimes \sigma^{*2'} \end{bmatrix} \tag{14}$$

and $\nu^{*'} = (\nu^*_1, \ldots, \nu^*_J)$, $\sigma^{*2'} = (\sigma^{*2}_1, \ldots, \sigma^{*2}_J)$, the conditional distribution of x_{t+1} given $\underline{x_t}$ and $\underline{z_{t+1}}$ is the one corresponding to the switching AR(p) model defined by:

$$x_{t+1} - \nu^{*'} z_t = \varphi_1\,(x_t - \nu^{*'} z_{t-1}) \,+\, \ldots \,+\, \varphi_p\,(x_{t+1-p} - \nu^{*'} z_{t-p}) \,+\, (\sigma^{*'} z_t)\varepsilon_{t+1} . \tag{15}$$

b. Gamma case

Let us now start from the ARG(p) process associated with the conditional Laplace transform (6):

$$E\left[\exp(ux_{t+1})\,|\,\underline{x_t}\right] = \exp\left[\frac{u}{1-u\mu}\varphi' X_t - \nu\log(1-u\mu)\right] .$$

Here we have $\tilde{b}(u) = -\log(1-u\mu)$ and $\lambda = \nu$. If ν is replaced by ΛZ_t, where $\Lambda Z_t > 0$, the process x_t has a weak AR(p) representation given by:

$$x_{t+1} = \mu \Lambda Z_t + \varphi_1 \, x_t \; + \; \ldots \; + \; \varphi_p \, x_{t+1-p} \; + \; \zeta_{t+1} \, , \tag{16}$$

where ζ_{t+1} is a conditionally heteroscedastic martingale difference. For instance, we can take

$$\Lambda = e_1' \otimes \frac{\tilde{\nu}\,'}{\mu} \tag{17}$$

where $\tilde{\nu}\,' = (\tilde{\nu}_1, \ldots, \tilde{\nu}_J)$, $\tilde{\nu}_j \geq 0$. We have $\Lambda Z_t = \frac{\tilde{\nu}\,'}{\mu} z_t$ and, conditionally to the process z_t, the process x_t has a weak AR(p) representation given by:

$$x_{t+1} = \tilde{\nu}\,' z_t + \varphi_1 \, x_t \; + \; \ldots \; + \; \varphi_p \, x_{t+1-p} \; + \; \zeta_{t+1} \, . \tag{18}$$

It is also possible to consider a Λ of the form $(1, -\varphi_1, \ldots, -\varphi_p) \otimes \frac{\tilde{\nu}\,'}{\mu}$ if $\min(\tilde{\nu}_i) > \max(\tilde{\nu}_i) \sum_{i=1}^{J} \varphi_j$, since in this case $\Lambda Z_t = \frac{1}{\mu}\left(\tilde{\nu}\,' z_t - \sum_{i=1}^{J} \varphi_j \, \tilde{\nu}\,' z_{t-i}\right)$ ≥ 0. The weak conditional AR(p) process is then given by:

$$x_{t+1} - \tilde{\nu}\,' z_t = \varphi_1 \,(x_t - \tilde{\nu}\,' z_{t-1}) \; + \; \ldots \; + \; \varphi_p \,(x_{t+1-p} - \tilde{\nu}\,' z_{t-p}) \; + \; \zeta_{t+1} \, . \tag{19}$$

2.6 Specification of multivariate Car(1) processes

In order to have simple notations we will consider the bivariate case, but all the results are easily extended to the general case. A bivariate Car(1) process $y_t = (y_{1,t}, y_{2,t})'$ will be defined in a recursive way. We consider two univariate exponential affine Laplace transform

$$\begin{aligned} &\exp\left[a_1(u_1) w_{1,t} + b_1(u_1)\right] \, , \\ \text{and} \qquad &\exp\left[a_2(u_2) w_{2,t} + b_2(u_2)\right] \, . \end{aligned} \tag{20}$$

Then, we assume that the conditional distribution of $y_{1,t+1}$ given $(y_{2,t+1}, \underline{y_{1,t}}, \underline{y_{2,t}})$ has a Laplace transform given by

$$\begin{aligned} &E_t[\exp(u_1 y_{1,t+1}) \,|\, y_{2,t+1}, \underline{y_{1t}}, \underline{y_{2t}}] \\ &\quad = \exp\left[a_1(u_1)(\beta_o y_{2,t+1} + \beta_{11} y_{1,t} + \beta_{12} y_{2,t}) + b_1(u_1)\right] \end{aligned} \tag{21}$$

and the conditional distribution of $y_{2,t+1}$, given $(\underline{y_{1,t}}, \underline{y_{2,t}})$, has a Laplace transform given by

$$E_t[\exp(u_2 y_{2,t+1}) \,|\, \underline{y_{1,t}}, \underline{y_{2,t}}] = \exp\left[a_2(u_2)(\beta_{21} y_{1,t} + \beta_{22} y_{2,t}) + b_2(u_2)\right] . \tag{22}$$

Note that, if the Laplace transforms (20) correspond to positive variables and if the parameters $\beta_o, \beta_{11}, \beta_{12}, \beta_{21}, \beta_{22}$ are positive the bivariate process y_t has positive components. Moreover, the joint conditional distribution of y_{t+1} given $\underline{y_t}$ has a Laplace transform given by:

$$\begin{aligned}
& E[\exp(u_1 y_{1,t+1} + u_2 y_{2,t+1}) | \underline{y_{1,t}}, \underline{y_{2,t}}] \\
= \; & E\left[\exp(u_2 y_{2,t+1}) E\left(\exp(u_1 y_{1,t+1}) | \underline{y_{1,t}}, \underline{y_{2,t+1}}\right) \,|\, \underline{y_{1,t}}, \underline{y_{2,t}}\right] \\
= \; & \exp\left[a_1(u_1)(\beta_{11} y_{1,t} + \beta_{12} y_{2,t}) + b_1(u_1)\right] E_t\left[(u_2 + a_1(u_1)\beta_o) y_{2,t+1} \,|\, \underline{y_{1,t}}, \underline{y_{2,t}}\right] \\
= \; & \exp\left[a_1(u_1)(\beta_{11} y_{1,t} + \beta_{12} y_{2,t}) + b_1(u_1)\right. \\
& \qquad \left. + a_2(u_2 + a_1(u_1)\beta_o)(\beta_{21} y_{1,t} + \beta_{22} y_{2,t}) + b_2(u_2 + a_1(u_1)\beta_o)\right] \\
= \; & \exp\left\{[a_1(u_1)\beta_{11} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{21}] y_{1,t}\right. \\
& \qquad \left. + [a_1(u_1)\beta_{12} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{22}] y_{2,t} + b_1(u_1) + b_2(u_2 + a_1(u_1)\beta_o)\right\} .
\end{aligned} \tag{23}$$

So, we have the following proposition.

Proposition 3 : The bivariate process y_t defined by the conditional dynamics (21), (22) is a bivariate Car(1) process with a conditional Laplace transform given by (23).

2.7 Specification of multivariate Index-Car(p) processes

We consider a bivariate process $\tilde{x}_t = (x_{1,t}, x_{2,t})'$ and we introduce the notations : $X_{1t} = (x_{1,t}, \ldots, x_{1,t+1-p})'$, $X_{2t} = (x_{2,t}, \ldots, x_{2,t+1-p})'$. Given the univariate Laplace transforms like (20), a bivariate Index-Car(p) is defined in the following way.

Definition 3 : A bivariate Index-Car(p) dynamics is defined by the conditional Laplace transforms:

$$E_t[\exp(u_1 x_{1,t+1}) \,|\, x_{2,t+1}, \underline{x_{1,t}}, \underline{x_{2,t}}]$$

$$= \exp\left[a_1(u_1)(\beta_o x_{2,t+1} + \beta'_{11} X_{1t} + \beta'_{12} X_{2t}) + b_1(u_1)\right] ,$$

$$E_t[\exp(u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}] = \exp\left[a_2(u_2)(\beta'_{21} X_{1t} + \beta'_{22} X_{2t}) + b_2(u_2)\right] , \tag{24}$$

where the β_{ij} are p-vectors. It is easily seen that the process $\tilde{x}_t$ is a Car(p) process with a conditional Laplace transform given by (23) in which $y_{1,t}$ is replaced by X_{1t} and $y_{2,t}$ by X_{2t} and the β_{ij} by the β'_{ij}, i.e.

$$\begin{aligned} & E\left[\exp(u'\tilde{x}_{t+1}) \,|\, \underline{\tilde{x}_t}\right] \\ = \quad & \exp\{[a_1(u_1)\beta_{11} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{21}]' X_{1t} \\ & +[a_1(u_1)\beta_{12} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{22}]' X_{2t} \\ & + b_1(u_1) + b_2(u_2 + a_1(u_1)\beta_o)\} \,. \end{aligned} \tag{25}$$

From the properties of Car(p) processes we get a representation of the form:

$$\begin{cases} x_{1,t+1} = \alpha_1 + \alpha_o x_{2,t+1} + \alpha'_{11} X_{1t} + \alpha'_{12} X_{2t} + \varepsilon_{1,t+1} \\ x_{2,t+1} = \alpha_2 + \alpha'_{21} X_{1t} + \alpha'_{22} X_{2t} + \varepsilon_{2,t+1} \end{cases} \tag{26}$$

where the errors terms satisfy :

$$\begin{aligned} E[\varepsilon_{1,t+1} \,|\, x_{2,t+1}, \underline{\tilde{x}_t}\,] &= 0 \\ E[\varepsilon_{2,t+1} \,|\, \underline{\tilde{x}_t}] &= 0\,; \end{aligned} \tag{27}$$

in particular, we get

$$\begin{aligned} E[\varepsilon_{1,t+1} \,|\, \underline{\tilde{x}_t}\,] &= 0 \\ E[\varepsilon_{2,t+1} \,|\, \underline{\tilde{x}_t}] &= 0 \\ Cov(\varepsilon_{1,t+1}, \varepsilon_{2,t+1}) &= E(\varepsilon_{1,t+1}\varepsilon_{2,t+1} \,|\, \underline{\tilde{x}_t}) \\ &= E\left[\varepsilon_{2,t+1} E(\varepsilon_{1,t+1} \mid x_{2,t+1}, \underline{\tilde{x}_t}) \,|\, \underline{\tilde{x}_t}\right] \\ &= 0\,. \end{aligned} \tag{28}$$

So, the error terms are non correlated, conditionally heteroscedastic, martingale differences. In particular, in the stationary case, $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are uncorrelated weak white noises and (26) is a weak recursive VAR(p) representation of the process $\tilde{x}_t$.

In the rest of the paper we will consider two important particular cases.

a) Normal VAR(p) or VARN(p) processes

In this case the conditional distributions defined by (20) are gaussian, with affine expectations and fixed variances. In other words:

$$\begin{aligned} a_1(u_1) = \rho_1 u_1 \, , \, b_1(u_1) = \nu_1 u_1 + \tfrac{\sigma_1^2 u_1^2}{2} \\ a_2(u_2) = \rho_2 u_2 \, , \, b_2(u_2) = \nu_2 u_2 + \tfrac{\sigma_2^2 u_2^2}{2} \, . \end{aligned} \tag{29}$$

Using the notations $\varphi_o = \rho_1\beta_o$, $\varphi_{11} = \rho_1\beta_{11}$, $\varphi_{12} = \rho_1\beta_{12}$, $\varphi_{21} = \rho_2\beta_{21}$, $\varphi_{22} = \rho_2\beta_{22}$, we have the following strong VAR(p) recursive representation for the process $\tilde{x}_t = (x_{1,t}, x_{2,t})'$:

$$\left\{ \begin{array}{lcl} x_{1,t+1} & = & \nu_1 + \varphi_o x_{2,t+1} + \varphi_{11}' X_{1t} + \varphi_{12}' X_{2t} + \sigma_1 \eta_{1,t+1} \\ x_{2,t+1} & = & \nu_2 + \varphi_{21}' X_{1t} + \varphi_{22}' X_{2t} + \sigma_2 \eta_{2,t+1} \, , \end{array} \right. \tag{30}$$

where $\eta_t = (\eta_{1,t}, \eta_{2,t})'$ is a bivariate gaussian white noise distributed as $\mathcal{N}(0, I_2)$, where I_2 denotes the (2×2) identity matrix.

b) Gamma VAR(p) or VARG(p) processes

In this case we have:

$$\begin{aligned} a_1(u_1) = \tfrac{\rho_1 u_1}{1 - u_1 \mu_1} \, , \, b_1(u_1) = -\nu_1 \log(1 - u_1 \mu_1) \\ a_2(u_2) = \tfrac{\rho_2 u_2}{1 - u_2 \mu_2} \, , \, b_2(u_2) = -\nu_2 \log(1 - u_2 \mu_2) \, , \end{aligned} \tag{31}$$

and the process $\tilde{x}_t = (x_{1,t}, x_{2,t})'$ has the following weak VAR(p) representation (using the same notation as above, and where all the parameters are positive):

$$\left\{ \begin{array}{lcl} x_{1,t+1} & = & \nu_1 \mu_1 + \varphi_o x_{2,t+1} + \varphi_{11}' X_{1t} + \varphi_{12}' X_{2t} + \xi_{1,t+1} \\ x_{2,t+1} & = & \nu_2 \mu_2 + \varphi_{21}' X_{1t} + \varphi_{22}' X_{2t} + \xi_{2,t+1} \, , \end{array} \right. \tag{32}$$

where $\xi_{1,t}$ and $\xi_{2,t}$ are non correlated, conditionally heteroscedastic, martingale differences. The conditional variances of $\xi_{1,t+1}$ and $\xi_{2,t+1}$ are given by:

$$
\begin{aligned}
V[\xi_{1,t+1} \,|\, \underline{\tilde{x_t}}\,] &= \nu_1\mu_1^2 + 2\mu_1[\varphi_o(\nu_2\mu_2 + \varphi_{21}' X_{1t} + \varphi_{22}' X_{2t}) \\
&\qquad + \varphi_{11}' X_{1t} + \varphi_{12}' X_{2t}] \qquad (33) \\
V[\xi_{2,t+1} \,|\, \underline{\tilde{x_t}}] &= \nu_2\mu_2^2 + 2\mu_2(\varphi_{21}' X_{1t} + \varphi_{22}' X_{2t})\,.
\end{aligned}
$$

It is important to stress that the components of this VARG(p) process are positive.

2.8 Switching Multivariate Index-Car processes

Switching regimes can be introduced in a multivariate Index-Car(p) model using a method extending the one retained in the univariate case. If we assume that the functions $b_1(u_1)$, $b_2(u_2)$ appearing in definition 3 can be written, respectively, as $\tilde{b_1}(u_1)'\lambda_1$ and $\tilde{b_2}(u_2)'\lambda_2$, and if we replace λ_1 and λ_2, respectively by $\Lambda_1 Z_t$ and $\Lambda_2 Z_t$, we obtain the following conditional Laplace transform for the distribution of $(x_{1,t+1}, x_{2,t+1}, z_{t+1})$ given $(\underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t})$:

$$
\begin{aligned}
& E[\exp(u_1 x_{1,t+1} + u_2 x_{2,t+1} + v' z_{t+1}) |\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}] \\
= \;& \exp\{[a_1(u_1)\beta_{11} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{21}]' X_{1t} \\
& \qquad + [a_1(u_1)\beta_{12} + a_2(u_2 + a_1(u_1)\beta_o)\beta_{22}]' X_{2t} \\
& \qquad + [\, e_1' \otimes a_z(v,\pi)' + \tilde{b_1}(u_1)'\Lambda_1 + \tilde{b_2}(u_2 + a_1(u_1)\beta_o)'\Lambda_2] Z_t \Big\}\,,
\end{aligned} \qquad (34)
$$

where $a_z(v,\pi)$ is given in proposition 1. So we obtain a multivariate Car(p+1) process.

Proposition 4 : The Laplace transform of $(x_{1,t+1}, x_{2,t+1}, z_{t+1})$, conditionally to $(\underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t})$, has the form given in (34) and the process $(x_{1,t}, x_{2,t}, z_t)$ is Car($p+1$).

2.9 Examples of Switching Multivariate Index-Car processes

a. Gaussian case

Taking

$$a_1(u_1) = \rho_1 u_1,\ b_1(u_1) = \nu_1 u_1 + \tfrac{\sigma_1^2}{2} u_1^2,\ \tilde{b_1}(u_1)' = \left(u_1, \tfrac{u_1^2}{2}\right),$$

$$a_2(u_2) = \rho_2 u_2,\ b_2(u_2) = \nu_2 u_2 + \tfrac{\sigma_2^2}{2} u_2^2,\ \tilde{b_2}(u_2)' = \left(u_2, \tfrac{u_2^2}{2}\right),$$

$$\Lambda_1 = \begin{pmatrix} \lambda'_{11} \\ \lambda'_{12} \end{pmatrix},\ \Lambda_2 = \begin{pmatrix} \lambda'_{21} \\ \lambda'_{22} \end{pmatrix},$$

and denoting $\varphi_o = \rho_1 \beta_o,\ \varphi_{11} = \rho_1 \beta_{11},\ \varphi_{12} = \rho_1 \beta_{12},\ \varphi_{21} = \rho_2 \beta_{21},\ \varphi_{22} = \rho_2 \beta_{22}$, we obtain the Switching VARN(p) model:

$$\left\{ \begin{array}{lcl} x_{1,t+1} & = & \lambda'_{11} Z_t + \varphi_o x_{2,t+1} + \varphi'_{11} X_{1t} + \varphi'_{12} X_{2t} + (\lambda'_{12} Z_t)^{1/2} \eta_{1,t+1} \\ x_{2,t+1} & = & \lambda'_{21} Z_t + \varphi'_{21} X_{1t} + \varphi'_{22} X_{2t} + (\lambda'_{22} Z_t)^{1/2} \eta_{2,t+1}\,, \end{array} \right. \tag{35}$$

where $\eta_t = (\eta_{1,t}, \eta_{2,t})'$ is a gaussian white noise distributed as $\mathcal{N}(0, I_2)$, $Z_t = (z'_t, \ldots, z'_{t-p})'$, and where z_t is a homogeneous J-states Markov chain with transition probability $\pi(e_i, e_j)$. Note that (35) can also be written as:

$$\left\{ \begin{array}{lcl} x_{1,t+1} & = & \tilde{\lambda}'_{11} Z_t + \tilde{\varphi}'_{11} X_{1t} + \tilde{\varphi}'_{12} X_{2t} + \varphi_o (\lambda'_{22} Z_t)^{1/2} \eta_{2,t+1} + (\lambda'_{12} Z_t)^{1/2} \eta_{1,t+1} \\ x_{2,t+1} & = & \lambda'_{21} Z_t + \varphi'_{21} X_{1t} + \varphi'_{22} X_{2t} + (\lambda'_{22} Z_t)^{1/2} \eta_{2,t+1}\,, \end{array} \right. \tag{36}$$

with $\tilde{\lambda}_{11} = \lambda_{11} + \varphi_o \lambda_{21},\ \tilde{\varphi}_{11} = \varphi_{11} + \varphi_o \varphi_{21},\ \tilde{\varphi}_{12} = \varphi_{12} + \varphi_o \varphi_{22}$ or, with obvious notations

$$\tilde{x}_{t+1} = \tilde{\lambda}' Z_t + \tilde{\Phi}' \tilde{X}_t + \begin{bmatrix} (\lambda'_{12} Z_t)^{1/2} & \varphi_o (\lambda'_{22} Z_t)^{1/2} \\ 0 & (\lambda'_{22} Z_t)^{1/2} \end{bmatrix} \eta_{t+1}\,. \tag{37}$$

b. Gamma case

If we take

$$a_1(u_1) = \tfrac{\rho_1 u_1}{1 - u_1 \mu_1},\ b_1(u_1) = -\nu_1 \log(1 - u_1 \mu_1),\ \tilde{b_1}(u_1) = \log(1 - u_1 \mu_1),$$

$$a_2(u_2) = \tfrac{\rho_2 u_2}{1 - u_2 \mu_2},\ b_2(u_2) = -\nu_2 \log(1 - u_2 \mu_2),\ \tilde{b_2}(u_2) = \log(1 - u_2 \mu_2)\,,$$

we obtain the positive Switching VARG(p) model

$$\left\{\begin{array}{rcl} x_{1,t+1} & = & \mu_1\Lambda_1' Z_t + \varphi_o x_{2,t+1} + \varphi_{11}' X_{1t} + \varphi_{12}' X_{2t} + \xi_{1,t+1} \\ x_{2,t+1} & = & \mu_2\Lambda_2' Z_t + \varphi_{21}' X_{1t} + \varphi_{22}' X_{2t} + \xi_{2,t+1}\,, \end{array}\right. \tag{38}$$

where $\xi_{1,t}$ and $\xi_{2,t}$ are non correlated, conditionally heteroscedastic, martingale differences, the conditional variances being respectively given by:

$$\begin{array}{rcl} V[\xi_{1,t+1}\,|\,\underline{\tilde{x_t}}\,] & = & \Lambda_1' Z_t \mu_1^2 + 2\mu_1[\varphi_o(\Lambda_2' Z_t \mu_2 + \varphi_{21}' X_{1t} + \varphi_{22}' X_{2t}) \\ & & + \varphi_{11}' X_{1t} + \varphi_{12}' X_{2t}] \qquad (39) \\ V[\xi_{2,t+1}\,|\,\underline{\tilde{x_t}}] & = & \Lambda_2' Z_t \mu_2^2 + 2\mu_2(\varphi_{21}' X_{1t} + \varphi_{22}' X_{2t})\,. \end{array}$$

3 SWITCHING AUTOREGRESSIVE NORMAL (SAN) TERM STRUCTURE MODEL OF ORDER p

We first consider the case of univariate exogenous factor; the endogenous case and the multivariate case will be discussed, respectively, in sections 3.7 and 3.8.

3.1 The historical dynamics

The first set of assumptions of a SAN(p) Term Structure Model deals with the historical dynamics. We assume that the historical dynamics of the exogenous factor x_t is given by

$$x_{t+1} = \nu(Z_t) + \varphi_1(Z_t)x_t + \ldots + \varphi_p(Z_t)x_{t+1-p} + \sigma(Z_t)\varepsilon_{t+1}\,, \tag{40}$$

where ε_{t+1} is a gaussian white noise with $\mathcal{N}(0,1)$ distribution, $Z_t = (z_t', \ldots, z_{t-p}')'$, and z_t is a J-states non-homogeneous Markov chain such that $P(z_{t+1} = e_j\,|\,z_t = e_i; \underline{x_t}) = \pi(e_i, e_j\,; X_t)$ (e_i is the i^{th} column of the identity matrix I_J). Equation (40) will be also written

$$x_{t+1} = \nu(Z_t) + \varphi(Z_t)' X_t + \sigma(Z_t)\varepsilon_{t+1}\,, \tag{41}$$

where $X_t = (x_t, \ldots, x_{t+1-p})'$, $\varphi(Z_t) = (\varphi_1(Z_t), \ldots, \varphi_p(Z_t))'$. This model can also be rewritten in the following vectorial form:

$$X_{t+1} = \Phi(Z_t)X_t + [\nu(Z_t) + \sigma(Z_t)\varepsilon_{t+1}]\,e_1 \tag{42}$$

where

$$\Phi(Z_t) = \begin{bmatrix} \varphi_1(Z_t) & \dots & \dots & \varphi_p(Z_t) \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}$$

is a $(p \times p)$-matrix, and where e_1 is the first column of the identity matrix I_p. Note that, since the coefficients φ_i are allowed to depend on Z_t and since the Markov chain z_t may not be homogeneous, the dynamics of (x_t, z_t) is not Car in general.

3.2 The Stochastic Discount Factor

The second element of a SAN(p) modeling is the SDF. We denote by $M_{t,t+1}$ the stochastic discount factor (SDF) between the date t and $t+1$ and in order to get time-varying risk-premia we specify it as an exponential affine function of the variables $(x_{t+1},\, z_{t+1})$ but with coefficients depending on the information at time t. More precisely we assume that:

$$\begin{aligned} M_{t,t+1} \quad = \quad & \exp\left[-c'X_t - d'Z_t + \Gamma(Z_t, X_t)\,\varepsilon_{t+1}\right. \\ & \left. -\tfrac{1}{2}\Gamma(Z_t, X_t)^2 - \delta(Z_t, X_t)' z_{t+1}\right] \,, \end{aligned} \tag{43}$$

where $\Gamma(Z_t, X_t) = \gamma(Z_t) + \tilde{\gamma}'(Z_t)X_t$. Observe that this specification extends to the multi-lag case the one proposed by Dai, Singleton, Yang (2003). It is well known that the existence of a positive stochastic discount factor is equivalent to the absence of arbitrage opportunity condition and that the price p_t at t of a payoff W_{t+1} at $t+1$ is given by:

$$\begin{aligned} p_t \quad &= \quad E[M_{t,t+1}W_{t+1} \,|\, I_t] \\ &= \quad E_t[M_{t,t+1}W_{t+1}]\,, \end{aligned}$$

where the information I_t, available for the investors at the date t, is given by $(\underline{x_t}, \underline{z_t})$. More generally, the price $p_{t,h}$ at t of an asset paying W_{t+h} at $t+h$ is:

$$p_{t,h} \quad = \quad E_t\left[M_{t,t+1} \cdot \ldots \cdot M_{t+h-1,t+h}W_{t+h}\right] \,.$$

Now, using the absence of arbitrage assumption for the short-term interest rate between t and $t+1$, denoted by r_{t+1} and known at t, we get:

$$\begin{aligned} \exp(-r_{t+1}) \quad &= \quad E_t\left(M_{t,t+1}\right) \\ &= \quad \exp\left[-c'X_t - d'Z_t\right] \times \textstyle\sum_{j=1}^{J} \pi\left(e_i, e_j\,; X_t\right) \exp\left[-\delta(Z_t, X_t)' e_j\right] \,, \end{aligned}$$

and assuming the normalisation condition:

$$\sum_{j=1}^{J} \pi\left(e_i, e_j\, ; X_t\right) \exp\left[-\delta(Z_t, X_t)' e_j\right] = 1 \quad \forall Z_t, X_t\,, \tag{44}$$

we obtain:

$$r_{t+1} = c' X_t + d' Z_t\,. \tag{45}$$

3.3 Risk premia

In this paper we will use the following definition of a risk premium.

Definition 4 : Let p_t the price of a given asset at time t. The risk premium of this asset between t and $t+1$ is $\omega_t = \log(E_t p_{t+1}) - \log p_t - r_{t+1}$.

Using this definition we obtain interpretations of the Γ and δ functions appearing in the SDF which are similar to that obtained by Dai, Singleton and Yang (2003). Let us first consider an asset providing the payoff $\exp(-\theta x_{t+1})$ at $t+1$; the price at t of this asset is

$$\begin{aligned}
p_t &= E_t[M_{t,t+1} \exp(-\theta x_{t+1})] \\
&= \exp\left[-r_{t+1} - \theta\nu(Z_t) - \theta\varphi(Z_t)' X_t - \tfrac{1}{2}\Gamma(X_t, Z_t)^2\right] \times \\
&\qquad E_t\left\{\exp\left[\left[\Gamma(X_t, Z_t) - \theta\sigma(Z_t)\right] \varepsilon_{t+1}\right]\right\} \\
&= \exp\left[-r_{t+1} - \theta\nu(Z_t) - \theta\varphi(Z_t)' X_t - \theta\Gamma(X_t, Z_t)\sigma(Z_t) + \tfrac{\theta^2}{2}\sigma^2(Z_t)\right]\,,
\end{aligned}$$

and

$$\begin{aligned}
E_t p_{t+1} &= E_t[\exp(-\theta x_{t+1})] \\
&= \exp\left[-\theta\nu(Z_t) - \theta\varphi(Z_t)' X_t\right] \times \\
&\qquad E_t\left\{\exp\left[\left[-\theta\sigma(Z_t)\right] \varepsilon_{t+1}\right]\right\} \\
&= \exp\left[-\theta\nu(Z_t) - \theta\varphi(Z_t)' X_t + \tfrac{\theta^2}{2}\sigma^2(Z_t)\right]\,.
\end{aligned}$$

Finally, the risk premium is:

$$\omega_t(\theta) = \theta\Gamma(X_t, Z_t)\sigma(Z_t)\,. \tag{46}$$

Therefore, θ, Γ and σ can be seen respectively as a risk sensitivity of the asset, a risk price and a risk measure.

Similarly, if we consider a digital asset providing one money unit at $t+1$ if $z_{t+1} = e_j$, we get:

$$\begin{aligned} p_t &= E_t[M_{t,t+1}\mathbb{I}_{(e_j)}(z_{t+1})] \\ &= \exp[-r_{t+1}]\exp[-\delta_j(X_t, Z_t)]\pi\left(z_t, e_j\,; X_t\right)\ , \end{aligned}$$

and

$$\begin{aligned} E_t p_{t+1} &= E_t[\mathbb{I}_{(e_j)}(z_{t+1})] \\ &= \pi\left(z_t, e_j\,; X_t\right)\ . \end{aligned}$$

Therefore, the risk premium is

$$\omega_t(\theta) = \delta_j(X_t, Z_t)\,, \tag{47}$$

and the j^{th} component of δ can be seen as the risk premium associated with the digital asset.

3.4 Risk-Neutral dynamics

The assumptions on the historical dynamics and on the SDF imply a risk-neutral dynamics. The probability density function of the one-period conditional risk-neutral probability with respect to the corresponding historical probability is $\frac{M_{t,t+1}}{E_t(M_{t,t+1})} = \exp(r_{t+1})M_{t,t+1}$. Note that using $E_t^{\mathbb{Q}}$ as the conditional expectation with respect to this risk-neutral distribution, the risk-premium ω_t can be written $\log(E_t p_{t+1}) - \log(E_t^{\mathbb{Q}} p_{t+1})$. The Laplace transform of the one-period conditional risk-neutral probability is:

$$\begin{aligned} & E_t^{\mathbb{Q}}[\exp(ux_{t+1} + v'z_{t+1})] \\ = \ & E_t\{\exp[\Gamma(X_t, Z_t)\,\varepsilon_{t+1} - \tfrac{1}{2}\Gamma(X_t, Z_t)^2 - \delta'(Z_t, X_t)z_{t+1} \\ & \qquad +u[\nu(Z_t) + \varphi(Z_t)'X_t + \sigma(Z_t)\varepsilon_{t+1}] + v'z_{t+1}]\} \\ = \ & \exp\left\{\, u[\,\varphi'(Z_t)X_t + \Gamma(X_t, Z_t)\sigma(Z_t)] + u\nu(Z_t) + \tfrac{1}{2}u^2\sigma(Z_t)^2\right\}\ \times \\ & \qquad \textstyle\sum_{j=1}^{J} \pi\left(z_t, e_j\,; X_t\right)\exp\left[(v - \delta(Z_t, X_t))'e_j\right] \\ = \ & \exp\left\{\, u[\,\varphi(Z_t) + \tilde{\gamma}(Z_t)\sigma(Z_t)]'X_t + u[\nu(Z_t) + \gamma(Z_t)\sigma(Z_t)] + \tfrac{1}{2}u^2\sigma(Z_t)^2\right\}\ \times \\ & \qquad \textstyle\sum_{j=1}^{J} \pi\left(z_t, e_j\,; X_t\right)\exp\left[(v - \delta(Z_t, X_t))'e_j\right]\ . \end{aligned} \tag{48}$$

Therefore, we get the following result.

Proposition 5 : The risk-neutral dynamics of the process (x_t, z_t) is given by:

$$x_{t+1} \overset{\mathbb{Q}}{=} \nu(Z_t) + \gamma(Z_t)\sigma(Z_t) + [\varphi(Z_t) + \tilde{\gamma}(Z_t)\sigma(Z_t)]' X_t + \sigma(Z_t)\xi_{t+1} \,, \tag{49}$$

where $\overset{\mathbb{Q}}{=}$ denotes the equality in distribution (associated to the probability $\mathbb{Q}$), ξ_{t+1} is (under $\mathbb{Q}$) a gaussian white noise with $\mathcal{N}(0,1)$ distribution, and where $Z_t = (z_t', \ldots, z_{t-p}')'$, z_t being a Markov chain such that:

$$\mathbb{Q}(z_{t+1} = e_j \mid \underline{z_t}; \underline{x_t}) = \pi\,(z_t, e_j\,; X_t) \exp\left[(-\delta(Z_t, X_t))' e_j\right] \,.$$

Note that, from (44), these probabilities add to one.

Now, in order to get a generalized linear term structure we impose that the risk-neutral dynamics is switching regime gaussian Car(p). Using (13), this impose that the dynamics has to satisfy the following specification:

$$x_{t+1} \overset{\mathbb{Q}}{=} {\nu^*}' Z_t + {\varphi^*}' X_t + ({\sigma^*}' Z_t)\xi_{t+1} \,, \tag{50}$$

where $Z_t = (z_t', \ldots, z_{t-p}')'$, with z_t a J-states Markov chain such that

$$\mathbb{Q}(z_{t+1} = e_j \mid z_t = e_i) = \pi^*\,(e_i, e_j) \,. \tag{51}$$

From proposition 5, this implies the following restrictions on the historical dynamics and on the SDF:

i) $\sigma(Z_t) = {\sigma^*}' Z_t$: the historical stochastic volatility must be linear in Z_t;

ii)

$$\gamma(Z_t) = \frac{{\nu^*}' Z_t - \nu(Z_t)}{{\sigma^*}' Z_t} :$$

for a given historical stochastic drift $\nu(Z_t)$ and stochastic volatility ${\sigma^*}' Z_t$, the coefficient $\gamma(Z_t)$ belongs to the previous family indexed by the free parameter vector ν^*.

iii)

$$\tilde{\gamma}(Z_t) = \frac{\varphi^* - \varphi(Z_t)}{{\sigma^*}' Z_t} :$$

for a given historical stochastic slope parameter $\varphi(Z_t)$ and stochastic volatility ${\sigma^*}' Z_t$ the coefficient vector $\tilde{\gamma}(Z_t)$ belongs to the previous family indexed by the free parameter vector φ^*.

iv)

$$\delta_j(X_t, Z_t) \quad = \quad \log \left[\frac{\pi\,(z_t, e_j\,;X_t)}{\pi^*(z_t, e_j)} \right] :$$

for a given historical transition matrix $\pi\,(z_t, e_j\,;X_t)$, the coefficient $\delta_j(X_t, Z_t)$ depend on z_t only and belongs to the previous family indexed by the entries $\pi^*(z_t, e_j)$ of a transition matrix.

Note that condition *iv*) implies that the risk premia coefficients δ_j, $j \in \{1, \ldots, J\}$, cannot be all positive [or all negative] since this would imply $\pi\,(z_t, e_j\,;X_t) > \pi^*(z_t, e_j)$, $\forall j$ [or $\pi\,(z_t, e_j\,;X_t) < \pi^*(z_t, e_j)$, $\forall j$], which is impossible since $\sum_{j=1}^{J} \pi\,(z_t, e_j\,;X_t) = \sum_{j=1}^{J} \pi^*(z_t, e_j) = 1$. Also note that condition *iv*) implies the normalisation condition (44).

3.5 The Generalised Linear Term Structure

We have seen in the previous section that the risk-neutral dynamics is defined by relations (50), (51); now, relation (50) can be rewritten:

$$X_{t+1} \overset{\mathbb{Q}}{=} \Phi^* X_t + \left[\nu^{*'} Z_t + (\sigma^{*'} Z_t)\xi_{t+1} \right] e_1 \tag{52}$$

where

$$\Phi^* = \begin{bmatrix} \varphi_1^* & \ldots & \ldots & \varphi_p^* \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & 1 & 0 \end{bmatrix} \text{ is a } (p \times p) - \text{matrix}\,,$$

$$X_t = (x_t, \ldots, x_{t+1-p})'\,,$$

and where e_1 is the first column of the identity matrix I_p.

Denoting by $B(t, h)$ the price at t of a zero-coupon with residual maturity h, we have the following result.

Proposition 6 : In the univariate SAN(p) model the price at date t of the zero-coupon bond with residual maturity h is :

$$B(t, h) = \exp\left(C_h'\, X_t + D_h'\, Z_t\right)\,,\ \text{for } h \geq 1\,, \tag{53}$$

where the vectors C_h and D_h satisfy the following recursive equations :

$$\begin{cases} C_h = \Phi^{*'} C_{h-1} - c \\ \\ D_h = -d \,+\, C_{1,h-1}\,\nu^* \,+\, \frac{1}{2}\,C_{1,h-1}^2\,\sigma^{*2} \,+ \tilde{D}_{h-1} + F(D_{1,h-1})\,, \end{cases} \tag{54}$$

where $C_{1,h-1}$ denotes the first component of the p-dimensional vector C_{h-1}, $D_{1,h-1}$ and $D_{2,h-1}$ are, respectively, the first J-dimensional component and the remaining (pJ)-dimensional component of D_{h-1}, i.e. $D_{h-1} = (D'_{1,h-1}, D'_{2,h-1})'$, $\tilde{D}_{h-1} = (D'_{2,h-1}, 0)'$, and where $F(D_{1,h-1}) = e_1 \otimes a_z(D_{1,h-1}, \pi^*)$, e_1 being the vector $(1, 0, \ldots, 0)'$ of size $(p+1)$ and a_z is the J-vector given in proposition 1; σ^{*2} is the vector whose components are the squares of the entries of σ^*. The initial conditions are $C_0 = 0$, $D_0 = 0$ (or $C_1 = -c$, $D_1 = -d$). [Proof : see Appendix 1.]
For clarity we give again the expression of $a_z(D_{1,h-1}, \pi^*)$:

$$
\begin{aligned}
& a_z(D_{1,h-1}, \pi^*) \\
= \quad & \left[\log \left(\sum_{j=1}^{J} \exp(D'_{1,h-1} e_j) \pi^*(e_1,\, e_j) \right), \ldots, \log \left(\sum_{j=1}^{J} \exp(D'_{1,h-1} e_j) \pi^*(e_J,\, e_j) \right) \right]' .
\end{aligned}
$$

From proposition 6 we see that the yields to maturity are:

$$
\begin{aligned}
R(t,h) &= -\frac{1}{h} \log B(t,h) \\
&= -\frac{C'_h}{h} X_t - \frac{D'_h}{h} Z_t \,, \quad h \geq 1 \,.
\end{aligned} \tag{55}
$$

So, they are linear functions of X_t, Z_t, i.e. of the present and past values of x_t and z_t. We observe that there is, in general, instantaneous causality between x_t and z_t.

3.6 The yield curve process

The result presented in Proposition 6 describes, conditionally to X_t and Z_t, the yields as a deterministic function of the time to maturity h, for a fixed date t. Nevertheless, in many financial and economic contexts one needs, for instance, also to study which are the effects of a shock, in the state variables, on the yield curve at different future instant times and for several maturities (e.g.: a Central Bank that needs to set a monetary policy). This means that we are interested in the dynamics of the process $R_{\mathcal{H}} = [\, R(t,h),\, 0 \leq t < T,\, h \in \mathcal{H} \,]$, for a given set of residual time to maturities $\mathcal{H} = (1, \ldots, H)$.

Now, if we consider a fixed h, we have that the process $R = [\, R(t,h),\, 0 \leq t < T \,]$ can be described by the following proposition.

Proposition 7 : For a fixed time to maturity h, the process $R = [\,R(t,h),\ 0 \leq t < T\,]$ is, under the historical probability, a switching ARMA(p, $p-1$) process of the following type :

$$\begin{aligned}\Psi(L, Z_t)\, R(t+1,h) &= D_h(L)\,\Psi(L,Z_t)\, z_{t+1} + C_h(L)\,\nu(Z_t) \\ &\qquad + C_h(L)[(\sigma^{*'} Z_t)\,\varepsilon_{t+1}]\,. \end{aligned} \tag{56}$$

where

$$\begin{aligned} C_h(L) &= -\frac{1}{h}\,(C_{1,h} + C_{2,h}L + \ldots + C_{p,h}L^{p-1}) \\ D_h(L) &= -\frac{1}{h}\,(D_{1,h} + D_{2,h}L + \ldots + D_{p+1,h}L^{p}) \\ \Psi(L, Z_t) &= 1 - \varphi_1(Z_t)L - \ldots - \varphi_p(Z_t)L^p\,, \end{aligned}$$

are lag polynomials in the backward shift operator L, and where the AR polynomial $\Psi(L, Z_t)$ applies to t. [Proof : see Appendix 2].

Proposition 8 : For a given set of residual time to maturities $\mathcal{H} = (1, \ldots, H)$, the stochastic evolution of the yield curve process $R_{\mathcal{H}} = [\,R(t,h),\ 0 \leq t < T, h \in \mathcal{H}\,]$ takes the following particular switching H-variate VARMA(p, $p-1$) representation:

$$\begin{aligned}\Psi(L, Z_t)\begin{pmatrix} R(t+1,1) \\ R(t+1,2) \\ \vdots \\ R(t+1,H)\end{pmatrix} &= \begin{pmatrix} C_1(L) \\ C_2(L) \\ \vdots \\ C_H(L)\end{pmatrix}(\sigma^{*'} Z_t)\varepsilon_{t+1} \\ &\quad + \begin{pmatrix} D_1(L) \\ D_2(L) \\ \vdots \\ D_H(L)\end{pmatrix}\Psi(L, Z_t)\, z_{t+1} + \begin{pmatrix} C_1(L) \\ C_2(L) \\ \vdots \\ C_H(L)\end{pmatrix}\nu(Z_t)\,. \end{aligned} \tag{57}$$

Similar results are easily obtained in the risk-neutral world.

3.7 Endogenous case

In the previous sections the factor x_t was exogenous. It is often assumed, in term structure models, that the factor x_t is the short rate process r_{t+1}. In this case the previous results remain valid, and the only modification comes from the absence of arbitrage opportunity condition for r_{t+1}, which imposes:

$$c = e_1 \, , d = 0 \, , \tag{58}$$

with e_1 the first column of the identity matrix I_p; consequently, the initial conditions in the recursive equations of proposition 6 become:

$$C_1 = -e_1 \, , D_1 = 0 \, . \tag{59}$$

Moreover, the switching ARMA(p, $p-1$) representation (56), or its analogous in the risk-neutral world, could be used to analyse how a shock on ε_t, i.e. on $r_{t+1} = R(t, 1)$, is propagated on the surface $[\, R(t+\tau, h),\ \tau \in \mathcal{T},\ h \in \mathcal{H}\,]$, where $\mathcal{T} = \{0, \ldots, T-t-1\}$ and $\mathcal{H} = (1, \ldots, H)$ (for instance when the process z_t is exogenous).

3.8 Multi-Factor generalization [Switching VARN(p) model]

For sake of notational simplicity we consider the two factor case but an extension to more that two factors is straightforward. The historical dynamics of $\tilde{x}_t = (x_{1,t}, x_{2,t})'$ is a switching bivariate VARN(p) model given by:

$$\left\{ \begin{array}{lcl} x_{1,t+1} & = & \nu_1(Z_t) + \varphi_o(Z_t) x_{2,t+1} + \varphi_{11}(Z_t)' X_{1t} + \varphi_{12}(Z_t)' X_{2t} + \sigma_1(Z_t)\varepsilon_{1,t+1} \\ \\ x_{2,t+1} & = & \nu_2(Z_t) + \varphi_{21}(Z_t)' X_{1t} + \varphi_{22}(Z_t)' X_{2t} + \sigma_2(Z_t)\varepsilon_{2,t+1} \, , \end{array} \right. \tag{60}$$

where $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are independent standard normal white noises, $X_{1t} = (x_{1,t}, \ldots, x_{1,t+1-p})'$, $X_{2t} = (x_{2,t}, \ldots, x_{2,t+1-p})'$, $Z_t = (z_t', \ldots, z_{t-p}')'$, with z_t a J-states non-homogeneous Markov chain such that $P(z_{t+1} = e_j \,|\, z_t = e_i; \underline{\tilde{x}_t}) = \pi(e_i, e_j\,; \tilde{X}_t)$, and where $\tilde{X}_t = (X_{1t}', X_{2t}')'$. The recursive form (60) is equivalent to the canonical form :

$$\left\{ \begin{array}{lcl} x_{1,t+1} & = & \tilde{\nu}_1(Z_t) + \tilde{\varphi}_{11}(Z_t)' X_{1t} + \tilde{\varphi}_{12}(Z_t)' X_{2t} + \sigma_1(Z_t)\varepsilon_{1,t+1} + \varphi_o(Z_t)\sigma_2(Z_t)\varepsilon_{2,t+1} \\ \\ x_{2,t+1} & = & \nu_2(Z_t) + \varphi_{21}(Z_t)' X_{1t} + \varphi_{22}(Z_t)' X_{2t} + \sigma_2(Z_t)\varepsilon_{2,t+1} \, , \end{array} \right. \tag{61}$$

where $\tilde{\nu}_1 = \nu_1 + \varphi_o \nu_2$, $\tilde{\varphi}_{11} = \varphi_{11} + \varphi_o \varphi_{21}$, $\tilde{\varphi}_{12} = \varphi_{12} + \varphi_o \varphi_{22}$ or, with obvious notations:

$$\tilde{x}_{t+1} \quad = \quad \tilde{\nu}(Z_t) + \tilde{\Phi}(Z_t)\tilde{X}_t + S(Z_t)\varepsilon_{t+1} \, , \tag{62}$$

where

$$S(Z_t) = \begin{bmatrix} \sigma_1(Z_t) & \varphi_o(Z_t)\sigma_2(Z_t) \\ 0 & \sigma_2(Z_t) \end{bmatrix}$$

Using the notation

$$\Gamma(Z_t, \tilde{X}_t) = \Big[\Gamma_1(Z_t, \tilde{X}_t),\, \Gamma_2(Z_t, \tilde{X}_t)\Big]'$$

where $\Gamma_i(Z_t, \tilde{X}_t) = \gamma_i(Z_t) + \tilde{\gamma_i}(Z_t)'\tilde{X}_t,\ i \in \{1,2\}$ and $\Gamma(Z_t, \tilde{X}_t) = \gamma(Z_t) + \tilde{\Gamma}(Z_t, \tilde{X}_t)\tilde{X}_t$, with $\gamma(Z_t) = [\gamma_1(Z_t), \gamma_2(Z_t)]'$, $\tilde{\Gamma}(Z_t, \tilde{X}_t) = [\tilde{\gamma_1}(Z_t)', \tilde{\gamma_2}(Z_t)']'$, the SDF is defined as :

$$\begin{aligned} M_{t,t+1} \;=\; \exp\Big[&-c'\tilde{X}_t - d'Z_t + \Gamma(Z_t, \tilde{X}_t)'\,\varepsilon_{t+1} \\ &-\tfrac{1}{2}\Gamma(Z_t, \tilde{X}_t)'\Gamma(Z_t, \tilde{X}_t) - \delta(Z_t, \tilde{X}_t)'z_{t+1}\Big]\,. \end{aligned} \tag{63}$$

Assuming the normalisation condition (44) and the absence of arbitrage opportunity for r_{t+1} we get:

$$r_{t+1} = c'\tilde{X}_t + d'Z_t\,. \tag{64}$$

It is also easily seen that the risk premium for an asset providing the payoff $\exp(-\theta'\tilde{x}_{t+1})$ at $t+1$ is $\omega(\theta) = \theta' S(Z_t)\Gamma(Z_t, \tilde{X}_t)$ and that the risk premium associated with the digital payoff $\mathbb{I}_{(e_j)}(z_{t+1})$ is unchanged.

The Laplace transform of the one-period conditional risk-neutral distribution is :

$$\begin{aligned} &E_t^{\mathbb{Q}}[\exp(u'\tilde{x}_{t+1} + v'z_{t+1})] \\ =\;& E_t\{\exp[\Gamma(\tilde{X}_t, Z_t)'\,\varepsilon_{t+1} - \tfrac{1}{2}\Gamma(\tilde{X}_t, Z_t)'\Gamma(\tilde{X}_t, Z_t) - \delta'(Z_t, \tilde{X}_t)z_{t+1} \\ &\quad + u'[\tilde{\nu}(Z_t) + \tilde{\Phi}(Z_t)\tilde{X}_t + S(Z_t)\varepsilon_{t+1}] + v'z_{t+1}]\} \\ =\;& \exp\Big\{\, u'[\,\tilde{\Phi}(Z_t)\tilde{X}_t + S(Z_t)\Gamma(\tilde{X}_t, Z_t)] + u'\tilde{\nu}(Z_t) + \tfrac{1}{2}u'S(Z_t)S(Z_t)'u\Big\} \times \\ &\quad \textstyle\sum_{j=1}^{J} \pi(z_t, e_j\,; \tilde{X}_t) \exp\Big[(v - \delta(Z_t, \tilde{X}_t))'e_j\Big] \\ =\;& \exp\Big\{\, u'[\,\tilde{\Phi}(Z_t) + S(Z_t)\tilde{\Gamma}(Z_t, \tilde{X}_t)]\tilde{X}_t + u'[\,\tilde{\nu}(Z_t) + S(Z_t)\gamma(Z_t)] + \tfrac{1}{2}u'S(Z_t)S(Z_t)'u\Big\} \times \\ &\quad \textstyle\sum_{j=1}^{J} \pi(z_t, e_j\,; \tilde{X}_t) \exp\Big[(v - \delta(Z_t, \tilde{X}_t))'e_j\Big]\,. \end{aligned} \tag{65}$$

Therefore, we get:

Proposition 9 : The risk-neutral dynamics of the process $(\tilde{x}_t, z_t)$ is given by:

$$\tilde{x}_{t+1} \overset{\mathbb{Q}}{=} \tilde{\nu}(Z_t) + S(Z_t)\gamma(Z_t) + [\tilde{\Phi}(Z_t) + S(Z_t)\tilde{\Gamma}(Z_t, \tilde{X}_t)]\tilde{X}_t + S(Z_t)\xi_{t+1}\,, \tag{66}$$

where $\overset{\mathbb{Q}}{=}$ denotes the equality in distribution (associated to the probability $\mathbb{Q}$), ξ_{t+1} is (under $\mathbb{Q}$) a bivariate gaussian white noise with $\mathcal{N}(0, I_2)$ distribution, and where $Z_t = (z_t', \ldots, z_{t-p}')'$, with z_t a Markov chain such that:

$$\mathbb{Q}(z_{t+1} = e_j \,|\, \underline{z_t}; \underline{\tilde{x}_t}) = \pi(z_t, e_j\,; \tilde{X}_t) \exp\left[(-\delta(Z_t, \tilde{X}_t))' e_j\right] .$$

If we want to obtain a Switching bivariate Car process, we must have using (37) :

i)

$$\begin{aligned} \sigma_1(Z_t) &= \sigma_1^{*'} Z_t \\ \sigma_2(Z_t) &= \sigma_2^{*'} Z_t \\ \varphi_o(Z_t) &= \varphi_o^*\,, \end{aligned}$$

and, therefore,

$$S(Z_t) = \begin{bmatrix} \sigma_1^{*'} Z_t & \varphi_o^* \sigma_2^{*'} Z_t \\ 0 & \sigma_2^{*'} Z_t \end{bmatrix}$$

ii)

$$\gamma(Z_t) = [S(Z_t)]^{-1}[\nu^* Z_t - \tilde{\nu}(Z_t)]\,,$$

where ν^* is a $(2 \times (p+1)J)$-matrix.

iii)

$$\tilde{\Gamma}(Z_t, \tilde{X}_t) = [S(Z_t)]^{-1}\left[\Phi^* - \tilde{\Phi}(Z_t)\right]\,,$$

where Φ^* is a $(2 \times 2p)$-matrix.

iv)

$$\delta_j(\tilde{X}_t, Z_t) = \log\left[\frac{\pi(z_t, e_j\,; \tilde{X}_t)}{\pi^*(z_t, e_j)}\right] .$$

The risk-neutral dynamics can be written:

$$\left\{\begin{array}{lcl} x_{1,t+1} & \overset{\mathbb{Q}}{=} & \nu_1^* Z_t + \Phi_1^* \tilde{X}_t + S_1^*(Z_t)\xi_{t+1} \\ \\ x_{2,t+1} & \overset{\mathbb{Q}}{=} & \nu_2^* Z_t + \Phi_2^* \tilde{X}_t + S_2^*(Z_t)\xi_{t+1} \,, \end{array}\right. \tag{67}$$

where ν_i^*, Φ_i^*, S_i^* are the i^{th} row of ν^*, Φ^*, S^*, with $i \in \{1,2\}$, or

$$\tilde{X}_{t+1} \overset{\mathbb{Q}}{=} \tilde{\Phi}^* \tilde{X}_t + [\nu_1^* Z_t + S_1^*(Z_t)\xi_{t+1}]\, e_1 + [\nu_2^* Z_t + S_2^*(Z_t)\xi_{t+1}]\, e_{p+1} \,,$$

where e_1 (respectively, e_{p+1}) is of size $2p$, with entries equal to zero except the first (respectively, the $(p+1)^{th}$) one which is equal to one, and

$$\tilde{\Phi}^* = \begin{bmatrix} \Phi_{11}^* & \Phi_{12}^* \\ \\ \tilde{I} & \tilde{\mathbf{0}} \\ \\ \Phi_{21}^* & \Phi_{22}^* \\ \\ \tilde{\mathbf{0}} & \tilde{I} \end{bmatrix}$$

where $\Phi_1^* = (\Phi_{11}^*, \Phi_{12}^*)$, $\Phi_2^* = (\Phi_{21}^*, \Phi_{22}^*)$, and where $\tilde{\mathbf{0}}$ is a $[(p-1)\times p]$-matrix of zeros and $\tilde{I}$ is a $[(p-1)\times p]$-matrix equal to $(I_{p-1}, 0)$, where 0 is a vector of size $(p-1)$.
The term structure is given by the following proposition:

Proposition 10 : In the bivariate SAN(p) model the price at date t of the zero-coupon bond with residual maturity h is :

$$B(t,h) = \exp\left(C_h' \, \tilde{X}_t + D_h' \, Z_t \right) \,, \text{ for } h \geq 1 \tag{68}$$

where the vectors C_h and D_h satisfy the following recursive equations :

$$\left\{\begin{array}{l} C_h = \tilde{\Phi}^{*'} C_{h-1} - c \\ \\ D_h = -d \,+\, C_{1,h-1}\, \nu_1^{*'} \,+\, \frac{1}{2}\, C_{1,h-1}^2 (\sigma_1^{*2} + \varphi_o^{*2}\sigma_2^{*2}) \\ \\ \qquad\qquad + C_{p+1,h-1}\, \nu_2^{*'} + \frac{1}{2}\, C_{p+1,h-1}^2 \sigma_2^{*2} + \tilde{D}_{h-1} + F(D_{1,h-1}) \,, \end{array}\right. \tag{69}$$

where $\tilde{D}_{h-1}$ and $F(D_{1,h-1})$ have the same meaning as in proposition 6, and the initial conditions are $C_0 = 0$, $D_0 = 0$ (or $C_1 = -c$, $D_1 = -d$) [Proof : see Appendix 3].

So, proposition 10 shows that the yields to maturity are:

$$R(t,h) \;=\; -\frac{C_h'}{h}\tilde{X}_t - \frac{D_h'}{h}Z_t\,, \quad h \geq 1\,. \tag{70}$$

In the endogenous case we can take $x_{1t} = r_{t+1}$, and $x_{2t} = R(t,H)$ for a given time to maturity H. In this case the absence of arbitrage conditions for r_{t+1} and $R(t,H)$ imply:

$$(i)\; C_1 \;=\; -e_1,\; D_1 = 0\,,\text{ or } c = e_1,\; d = 0$$

$$(ii)\; C_H \;=\; -\,H\,e_{p+1},\; D_H = 0\,.$$

Using the notations $C_h = (C_{1,h}, C^*_{1,h}, C_{p+1,h}, C^*_{2,h})'$, $\tilde{C}_{1,h} = ({C^*_{1,h}}', 0)'$, $\tilde{C}_{2,h} = ({C^*_{2,h}}', 0)'$ (where the zeros are scalars), and $\tilde{C}_h = (\tilde{C}'_{1,h}, \tilde{C}'_{2,h})'$, it easily seen that the recursive equation $C_h = \tilde{\Phi}^{*'} C_{h-1} - c$ can be written :

$$C_h = \Phi_1^{*'} C_{1,h-1} + \Phi_2^{*'} C_{p+1,h-1} + \tilde{C}_{h-1} - c\,.$$

The first set of conditions is used as initial values in the recursive procedure of proposition 10; the second set of conditions implies restrictions on the parameters $\tilde{\Phi}^*, \nu_1^*, \nu_2^*, \sigma_1^*, \sigma_2^*, \varphi_o^*, \pi^*(z_t, e_j)$ which must be taken into account at the estimation stage.

4 SWITCHING AUTOREGRESSIVE GAMMA (SAG) TERM STRUCTURE MODEL OF ORDER p

Like for SAN(p) models we start the description of the SAG(p) modeling by the case of one exogenous factor.

4.1 The historical dynamics

We assume that the Laplace transform of the conditional distribution of x_{t+1}, given $(\underline{x_t}, \underline{z_t})$, is:

$$\begin{aligned} E\left[\exp(u x_{t+1}) \,|\, \underline{x_t}, , \underline{z_t}\right] \;&=\; \exp\left[\tfrac{u}{1-u\mu(X_t,Z_t)}\left[\varphi_1(Z_t)x_t + \ldots + \varphi_p(Z_t)x_{t-p+1}\right]\right. \\ &\qquad \left. -\,\nu(Z_t)\log(1-u\mu(X_t,Z_t))\right]\,, \end{aligned} \tag{71}$$

where $Z_t = (z_t', \ldots, z_{t-p}')'$, with z_t a J-states non-homogeneous Markov chain such that $P(z_{t+1} = e_j \,|\, z_t = e_i ; \underline{x_t}) = \pi(e_i, e_j\,; \tilde{X}_t)$, and where $X_t = (x_t, \ldots, x_{t+1-p})'$. Using the notation:

$$\begin{aligned} A[u; \varphi(Z_t), \mu(X_t, Z_t)] &= \tfrac{u}{1-u\mu(X_t,Z_t)}[\varphi_1(Z_t), \ldots, \varphi_p(Z_t)]' = \tfrac{u}{1-u\mu(X_t,Z_t)}\varphi(Z_t) \\ b[u; \nu(Z_t), \mu(X_t, Z_t)] &= -\,\nu(Z_t)\log(1 - u\mu(X_t, Z_t))\,, \end{aligned}$$

relation (71) can be written:

$$\begin{aligned} E\left[\exp(u x_{t+1}) \,|\, \underline{x_t}, , \underline{z_t}\right] &= \exp\{A[u; \varphi(Z_t), \mu(X_t, Z_t)]' X_t \\ &\qquad + b[u; \nu(Z_t), \mu(X_t, Z_t)]\}\ . \end{aligned} \tag{72}$$

The process (x_t) can also be written:

$$\begin{aligned} x_{t+1} &= \nu(Z_t)\mu(X_t, Z_t) + \varphi_1(Z_t)x_t + \ldots + \varphi_p(Z_t)x_{t+1-p} + \varepsilon_{t+1} \\ &= \nu(Z_t)\mu(X_t, Z_t) + \varphi(Z_t)' X_t + \varepsilon_{t+1}\,, \end{aligned} \tag{73}$$

where ε_{t+1} is a martingale difference sequence with conditional Laplace transform given by:

$$\begin{aligned} E\left[\exp(u\varepsilon_{t+1}) \,|\, \underline{x_t}, , \underline{z_t}\right] &= \exp\{-u[\nu(Z_t)\mu(X_t, Z_t) + \varphi(Z_t)' X_t] \\ &\qquad + A[u; \varphi(Z_t), \mu(X_t, Z_t)]' X_t \\ &\qquad\qquad + b[u; \nu(Z_t), \mu(X_t, Z_t)]\} \\ &= \exp\{[A[u; \varphi(Z_t), \mu(X_t, Z_t)] - u\varphi(Z_t)]' X_t \\ &\qquad + b[u; \nu(Z_t), \mu(X_t, Z_t)] - u\,\nu(Z_t)\mu(X_t, Z_t)\}\ . \end{aligned} \tag{74}$$

Note that the dynamics of (x_t, z_t) is in general not Car.

4.2 The Stochastic Discount Factor

In the SAG(p) model the SDF is specified in the following way:

$$\begin{aligned} M_{t,t+1} &= \exp\{-c' X_t - d' Z_t + \Gamma(Z_t, X_t)\varepsilon_{t+1} + \Gamma(Z_t, X_t)\,[\nu(Z_t)\mu(X_t, Z_t) + \varphi(Z_t)' X_t] \\ &\qquad - A[\Gamma(Z_t, X_t); \varphi(Z_t), \mu(X_t, Z_t)]' X_t \\ &\qquad\quad - b[\Gamma(Z_t, X_t); \nu(Z_t), \mu(X_t, Z_t)] - \delta(Z_t, X_t)' z_{t+1}\}\ , \end{aligned} \tag{75}$$

where $\Gamma(Z_t, X_t) = \gamma(Z_t) + \tilde{\gamma}'(Z_t)X_t$, or, equivalently

$$\begin{aligned} M_{t,t+1} &= \exp\{-c'X_t - d'Z_t + \Gamma(Z_t, X_t)x_{t+1} - A[\Gamma(Z_t, X_t); \varphi(Z_t), \mu(X_t, Z_t)]'X_t \\ &\qquad -b[\Gamma(Z_t, X_t); \nu(Z_t), \mu(X_t, Z_t)] - \delta(Z_t, X_t)'z_{t+1}\}\ , \end{aligned} \tag{76}$$

Assuming the normalisation condition (44), we get that:

$$r_{t+1} = c'X_t + d'Z_t\,. \tag{77}$$

4.3 A useful Lemma

In the subsequent sections we will use several times the following lemmas.
Let us consider the functions:

$$\tilde{a}(u; \rho, \mu) = \frac{\rho u}{1 - u\mu} \quad \text{and} \quad \tilde{b}(u; \nu, \mu) = -\nu \log(1 - u\mu)\,;$$

we have:

Lemma 1 :

$$\tilde{a}(u + \alpha; \rho, \mu) - \tilde{a}(\alpha; \rho, \mu) = \tilde{a}(u; \rho^*, \mu^*)$$

$$\tilde{b}(u + \alpha; \nu, \mu) - \tilde{b}(\alpha; \nu, \mu) = \tilde{b}(u; \nu, \mu^*)$$

$$\text{with } \rho^* = \frac{\rho}{(1 - \alpha\mu)^2},\ \mu^* = \frac{\mu}{1 - \alpha\mu}\,,$$

[Proof : see Appendix 4.]

Lemma 1 immediately implies lemma 2.

Lemma 2 :

$$A[u + \alpha; \varphi(Z_t), \mu(X_t, Z_t)] - A[\alpha; \varphi(Z_t), \mu(X_t, Z_t)] = A[u; \varphi^*(Z_t), \mu^*(X_t, Z_t)]$$

$$b[u + \alpha; \nu(Z_t), \mu(X_t, Z_t)] - b[\alpha; \nu(Z_t), \mu(X_t, Z_t)] = b[u; \nu(Z_t), \mu^*(Z_t, X_t)]$$

$$\text{with } \varphi^*(Z_t) = \frac{\varphi(Z_t)}{[1 - \alpha\mu(Z_t, X_t)]^2},\ \mu^*(Z_t, X_t) = \frac{\mu(X_t, Z_t)}{1 - \alpha\mu(X_t, Z_t)}\,.$$

4.4 Risk-neutral dynamics

The Laplace transform of the risk-neutral conditional distribution of (x_{t+1}, z_{t+1}) is, using the notation $\Gamma_t = \Gamma(X_t, Z_t)$:

$$
\begin{aligned}
& E_t^{\mathbb{Q}}[\exp(u x_{t+1} + v' z_{t+1})] \\
= \;& E_t\{\exp[(u+\Gamma_t)x_{t+1} - A[\Gamma_t; \varphi(Z_t), \mu(X_t, Z_t)]' X_t \\
& \qquad -b[Z_t; \nu(Z_t), \mu(X_t, Z_t)] + (v - \delta(X_t, Z_t))' z_{t+1}]\} \\
= \;& \exp\{[(A[u+\Gamma_t; \varphi(Z_t), \mu(X_t, Z_t)] - A[\Gamma_t; \varphi(Z_t), \mu(X_t, Z_t)])' X_t \\
& \qquad + b[u+\Gamma_t; \nu(Z_t), \mu(X_t, Z_t)] - b[\Gamma_t; \nu(Z_t), \mu(X_t, Z_t)]]\} \\
& \qquad \times \textstyle\sum_{j=1}^{J} \pi(z_t, e_j\,; X_t) \exp[(v - \delta(Z_t, X_t))' e_j]\ ;
\end{aligned}
\tag{78}
$$

now, using lemma 2, (79) can be written:

$$
\begin{aligned}
& E_t^{\mathbb{Q}}[\exp(u x_{t+1} + v' z_{t+1})] \\
= \;& \exp\{A[u; \varphi^*(Z_t), \mu^*(X_t, Z_t)]' X_t + b[u; \nu(Z_t), \mu^*(Z_t, X_t)]\} \\
& \qquad \times \textstyle\sum_{j=1}^{J} \pi(z_t, e_j\,; X_t) \exp[(v - \delta(Z_t, X_t))' e_j]\ ,
\end{aligned}
\tag{79}
$$

with $\varphi^*(Z_t) = \dfrac{\varphi(Z_t)}{[1 - \Gamma_t \mu(Z_t, X_t)]^2}$ and $\mu^*(Z_t, X_t) = \dfrac{\mu(X_t, Z_t)}{1 - \Gamma_t \mu(X_t, Z_t)}$.

So, from (72), we see that the risk-neutral conditional distribution of x_{t+1}, given $(\underline{x_t}, \underline{z_t})$, is in the same class as the historical one and obtained by replacing $\varphi(Z_t)$ with $\varphi^*(Z_t)$, and $\mu(X_t, Z_t)$ with $\mu^*(Z_t, X_t)$.

In order to get a generalize linear term structure we impose that the risk-neutral dynamics is a switching regime Gamma Car(p) process. So, using the results in section 2.5.b, we get that $\varphi^*(Z_t)$ and $\mu^*(Z_t, X_t)$ must be constant, $\nu(Z_t) = {\nu^*}' Z_t$ and $\pi(z_t, e_j\,; X_t) = \pi^*(z_t, e_j) \exp[(\delta(Z_t, X_t))' e_j\,]$. Also note that μ^* must be positive as well as the components of ν^* and φ^*. This implies the following constraint on the historical dynamics and on the

SDF:

$$
\begin{aligned}
\mu(X_t, Z_t) &= \mu^*[1 - \Gamma(X_t, Z_t)\mu(X_t, Z_t)] \\
\varphi(Z_t) &= \varphi^*[1 - \Gamma(X_t, Z_t)\mu(X_t, Z_t)]^2 \\
\nu(Z_t) &= \nu^{*'} Z_t \\
\delta_j(X_t, Z_t) &= \log\left[\tfrac{\pi(z_t, e_j\,; X_t)}{\pi^*(z_t, e_j)}\right].
\end{aligned}
$$

We see that $\varphi(Z_t) = \frac{\varphi^*}{\mu^{*2}}\,\mu(X_t, Z_t)^2$, so $\mu(X_t, Z_t)$ must depend only on Z_t, and therefore the same is true for $\Gamma(X_t, Z_t)$. Finally, we have the constraint:

i)

$$\mu(Z_t) = \mu^*[1 - \Gamma(Z_t)\mu(Z_t)]$$

ii)

$$\varphi(Z_t) = \varphi^*[1 - \Gamma(Z_t)\mu(Z_t)]^2$$

iii)

$$\nu(Z_t) = \nu^{*'} Z_t$$

iv)

$$\delta_j(X_t, Z_t) = \log\left[\frac{\pi\,(z_t, e_j\,; X_t)}{\pi^*(z_t, e_j)}\right];$$

In particular, since $\varphi(Z_t) = \frac{\varphi^*}{\mu^{*2}}\,\mu(Z_t)^2$, the random vector must be proportional to a deterministic vector.

Moreover, it is easily seen that the risk premium corresponding to the payoff $\exp(-\theta x_{t+1})$ at $t+1$ is:

$$
\begin{aligned}
\omega_t(\theta) &= \{A[-\theta; \varphi(Z_t), \mu(Z_t)] - A[-\theta; \varphi^*, \mu^*]\}' X_t \\
&\quad + b[-\theta; \nu^{*'} Z_t, \mu(Z_t)] - b[-\theta; \nu^{*'} Z_t, \mu^*]\,.
\end{aligned}
$$

Like in the gaussian case, we obtain an affine function in X_t also depending on Z_t. The risk premium associated with the digital asset providing one money unit at $t+1$ if $z_{t+1} = e_j$, is still given by (47).

4.5 The Generalised Linear Term Structure

Let us introduce the notations:

$$\begin{aligned} A^*(u) &= A(u;\varphi^*,\mu^*) \\ \tilde{C}_h &= (C_{2,h},\ldots,C_{p,h},0)'\,. \end{aligned} \tag{80}$$

As usual, $B(t,h)$ is the price at t of a zero-coupon bond with residual maturity h.

Proposition 11 : In the univariate SAG(p) model the price at date t of the zero-coupon bond with residual maturity h is :

$$B(t,h) = \exp\left(C_h'\,X_t + D_h'\,Z_t\right)\,,\ \text{for } h \geq 1\,, \tag{81}$$

where the vectors C_h and D_h satisfy the following recursive equations :

$$\begin{cases} C_h &= -c + A^*(C_{1,h-1}) + \tilde{C}_{h-1} \\ D_h &= -d\,-\,\nu^*\log(1-C_{1,h-1}\mu^*) +\,\tilde{D}_{h-1} + F(D_{1,h-1})\,, \end{cases} \tag{82}$$

where $\tilde{D}_{h-1}$ and $F(D_{1,h-1})$ have the same meaning as in proposition 6; the initial conditions are $C_0 = 0$, $D_0 = 0$ (or $C_1 = -c$, $D_1 = -d$) [Proof : see Appendix 5].

Again, we obtain a generalised linear term structure given by:

$$R(t,h) \;=\; -\frac{C_h'}{h}X_t - \frac{D_h'}{h}Z_t\,, \quad h \geq 1\,, \tag{83}$$

and, in the same spirit of propositions 7 and 8 for the univariate SAN(p) model [see section 3.6], it is easy to verify that the processes $R = [\,R(t,h),\ 0 \leq t < T\,]$ and $R_{\mathcal{H}} = [\,R(t,h),\ 0 \leq t < T,\, h \in \mathcal{H}\,]$ are, respectively, a weak switching ARMA(p, $p-1$) process and a weak H-variate switching VARMA(p, $p-1$) process.
In the endogenous case, where $x_t = r_{t+1}$, the previous results remains valid with $C_1 = -e_1$, $D_1 = 0$.

4.6 Positiveness of the yields

Since $r_{t+1} = R(t,1) = c'X_t + d'Z_t$, and since the components of X_t are positive, the short term process will be positive as soon as the components of c and d are nonnegative. The positiveness of r_{t+1} implies that of

$R(t,h)$, for every instant time t and time to maturity h, because $R(t,h) = -\frac{1}{h} \log E_t^{\mathbb{Q}} [\exp(-r_{t+1} - \ldots - r_{t+h})]$.

This positiveness can also be observed from the recursive equations of proposition 11. Indeed, using the fact that μ^* and the components of φ^* and ν^* are positive and that $0 < \pi^*_{ij} < 1$, it easily seen that, for any $u < 0$, the components of $A^*(u)$ and $-\nu^* \log(1 - C_{1,h-1}\mu^*)$ are negative and the result follows.

4.7 Multi-Factor generalizations [Switching VARG(p) model]

The bivariate process $\tilde{x}_t = (x_{1,t}, x_{2,t})$ is a switching VARG(p) model defined by the following conditional Laplace transforms:

$$\begin{aligned} & E_t[\exp(u_1 x_{1,t+1}) \,|\, \underline{x_{2,t+1}}, \underline{x_{1,t}}, \underline{z_t}\,] \\ = \; & \exp\left\{ \frac{u_1}{1 - u_1 \mu_1(Z_t)} \left[\varphi_o(Z_t) x_{2,t+1} + \varphi_{11}(Z_t)' X_{1t} + \varphi_{12}(Z_t)' X_{2t}\right] \right. \\ & \left. -\nu_1(Z_t) \log(1 - u_1 \mu_1(Z_t)) \right\} , \end{aligned} \tag{84}$$

$$\begin{aligned} & E_t[\exp(u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}\,] \\ = \; & \exp\left\{ \frac{u_2}{1 - u_2 \mu_2(Z_t)} \left[\varphi_{21}(Z_t)' X_{1t} + \varphi_{22}(Z_t)' X_{2t}\right] \right. \\ & \left. -\nu_2(Z_t) \log(1 - u_2 \mu_2(Z_t)) \right\} . \end{aligned} \tag{85}$$

We will use the notations:

$$\varphi_o(Z_t) = \varphi_{o,t} \,,$$

$$[\,\varphi_{11}(Z_t)', \varphi_{12}(Z_t)'\,] = \varphi'_{1,t} \,, \;\; [\,\varphi_{21}(Z_t)', \varphi_{22}(Z_t)'\,] = \varphi'_{2,t} \,,$$

$$\mu_i(Z_t) = \mu_{i,t} \,, \;\; \nu_i(Z_t) = \nu_{i,t} \,, \;\; i \in \{1,2\} \,,$$

and using the functions $\tilde{a}, \tilde{b}, A, B$ defined in lemmas 1 and in section 4.1, we will introduce the notations:

$$
\begin{aligned}
a_{1,t}(u_1) &= \tilde{a}(u_1; \varphi_{o,t}, \mu_{1,t}) \\
b_{1,t}(u_1) &= \tilde{b}(u_1; \nu_{1,t}, \mu_{1,t}), \ \ b_{2,t}(u_2) = \tilde{b}(u_2; \nu_{2,t}, \mu_{2,t}) \\
A_{1,t}(u_1) &= A(u_1; \varphi_{1,t}, \mu_{1,t}), \ \ A_{2,t}(u_2) = A(u_2; \varphi_{2,t}, \mu_{2,t}).
\end{aligned}
$$

With these notations, the Laplace transforms (84) and (85) become respectively:

$$
\begin{aligned}
& E_t[\exp(u_1 x_{1,t+1}) \,|\, \underline{x_{2,t+1}}, \underline{x_{1,t}}, \underline{z_t}] \\
= \ & \exp\left[a_{1,t}(u_1) x_{2,t+1} + A_{1,t}(u_1)' \tilde{X}_t + b_{1,t}(u_1)\right],
\end{aligned} \tag{86}
$$

$$
\begin{aligned}
& E_t[\exp(u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}] \\
= \ & \exp\left[A_{2,t}(u_2)' \tilde{X}_t + b_{2,t}(u_2)\right],
\end{aligned} \tag{87}
$$

where $\tilde{X}_t = (X'_{1t}, X'_{2t})'$. Moreover, the joint conditional Laplace transform of $(x_{1,t+1}, x_{2,t+1})$, given $(\underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t})$, is:

$$
\begin{aligned}
& E_t[\exp(u_1 x_{1,t+1} + u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}] \\
= \ & \exp\left\{[A_{1,t}(u_1) + A_{2,t}(u_2 + a_{1,t}(u_1))]' \tilde{X}_t + b_{1,t}(u_1) + b_{2,t}(u_2 + a_{1,t}(u_1))\right\}.
\end{aligned} \tag{88}
$$

The process z_t is assumed to be a non-homogeneous Markov chain such that $P(z_{t+1} = e_j \,|\, z_t = e_i; \underline{\tilde{x}_t}) = \pi(e_i, e_j\,; \tilde{X}_t)$.

We now introduce the SDF:

$$
\begin{aligned}
M_{t,t+1} = \exp\{ & -c'\tilde{X}_t - d'Z_t + \Gamma_{1t} x_{1,t+1} + \Gamma_{2t} x_{2,t+1} \\
& - [A_{1,t}(\Gamma_{1t}) + A_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t}))]' \tilde{X}_t \\
& - [b_{1,t}(\Gamma_{1t}) + b_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t}))] - \delta(Z_t, \tilde{X}_t)' z_{t+1}\},
\end{aligned} \tag{89}
$$

where $\Gamma_{1t} = \Gamma_1(Z_t)$ and $\Gamma_{2t} = \Gamma_2(Z_t)$.

4.8 Risk-neutral dynamics in the multifactor case

It can now be seen that the joint conditional Laplace transform of $(x_{1,t+1}, x_{2,t+1})$ in the risk-neutral world is:

$$
\begin{aligned}
& E_t^{\mathbb{Q}}[\exp(u_1 x_{1,t+1} + u_2 x_{2,t+1}) \,|\, \underline{x_{1,t}}, \underline{x_{2,t}}, \underline{z_t}\,] \\
= \; & \exp\Big\{ A_{2,t}\left[u_2 + \Gamma_{2t} + a_{1,t}(u_1 + \Gamma_{1t})\right]' \tilde{X}_t + b_{2,t}(u_2 + \Gamma_{2t} + a_{1,t}(u_1 + \Gamma_{1t})) \\
& + A_{1,t}(u_1 + \Gamma_{1t})' \tilde{X}_t + b_{1,t}(u_1 + \Gamma_{1t}) \\
& - A_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t}))' \tilde{X}_t - b_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t})) \\
& - A_{1,t}(\Gamma_{1t})' \tilde{X}_t - b_{1,t}(\Gamma_{1t}) \Big\} \; .
\end{aligned}
\tag{90}
$$

Using lemma 2 we get:

$$
\begin{aligned}
& A_{2,t}\left[u_2 + \Gamma_{2t} + a_{1,t}(u_1 + \Gamma_{1t})\right] - A_{2,t}(\Gamma_{2t} + a_{1,t}(\Gamma_{1t})) \\
= \; & A\left[u_2 + a_{1,t}(u_1 + \Gamma_{1t}) - a_{1,t}(\Gamma_{1t}); \varphi_{2t}^*, \mu_{2t}^*\right] \; ,
\end{aligned}
$$

with

$$
\begin{aligned}
\varphi_{2t}^* &= \frac{\varphi_{2t}}{\{1 - [\Gamma_{2t} + a_{1,t}(\Gamma_{1t})]\mu_{2t}\}^2} \\
\mu_{2t}^* &= \frac{\mu_{2t}}{\{1 - [\Gamma_{2t} + a_{1,t}(\Gamma_{1t})]\mu_{2t}\}} \; ,
\end{aligned}
$$

and using lemma 1

$$
\begin{aligned}
& A\left[u_2 + a_{1,t}(u_1 + \Gamma_{1t}) - a_{1,t}(\Gamma_{1t}); \varphi_{2t}^*, \mu_{2t}^*\right] \\
= \; & A\left[u_2 + \tilde{a}(u_1 + \Gamma_{1t}; \varphi_{ot}, \mu_{1,t}) - \tilde{a}(\Gamma_{1t}; \varphi_{ot}, \mu_{1,t}); \varphi_{2t}^*, \mu_{2t}^*\right] \\
= \; & A\left[u_2 + \tilde{a}(u_1; \varphi_{ot}^*, \mu_{1,t}^*); \varphi_{2t}^*, \mu_{2t}^*\right] \\
= \; & A_{2,t}^*[u_2 + a_{1,t}^*(u_1)] \text{ (say)}
\end{aligned}
$$

with

$$
\begin{aligned}
\varphi_{ot}^* &= \frac{\varphi_{ot}}{(1 - \Gamma_{1t}\,\mu_{1t})^2} \\
\mu_{1t}^* &= \frac{\mu_{1t}}{(1 - \Gamma_{1t}\mu_{1t})} \; .
\end{aligned}
$$

Similarly, we get:

$$\begin{aligned}
& b_{2,t}\left[u_2+\Gamma_{2t}+a_{1,t}(u_1+\Gamma_{1t})\right]-b_{2,t}(\Gamma_{2t}+a_{1,t}(\Gamma_{1t})) \\
= \;& \tilde{b}\left[u_2+\tilde{a}(u_1;\varphi^*_{ot},\mu^*_{1,t});\nu^*_{2t},\mu^*_{2t}\right] \\
= \;& b^*_{2,t}[u_2+a^*_{1t}(u_1)]\ (\text{say})\,,
\end{aligned}$$

$$\begin{aligned}
& b_{1,t}(u_1+\Gamma_{1t})-b_{1,t}(\Gamma_{1t}) \\
= \;& \tilde{b}_1(u_1;\nu^*_{1t},\mu^*_{1t}) \\
= \;& b^*_{1,t}(u_1)\ (\text{say})\,,
\end{aligned}$$

$$\begin{aligned}
& A_{1,t}(u_1+\Gamma_{1t})-\ A_{1,t}(\Gamma_{1t}) \\
= \;& A_1(u_1;\nu^*_{1t},\mu^*_{1t}) \\
= \;& A^*_{1,t}(u_1)\ (\text{say})\,,
\end{aligned}$$

with

$$\varphi^*_{1t} \;=\; \frac{\varphi_{1t}}{(1-\Gamma_{1t}\,\mu_{1t})^2}\,.$$

And finally, the joint conditional Laplace transform (90) becomes:

$$\begin{aligned}
& E^{\mathbb{Q}}_t[\exp(u_1x_{1,t+1}+u_2x_{2,t+1})\,|\,\underline{x_{1t}},\underline{x_{2t}},\underline{z_t}\,] \\
= \;& \exp\Big\{[A^*_{1,t}(u_1)+A^*_{2,t}[u_2+a^*_{1,t}(u_1)]]'\tilde{X}_t \\
& \qquad +b^*_{2,t}[u_2+a^*_{1,t}(u_1)]\;+\;b^*_{1,t}(u_1)\Big\}\,.
\end{aligned} \tag{91}$$

So, (91) has exactly the same form as (88) with different parameters. In other words the risk-neutral dynamics belongs to the same class as the historical one.

In order to have a Car process in the risk-neutral world, we know from section 2.9 that we must have the following constraint between the SDF and the historical dynamics:

i)

$$\frac{\mu_{1t}}{1-\Gamma_{1t}\mu_{1t}} = \mu_1^*$$

ii)

$$\frac{\varphi_{1t}}{(1-\Gamma_{1t}\mu_{1t})^2} \quad = \quad \varphi_1^*$$

iii)

$$\nu_1(Z_t) \quad = \quad \nu_1^{*'} Z_t$$

iv)

$$\frac{\varphi_{ot}}{(1-\Gamma_{1t}\mu_{1t})^2} \quad = \quad \varphi_o^*$$

v)

$$\frac{\mu_{2t}}{1-[\Gamma_{2t}+a_{1,t}(\Gamma_{1t})]\mu_{2t}} = \mu_2^*$$

vi)

$$\frac{\varphi_{2t}}{(1-[\Gamma_{2t}+a_{1,t}(\Gamma_{1t})]\mu_{2t})^2} \quad = \quad \varphi_2^*$$

vii)

$$\nu_2(Z_t) \quad = \quad \nu_2^{*'} Z_t \, .$$

Moreover, the constraint on the dynamics of the Markov chain are the same as in the gaussian case, namely:

viii)

$$\delta_j(\tilde{X}_t, Z_t) \quad = \quad \log\left[\frac{\pi(z_t, e_j\,; \tilde{X}_t)}{\pi^*(z_t, e_j)}\right] .$$

It is worth noting that, if there is no instantaneous causality between $x_{1,t+1}$ and $x_{2,t+1}$, that is if $\varphi_{ot} = 0$, function a_{1t} is also equal to zero and constraint v) and vi) are simpler and become similar to i) and ii).

4.9 The Generalised Linear Term Structure in the multifactor case

Using the notations:

$$\begin{aligned}
a_1^*(u_1) &= \tilde{a}(u_1; \varphi_o^*, \mu_1^*) \\
A_1^*(u_1) &= A(u_1; \varphi_1^*, \mu_1^*) \\
A_2^*(u_2) &= A(u_2; \varphi_2^*, \mu_2^*) \\
\tilde{C}_h &= (C_{2,h}, \ldots, C_{p,h}, 0, C_{p+2,h}, \ldots, C_{2p,h}, 0)',
\end{aligned}$$

we have

Proposition 12 : In the bivariate SAG(p) model the price at date t of the zero-coupon bond with residual maturity h is :

$$B(t,h) = \exp\left(C_h' \, \tilde{X}_t + D_h' \, Z_t\right) \,, \text{ for } h \geq 1 \tag{92}$$

where the vectors C_h and D_h satisfy the following recursive equations :

$$\begin{cases}
C_h = -c + A_1^*(C_{1,h-1}) + A_2^*[C_{p+1,h-1} + a_1^*(C_{1,h-1})] + \tilde{C}_{h-1} \\
D_h = -d - \nu_1^* \log(1 - C_{1,h-1}\mu_1^*) - \nu_2^* \log[1 - (C_{p+1,h-1} + a_1^*(C_{1,h-1}))\mu_2^*] \\
\qquad + \tilde{D}_{h-1} + F(D_{1,h-1}) \,,
\end{cases} \tag{93}$$

where $\tilde{D}_{h-1}$ and $F(D_{1,h-1})$ have the same meaning as in proposition 6; the initial conditions are $C_0 = 0$, $D_0 = 0$ (or $C_1 = -c$, $D_1 = -d$) [Proof : see Appendix 6].

So, proposition 12 shows that, also for the Switching VARG(p) model, yields to maturity are linear functions of $\tilde{X}_t$ and Z_t.

With regard to the endogenous case, where we can consider $x_{1t} = r_{t+1}$, and $x_{2t} = R(t, H)$ for a given time to maturity H, we have the same results as for the Switching VARN(p) case [see section 3.7].

It is also easily seen that the risk premium of the payoff $p_{t+1} = \exp(-\theta_1 x_{1,t+1}$

$-\theta_2 x_{2,t+1})$ is:

$$\begin{aligned}\omega_t(\theta_1,\theta_2) &= \{A_{2,t}[-\theta_2 + a_{1,t}(-\theta_1)] + A_{1,t}(-\theta_1) \\ &\quad - A_2^*[-\theta_2 + a_1^*(-\theta_1)] - A_1^*(-\theta_1)\}' X_t \\ &\quad + b_{2,t}[-\theta_2 + a_{1,t}(-\theta_1)] + b_{1,t}(-\theta_1) \\ &\quad - b_{2,t}^*[-\theta_2 + a_1^*(-\theta_1)] - b_{1,t}^*(-\theta_1)\,,\end{aligned}$$

with

$$\begin{aligned}b_{1,t}(u_1) &= -\nu_1^{*\prime} Z_t \log(1 - u_1\mu_1^*) \\ b_{2,t}(u_2) &= -\nu_2^{*\prime} Z_t \log(1 - u_2\mu_2^*)\,,\end{aligned}$$

and the risk premium of the digital asset is given once more by relation (47).

5 DERIVATIVE PRICING

5.1 Generalization of the recursive pricing formula

In the previous sections we have derived recursive formulas for the zero-coupon bond price $B(t,h)$ in various contexts which share the feature that the process $(\tilde{x}_t, z_t)$ is Car in the risk-neutral world. In fact the recursive approach can be generalized to other assets.

Let us consider a class of payoffs $g(\tilde{X}_{t+h}, Z_{t+h})$, (t,h) varying, for a given g function and let us assume that the price at t of this payoff is of the form:

$$P_t(g,h) = \exp\left[C_h(g)'\tilde{X}_t + D_h(g)'Z_t\right]\,. \tag{94}$$

It is clear that:

$$\begin{aligned}&\exp\left[C_h(g)'\tilde{X}_t + D_h(g)'Z_t\right] \\ =\ & E_t\left[M_{t,t+1}\exp\left(C_{h-1}(g)'\tilde{X}_{t+1} + D_{h-1}(g)'Z_{t+1}\right)\right] \\ =\ & \exp(-c'\tilde{X}_t - d'Z_t)E_t^{\mathbb{Q}}\left[\exp\left(C_{h-1}(g)'\tilde{X}_{t+1} + D_{h-1}(g)'Z_{t+1}\right)\right]\ ;\end{aligned}$$

so, for a given g function, the sequences $C_h(g), D_h(g), h \geq 1$, follow recursive equations which does not depend on g and, therefore, are identical to the case $g = 1$, that is to say to the zero-coupon bond pricing formulas given in the previous sections. The only condition for (94) to be true is to hold for $h = 1$ and, of course, this initial condition depends on g.

Formula (94) is valid for $h = 1$ if $g(\tilde{X}_{t+h}, Z_{t+h}) = \exp(\tilde{u}'\tilde{X}_{t+h} + \tilde{v}'Z_{t+h})$ for some vector $\tilde{u}$ and $\tilde{v}$. Indeed, using the notations

$$\begin{aligned} \tilde{u}'\tilde{X}_{t+1} &= u_1'\tilde{x}_{t+1} + u_{-1}'\tilde{X}_t \\ \tilde{v}'Z_{t+1} &= v_1'z_{t+1} + v_{-1}'Z_t\,, \end{aligned}$$

with $u_{-1}' = (u_2', \ldots, u_p', 0)$, $v_{-1}' = (v_2', \ldots, v_p', 0)$, we get:

$$\begin{aligned} P_t(\tilde{u}, \tilde{v}; 1) &= \exp(-c'\tilde{X}_t - d'Z_t + u_{-1}'\tilde{X}_t + v_{-1}'Z_t) \\ &\qquad \times E_t^{\mathbb{Q}}\left[\exp\left(u_1'\tilde{x}_{t+1} + v_1'z_{t+1}\right)\right]\,, \end{aligned} \tag{95}$$

which, using the Car representation of $(\tilde{x}_{t+1}, z_{t+1})$ under the probability $\mathbb{Q}$, has obviously the exponential linear form (94) and provides the initial conditions of the recursive equations. The standard recursive equation provide the price $P_t(\tilde{u}, \tilde{v}; h)$ at date t for the payoff $\exp(\tilde{u}'\tilde{X}_{t+h} + \tilde{v}'Z_{t+h})$. So we have the following proposition.

Proposition 13 : The price $P_t(\tilde{u}, \tilde{v}; h)$ at time t of the payoff $g(\tilde{X}_{t+h}, Z_{t+h}) = \exp(\tilde{u}'\tilde{X}_{t+h} + \tilde{v}'Z_{t+h})$ has the exponential form (94) where $C_h(g)$ and $D_h(g)$ follow the *same* recursive equations as in the zero-coupon bond case with initial values $C_1(g)$ and $D_1(g)$ given by the coefficients of $\tilde{X}_t$ and Z_t in the equation (95).

When $\tilde{u}$ and $\tilde{v}$ have complex components, $P_t(\tilde{u}, \tilde{v}; h)$ provides the complex Laplace transform $E_t[M_{t,t+h}\exp(\tilde{u}'\tilde{X}_{t+h} + \tilde{v}'Z_{t+h})]$.

5.2 Explicit and quasi explicit pricing formulas

The explicit formulas for zero-coupon bond prices also immediately provide explicit formulas for some derivatives like swaps. Moreover, the result of section 5.1, where $\tilde{u}$ and $\tilde{v}$ have complex components, can be used to price payoffs of the form:

$$\left[\exp(\tilde{u}_1'\tilde{X}_{t+h} + \tilde{v}_1'Z_{t+h}) - \exp(\tilde{u}_2'\tilde{X}_{t+h} + \tilde{v}_2'Z_{t+h})\right]^+\,,$$

like caps, floors or options on zero-coupon bonds. Let us consider, for instance, the problem to price, at date t, a European call option on the zero-coupon bond $B(t+h, H-h)$, then the pricing relation is :

$$\begin{aligned} p_t(K,h) &= E_t\left[M_{t,t+h}\left(B(t+h,H-h)-K\right)^+\right] \\ &= E_t\left[M_{t,t+h}\left(\exp[-(H-h)R(t+h,H-h)]-K\right)^+\right] \ , \end{aligned} \tag{96}$$

and, substituting here the yield to maturity formula (70), for the Switching VARN(p) model, or formula (92), for the Switching VARG(p) model, we can write :

$$\begin{aligned} p_t(K,h) &= E_t\left[M_{t,t+h}\left(\exp[C'_{H-h}\tilde{X}_{t+h}+D'_{H-h}Z_{t+h}]-K\right)^+\right] \\ &= E_t\left[M_{t,t+h}\left(\exp[C'_{H-h}\tilde{X}_{t+h}+D'_{H-h}Z_{t+h}]-K\right)\mathbb{I}_{[-C'_{H-h}\tilde{X}_{t+h}-D'_{H-h}Z_{t+h}<-\log K]}\right] \\ &= E_t\left[M_{t,t+h}\left(\exp[C'_{H-h}\tilde{X}_{t+h}+D'_{H-h}Z_{t+h}]\right)\mathbb{I}_{[-C'_{H-h}\tilde{X}_{t+h}-D'_{H-h}Z_{t+h}<-\log K]}\right] \\ &\qquad -KE_t\left[M_{t,t+h}\mathbb{I}_{[-C'_{H-h}\tilde{X}_{t+h}-D'_{H-h}Z_{t+h}<-\log K]}\right] \\ &= G_t(C_{H-h},D_{H-h},-C_{H-h},-D_{H-h},-\log K;h) \\ &\qquad -KG_t(\,0,0,-C_{H-h},-D_{H-h},-\log K;h)\,, \end{aligned} \tag{97}$$

where $\mathbb{I}$ denotes the indicator function, and where

$$\begin{aligned} & G_t(\tilde{u}_0,\tilde{v}_0,\tilde{u}_1,\tilde{v}_1,K;h) \\ = \ & E_t\left[M_{t,t+h}\left(\exp[\tilde{u}'_0\tilde{X}_{t+h}+\tilde{v}'_0 Z_{t+h}]\right)\mathbb{I}_{[-\tilde{u}'_1\tilde{X}_{t+h}-\tilde{v}'_1 Z_{t+h}<K]}\right] \end{aligned}$$

denotes the truncated real Laplace transform that we can deduce from the (untruncated) complex Laplace transform. In particular, we have the following formula :

$$\begin{aligned} G_t(\tilde{u}_0,\tilde{v}_0,\tilde{u}_1,\tilde{v}_1,K;h) &= \frac{P_t(\tilde{u}_0,\tilde{v}_0,h)}{2} \\ &\quad -\frac{1}{\pi}\int_0^{+\infty}\left[\frac{Im[P_t(\tilde{u}_0+i\tilde{u}_1 y,\tilde{v}_0+i\tilde{v}_1 y;h)]\exp(-iyK)}{y}\right]dy \end{aligned} \tag{98}$$

where $Im(z)$ denotes the imaginary part of the complex number z. So, formula (97) is quasi explicit since it only requires a simple (one-dimensional) integration to derives the values of G_t, which is equivalent to the computation of cumulative gaussian distribution function in the Black-Scholes model [see Duffie, Pan, Singleton (2000) for details].

6 Applications

7 Conclusions

Appendix 1

Proof of Proposition 6

$$
\begin{aligned}
B(t,h) &= \exp(C_h' X_t + D_h' Z_t) \\
&= \exp\left(-r_{t+1}\right) E_t^Q \left[B(t+1,h-1)\right] \\
&= \exp\left[-c' X_t - d' Z_t\right] \, E_t^Q \left[\exp\left(C_{h-1}' X_{t+1} + D_{h-1}' Z_{t+1}\right)\right] \\
&= \exp\left[-c' X_t - d' Z_t\right] \times \\
& \qquad E_t^Q \left[\exp\left(C_{h-1}' \left[\Phi^* X_t + \left(\nu^{*'} Z_t + \sigma^{*'} Z_t \, \xi_{t+1}\right) e_1\right] + D_{1,h-1}' z_{t+1} + \tilde{D}_{h-1}' Z_t\right)\right] \\
&= \exp\left[\left(\Phi^{*'} C_{h-1} - c\right)' X_t + \left(-d + C_{1,h-1}\nu^* + \tfrac{1}{2} C_{1,h-1}^2 \sigma^{*2} + \tilde{D}_{h-1}\right)' Z_t\right] \times \\
& \qquad E_t^Q \left[\exp\left(D_{1,h-1}' z_{t+1}\right)\right] \\
&= \exp\left\{\left(\Phi^{*'} C_{h-1} - c\right)' X_t + \right. \\
& \qquad \left. \left[-d + C_{1,h-1}\nu^* + \tfrac{1}{2} C_{1,h-1}^2 \sigma^{*2} + \tilde{D}_{h-1} + F(D_{1,h-1})\right]' Z_t\right\} ,
\end{aligned}
$$

and the result follows by identification.

Appendix 2

Proof of Proposition 7

Using the lag polynomials:

$$\begin{aligned} C_h(L) &= -\frac{1}{h}\,(C_{1,h} + C_{2,h}L + \ldots + C_{p,h}L^{p-1}) \\ D_h(L) &= -\frac{1}{h}\,(D_{1,h} + D_{2,h}L + \ldots + D_{p+1,h}L^{p}) \\ \Psi(L, Z_t) &= 1 - \varphi_1(Z_t)L - \ldots - \varphi_p(Z_t)L^p\,, \end{aligned}$$

we get from (55):

$$R(t,h) \;=\; C_h(L)x_t + D_h(L)' z_t\,,$$

and

$$\begin{aligned} \Psi(L, Z_t)\,R(t+1,h) &= C_h(L)\,\Psi(L, Z_t)\,x_{t+1} + D_h(L)\,\Psi(L, Z_t)\,z_{t+1}, \\ &= D_h(L)\,\Psi(L, Z_t)\,z_{t+1} + C_h(L)\,\nu(Z_t) + C_h(L)[(\sigma^{*\prime} Z_t)\,\varepsilon_{t+1}]\,. \end{aligned}$$

Appendix 3

Proof of Proposition 10

$$
\begin{aligned}
B(t,h) &= \exp(C_h'\tilde{X}_t + d_h' Z_t) \\
&= \exp\left(-r_{t+1}\right) E_t^Q\left[B(t+1,h-1)\right] \\
&= \exp\left[-c'\tilde{X}_t - d'Z_t\right]\ E_t^Q\left[\exp\left(C_{h-1}'\tilde{X}_{t+1} + D_{h-1}'Z_{t+1}\right)\right] \\
&= \exp\left[-c'\tilde{X}_t - d'Z_t\right] \times \\
&\quad E_t^Q\left[\exp\left(C_{h-1}'\tilde{\Phi}^*\tilde{X}_t + C_{1,h-1}(\nu_1^* Z_t + S_1^*(Z_t)\xi_{t+1})\right.\right. \\
&\quad\quad \left.\left. + C_{p+1,h-1}(\nu_2^* Z_t + S_2^*(Z_t)\xi_{t+1}) + D_{1,h-1}' z_{t+1} + \tilde{D}_{h-1}' Z_t\right)\right] \\
&= \exp\left[\left(\tilde{\Phi}^{*'} C_{h-1} - c\right)' X_t + \left[-d + C_{1,h-1}\nu_1^{*'} + \tfrac{1}{2}C_{1,h-1}^2(\sigma_1^{*2} + \varphi_o^{*2}\sigma_2^{*2})\right.\right. \\
&\quad\quad \left.\left. + C_{p+1,h-1}\nu_2^{*'} + \tfrac{1}{2}C_{p+1,h-1}^2\sigma_2^{*2} + \tilde{D}_{h-1} + F(D_{1,h-1})\right]' Z_t\right],
\end{aligned}
$$

and the result follows by identification.

Appendix 4

Proof of Lemma 1

$$
\begin{aligned}
\tilde{a}(u+\alpha;\rho,\mu) - \tilde{a}(\alpha;\rho,\mu) &= \frac{\rho(u+\alpha)}{1-(u+\alpha)\mu} - \frac{\rho\alpha}{1-\alpha\mu} \\
&= \rho\frac{u}{(1-\alpha\mu)^2 - u\mu(1-\alpha\mu)} \\
&= \frac{\rho}{(1-\alpha\mu)^2}\,\frac{u}{1-\frac{u\mu}{1-\alpha\mu}} \\
&= \frac{\rho^* u}{1-u\mu^*} = \tilde{a}(u;\rho^*,\mu^*)\,;
\end{aligned}
$$

$$
\begin{aligned}
\tilde{b}(u+\alpha;\nu,\mu) - \tilde{b}(\alpha;\nu,\mu) &= -\nu\log(1-(u+\alpha)\mu) + -\nu\log(1-\alpha\mu) \\
&= -\nu\log\left[\frac{1-(u+\alpha)\mu}{1-\alpha\mu}\right] \\
&= -\nu\log\left[1-\frac{u\mu}{1-\alpha\mu}\right] \\
&= -\nu\log(1-u\mu^*) \\
&= \tilde{b}(u;\nu,\mu^*)\,.
\end{aligned}
$$

Appendix 5

Proof of Proposition 11

$$
\begin{aligned}
B(t,h) &= \exp(C_h' X_t + D_h' Z_t) \\
&= \exp\left[-c' X_t - d' Z_t\right] E_t^Q \left[\exp\left(C_{h-1}' X_{t+1} + D_{h-1}' Z_{t+1}\right)\right] \\
&= \exp\Big(-c' X_t - d' Z_t + \tilde{C}_{h-1}' X_t + \tilde{D}_{h-1}' Z_t\Big) \\
&\qquad E_t^Q \Big[\exp\Big(C_{1,h-1} x_{t+1} + D_{1,h-1}' z_{t+1}\Big)\Big] \\
&= \exp\Big[-c' X_t - d' Z_t + \tilde{C}_{h-1}' X_t + \tilde{D}_{h-1}' Z_t + A^*(C_{1,h-1})' X_t \\
&\qquad -\nu^{*'} Z_t \log(1 - C_{1,h-1}\mu^*) + F'(D_{1,h-1}) Z_t\Big] ,
\end{aligned}
$$

and the result follows by identification.

Appendix 6

Proof of Proposition 12

$$
\begin{aligned}
B(t,h) &= \exp(C_h'\tilde{X}_t + D_h'Z_t) \\
&= \exp\left[-c'\tilde{X}_t - d'Z_t\right]\; E_t^Q\left[\exp\left(C_{h-1}'\tilde{X}_{t+1} + D_{h-1}'Z_{t+1}\right)\right] \\
&= \exp\left(-c'\tilde{X}_t - d'Z_t + \tilde{C}_{h-1}'\tilde{X}_t + \tilde{D}_{h-1}'Z_t\right) \\
&\qquad E_t^Q\left[\exp\left(C_{1,h-1}'x_{1,t+1} + C_{p+1,h-1}'x_{2,t+1} + D_{1,h-1}'z_{t+1}\right)\right] \\
&= \exp\Big[-c'\tilde{X}_t - d'Z_t + \tilde{C}_{h-1}'\tilde{X}_t + \tilde{D}_{h-1}'Z_t + A_1^*(C_{1,h-1})'\tilde{X}_t \\
&\qquad -\nu_1^{*'}Z_t\log(1 - C_{1,h-1}\mu_1^*) + A_2^*[C_{p+1,h-1} + a_1^*(C_{1,h-1})]'\tilde{X}_t \\
&\qquad -\nu_2^{*'}Z_t\log[1 - (C_{p+1,h-1} + a_1^*(C_{1,h-1}))\mu_2^*] + F'(D_{1,h-1})Z_t\Big]\;,
\end{aligned}
$$

and the result follows by identification.

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