

# Bivariate Generalizations of the ACD Models\*

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## Abstract

In this paper, several bivariate generalizations of the Autoregressive Conditional Duration model are proposed. An important feature shared by all the proposed models is that all of them allow for Granger non-causality analysis in a simple way. The models are illustrated using NYSE data on trades and quotes. Our results suggest that transactions Granger cause quotes, but not vice versa, which seems to confirm information-based microstructural theories

KEY WORDS: Tick by Tick Data, Non-Causality Analysis, Point Process

JEL Classification: C32, C41, G10

## 1 Introduction

Since the early nineties, automated electronic systems of trading have been introduced in most financial markets. These systems allow to record all the quotes posted and all the trades executed, along with their characteristics. The availability of these data has stimulated the development of new tools for the econometric analysis of the trading mechanisms, the intraday characteristics of the markets and the price formation process.

One relevant feature of these ultra high frequency data is that they are irregularly spaced over time, so that the usual time series methods can not be used. Engle and Russel (1998) introduce the ACD model as a statistical device to extract meaningful information out of the duration between two financial events. Different kinds of financial durations can be defined, such as the distance between trades or quotes or the distance between price changes. The duration of interest is regarded as a random variable, which can be studied conditionally on relevant related information by employing the tools of duration analysis. The ACD model assumes that the relevant information for the probability structure

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of future durations is given by the past evolution of the process. The basic version of the model assumes that durations are conditionally exponentially distributed with a mean that follows an ARMA process.

The Exponential distribution is extremely rigid, and therefore other versions of the ACD model have been also proposed, where other specifications of the conditional distribution of durations are assumed (generalized gamma, gamma, Burr, Weibull, exponential, Pareto).

The simple linear ARMA assumption for the conditional expectation has also been criticized and relaxed in the literature. The Log-ACD model is a logarithmic version which have been introduced by Bauwens and Giot (2000) to deal with the problem that, when explanatory variables are added linearly, the expected value may become negative, which is not admissible. Another variant is the threshold ACD (TACD by Zhang, Russel and Tsay, 1999), where instead of the simple linear specifications of the expected value assumed in the standard ACD setting, a more flexible specification is assumed, with different regimes which are allowed to have different duration persistence and error distribution. Fernandes and Grammig (2001) develop a family of ACD models based on Box and Cox (1964) transformation which encompasses many existing formulations existing in the literature. Jasiak (1998) introduces a class of fractionally integrated ACD models (FIACD) aimed at accounting for the highly persistent pattern of the autocorrelations of intertrade durations, displaying a slow, hyperbolic rate of decay, which is inconsistent with the exponential decline pattern implied by the ACD model and most of its derivations. Other specifications are derived by introducing a latent factor; this gives rise to the Stochastic Volatility Duration (SVD) model proposed by Ghysels, Gouriéroux and Jasiak (1997) and the Stochastic Conditional Duration (SCD) model proposed by Bauwens and Veredas (1999). Finally, Galli (2003) proposes a nonparametric approach to estimate the conditional expectation.

Other generalizations of the ACD models are aimed at extending the analysis towards a joint model of price dynamics and duration between financial events. Engle and Russell (1998b) develop a marked point process where price changes play the role of marks; the authors propose to decompose the joint distributions of arrival times and price changes into the products of the conditional distribution of price changes and the marginal distribution of arrival time, the latter being modelled as an ACD. Engle (2000) applies the ACD model to develop semiparametric hazard estimates and conditional intensities; combining these intensities with a GARCH model of prices gives rise to the UHF-GARCH model, where the dynamics of volatility are conditioned on transaction times. Ghysels and Jasiak (1998) develop a class of ARCH models for series sampled at unequal time intervals set by trade or quote arrivals. The class of models they introduce is called ACD-GARCH and can be described as a random coefficient GARCH, where the durations between transaction determine the parameter dynamics. Dufour and Engle (1999) use an ACD model to study the effect of trading intensity in the price formation process.

This paper tries to contribute to another niche of the literature on ultra high frequency financial data, namely multivariate ACD models. While the

literature on volatility processes abounds with multivariate specifications, in the ACD analysis of point processes the extensions in this direction are still a limited. The difficulties inherent to this analysis are mainly due to the fact that, while in an univariate point process durations can be easily ordered, so that only the duration of the spells and their relative positions need to be considered, in a multivariate setup a clear ordering of the durations of different processes is difficult to obtain and the study of the density of durations needs to be performed conditionally to calendar time too. As it is pointed out in Cox and Isham (1980), a point process can be analyzed under three main different perspectives: instantaneous probability of an occurrence (termed as intensity of the process), distribution of the number of occurrence in an arbitrarily fixed span of time, and distribution of the duration between events. The attempts to obtain a joint model of financial point processes have followed these three lines of research.<sup>1</sup>

The intensity of the multivariate process is the object of the analysis of Russell (1999), who proposes to model the instantaneous arrival rate conditionally of the multivariate filtration of arrival times and associated marks. Though the ensuing Autocorrelated Conditional Intensity (ACI) model is conceptually simple and appealing, it is analytically difficult and its estimation requires a great amount of computations. An interesting extension of this approach is provided by Bauwens and Hautsch (2002) who, in the Stochastic Conditional Intensity (SCI) model, add a latent factor to the specification of the conditional intensity function. This stochastic parametrization seems to improve the descriptive power of the model, though it obliges to recur to numerical methods in order to study the likelihood.

The Multivariate Autoregressive Conditional Poisson (MACP) model, a multivariate counting specification is proposed by Heinen and Rengifo (2003), who employ a Poisson and a double Poisson (with an additional parameter) distributions to model the number of financial events that take place during a fixed span of time. Conditionally on past observations, the vector of the means of the occurrences follow a VAR process. The interdependence between the different univariate processes is furtherly modelled via a multivariate normal copula, which introduces contemporaneous correlation between the series. The model can be estimated by maximum likelihood. This framework of analysis seems to be very flexible, as it can be applied to a series of point processes representing in principle any financial event.

For what concerns the direct study of the distribution of the durations between events (the very same perspective employed in the most of the literature on univariate processes), Engle and Lunde (1999) propose a censored bivariate ACD model. The goal of their work us to assess how quickly information in the transaction process impacts the prices via quote adjustments. Their primary interest lies therefore in the time between transactions, considered as the "driving process", and subsequent quote revisions. The two processes are not treated

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<sup>1</sup>Remember to quote Davis, Rydberg and Shephard (2001) and Spierdijk, Nijman and van Soest (2002).

symmetrically and some information is lost if multiple quote revisions occur without intervening transactions. This paper is in the same line, but avoids this asymmetry by extending to the dynamic framework the bivariate competing risks approach: whenever either event occurs, two spells (residual durations) start, which are referred to as *latent*: in fact they can not be both completely observed, since they censor each other. One key feature of the proposed model is that non-causality may be easily defined and tested for. Moreover, in the absence of causality in either direction, the model collapses into two univariate ACD models. Finally, our approach allows for a natural generalization to the  $k$ -variate setting ( $k > 2$ ). The model can also be extended in many of the directions illustrated above, by replacing the linear ARMA framework with non linear specifications of the conditional expectations or by considering joint models for price dynamics and bivariate durations between financial events: an attempt in the latter direction, in the line of Engle and Russell (1998b), is in Mosconi-Olivetti (2000).

The paper is organized as follows. Section 2 briefly describes the univariate ACD model, to set up the notation used in this paper. After a general introduction to the problems involved in the bivariate generalization of the ACD model, Section 3 introduces our basic bivariate ACD model, which is based on the simplifying assumptions of constant hazard and independence of the durations. These assumptions are relaxed in Section 4, where two alternative versions of the model are introduced based on the Weibull and Gumbel distributions respectively. Section 5 briefly illustrates how Engle and Russell (1998) approach to deal with the strong "time of the day" seasonality may be extended to the bivariate case. An empirical illustration of the three specifications is provided in Section 6, using data on trades and quotes extracted from the *TAQ* database, delivered by the *New York Stock Exchange*.

## 2 Univariate ACD modelling

Let  $X_i = T_i - T_{i-1}$  be the random variable which indicates the temporal interval between the  $i$ -th event (occurring at time  $T_i$ ) and the preceding one; we will refer to this variable with the name of *duration*. Engle and Russell (1998) define the model by the conditional intensity, conditioning on past realized durations. Let  $\psi_i$  be the  $i$ -th duration's conditional expected value and  $\mathcal{F}_i = \{X_i, X_{i-1}, \dots, X_1\}$  the filtration of the duration process up to time  $T_i$ :

$$\psi_i = E(X_i | \mathcal{F}_{i-1}; \theta) = \psi(x_{i-1}, x_{i-2}, \dots, x_1; \theta) \quad (1)$$

The dynamic specification can be generalized by introducing into (1) some marks associated to past realizations of the duration process.

Let us now assume:

$$X_i = \psi_i \varepsilon_i \quad (2)$$

where:

$$\varepsilon_i \sim i.i.d. \text{ with density } p_0(\varepsilon; \phi) \quad (3)$$

Given the multiplicative impact of the stochastic error  $\varepsilon_i$ ,  $p_0$  must have unit expectation.

In order to derive a general form for the conditional intensity, let us consider the hazard function  $\lambda_0(\cdot)$  associated with  $p_0(\cdot)$ ; since the model can be interpreted as an *accelerated life* model, it holds:

$$\lambda_i(x_i|F_{i-1}) = \lambda_0\left(\frac{x_i}{\psi_i}\right) \cdot \frac{1}{\psi_i}$$

where  $\lambda_i(\cdot)$  indicates the hazard function associated to the  $i$ -th duration. This expression is very useful since it allows to derive estimators for the parameters through the maximization of the log-likelihood function, which is given by:

$$\log L = \sum_{i=1}^{N(T)} \left( \log \lambda_i(x_i|F_{i-1}) - \int_0^{x_i} \lambda_i(u|F_i) du \right)$$

where  $N(t)$  is the number of transaction up to time  $t$ , and  $T$  is the end of the observation period.

Expressions (1)-(3) define the structure of the ACD model; the number of specifications is almost infinite, depending on both the functional form for  $\psi(\cdot)$  and the density for  $p_0(\cdot)$ . In the ACD( $m, q$ ) model introduced by Engle and Russell (1998), the following linear specification is assumed for (1):

$$\psi_i = \omega + \sum_{j=1}^m \alpha_j x_{i-j} + \sum_{j=1}^q \beta_j \psi_{i-j}$$

As shown by Engle and Russel, this is equivalent to the assumption of an ARMA( $\max(m, q), q$ ) specification for  $x_i$ . This is usually more parsimonious than the pure AR( $p$ ) specification

$$\psi_i = \omega + \sum_{j=1}^p \alpha_j x_{i-j}$$

The main problem with this specification is that it does not assure positivity of the conditional mean when at least one coefficient is negative. Actually, in many applications this is irrelevant, since values of coefficients and variables often keep the conditional mean far from zero. To address situations where negativity becomes a problem, Bauwens and Giot (2000) suggest either to restrict coefficients to be positive,<sup>2</sup> or to use a non-linear functional form, such as the exponential:

$$\psi_i = \exp \left( \omega + \sum_{j=1}^m \alpha_j \log x_{i-j} + \sum_{j=1}^q \beta_j \log \psi_{i-j} \right)$$

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<sup>2</sup>The major drawback of this solution is that it does not allow to test hypotheses on the coefficients' sign deriving from economic theory.

As for the distribution  $p_0(\cdot)$ , Engle and Russel propose both the Exponential and the Weibull distribution, which, combined with the linear form, define the EACD and the WACD model. It's easy to prove that also the conditional durations are Exponential and Weibull distributed respectively and that log-likelihood functions are:

$$\log L = - \sum_{i=1}^{N(T)} \left( \log \psi_i + \frac{x_i}{\psi_i} \right)$$

for the EACD model and

$$\log L = \sum_{i=1}^{N(T)} \left[ \log \frac{\gamma}{x_i} + \gamma \log \left( \frac{\Gamma(1 + 1/\gamma) x_i}{\psi_i} \right) - \left( \frac{\Gamma(1 + 1/\gamma) x_i}{\psi_i} \right)^\gamma \right]$$

for the WACD model, where  $\gamma$  is the shape parameter of the Weibull distribution and  $\Gamma(\cdot)$  is the gamma function. When  $\psi_i$  depends on its own past values, both the log-likelihood functions are recursive, and therefore have to be maximized numerically.

In many applications, durations behavior significantly depends on calendar time. For example, in stock markets intra-day seasonal effects cannot be ignored. In this case Engle and Russell propose to decompose the conditional mean into two components, one stochastic and one deterministic:

$$\psi_i = \phi_i(t_{i-1}; \theta_\phi) \tilde{\psi}_i(\tilde{x}_{i-1}, \tilde{x}_{i-2}, \dots, \tilde{x}_1; \theta_{\tilde{\psi}})$$

where  $\tilde{x}$  is the *diurnally adjusted* duration series ( $\tilde{x}_i = x_i / \phi_i(t_{i-1}; \theta_\phi)$ ) and  $\phi_i(\cdot)$  is usually specified by a cubic spline function (see Engle and Russell, 1998, for details).

### 3 Bivariate ACD modelling

In order to derive a bivariate generalization of the ACD model illustrated in the previous Section, let us introduce the following notation:

- $T_{1,i}$ : time of the  $i$ -th event of type 1 (e.g. transaction);
- $T_{2,i}$ : time of the  $i$ -th event of type 2 (e.g. quote);
- $X_{1,i} = T_{1,i} - T_{1,i-1}$ :  $i$ -th duration of the first process;
- $X_{2,i} = T_{2,i} - T_{2,i-1}$ :  $i$ -th duration of the second process;
- $N_1(t)$ : number of events of type 1 in  $(0; t]$ ;
- $N_2(t)$ : number of events of type 2 in  $(0; t]$ ;

It could seem natural to set up a bivariate model for  $X_{1,i}$  and  $X_{2,i}$  conditional on past observations of the same variables. Notice however that the average frequency of occurrence of the two events could be different, and therefore  $X_{1,i}$  and  $X_{2,i}$  could be far from each other in terms of calendar time. Such time mismatch makes any attempt to derive a joint model for  $X_{1,i}$  and  $X_{2,i}$  devoid of meaning. In order to find a more sound specification, we introduce the pooled process of events of type 1 or 2:

- $T_i$ : time of the  $i$ -th event of the pooled process (e.g. either transaction or quote);<sup>3</sup>
- $X_i = T_i - T_{i-1}$ :  $i$ -th duration of the pooled process (time between two consecutive events of any kind);
- $N(t)$ : number of events of the pooled process in  $(0; t]$ ;
- $N(t_1, t_2)$ : number of events of the pooled process in  $(t_1; t_2]$ ;
- $Y_i$ : dummy variable which is 1 if the  $i$ -th event of the pooled process is of type 1, and 0 otherwise;
- $\mathcal{F}_i = \{X_1, X_2, \dots, X_i, Y_1, Y_2, \dots, Y_i\}$ : filtration of the pooled process until  $T_i$ .

Next, in the line of the Competing Risks literature (see among others, Coxes and Oakes, 1984) we introduce the notion of *Latent Residual Duration* (LRD) for both events. The idea is that, when either event occurs, we start measuring the time to the next event of both types, i.e. the residual durations. The purpose is to measure such residual durations in the absence of the other event. The LRD is observable only for one of the two processes, namely the one occurring first. The other LRD is censored. More formally:

- $Z_{1,i}$ :  $i$ -th latent residual duration of the first process. If at  $T_i$  one event of the first type occurs,  $Z_{1,i}$  is observable and it takes on the value:  $Z_{1,i} = T_i - T_{i-1} = X_i$ . Conversely, if at  $T_i$  one event of the second type occurs, then  $Z_{1,i}$  is censored, since we observe only that  $Z_{1,i} > X_i$ ;
- $Z_{2,i}$ :  $i$ -th latent residual duration of the second process. If at  $T_i$  one event of the second type occurs,  $Z_{2,i}$  is observable and it takes on the value:  $Z_{2,i} = T_i - T_{i-1} = X_i$ . Conversely, if at  $T_i$  one event of the first type occurs, then  $Z_{2,i}$  is censored, since we observe that  $Z_{2,i} > X_i$ .

Figure 1 illustrates the relationship among the variables introduced above.

The reason for focusing on the unobserved LRD's rather than on the observed residual durations is that, whenever a new event of any type occurs, the filtration, and hence probabilistic structure of the process, is changed. This is

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<sup>3</sup>We assume that events of type 1 and 2 cannot occur simultaneously. Also, we exclude simultaneous occurrence of two or more events of the same kind. In other words, following Cox (...), we assume that both individual processes, as well as the pooled process, are orderly.

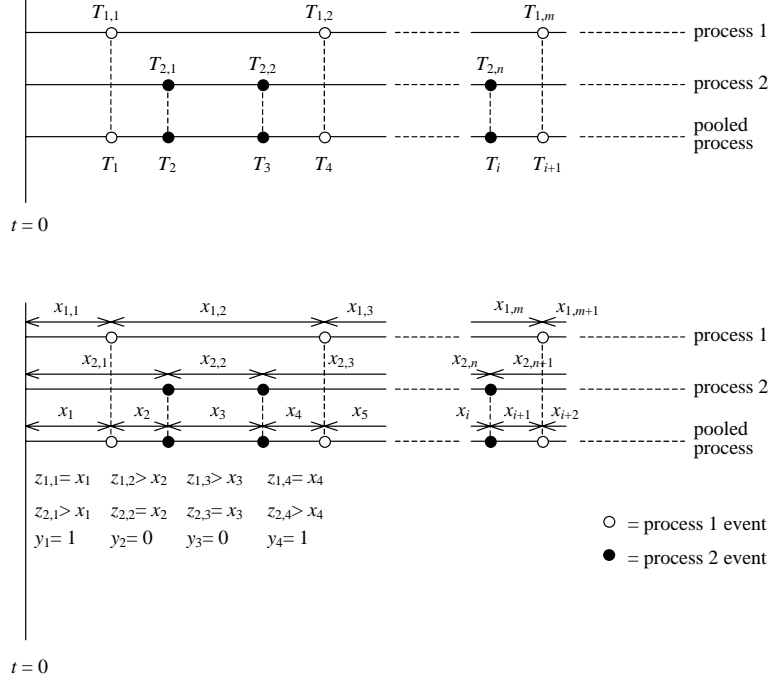


Figure 1: Illustration of the stochastic processes and notation

reflected in the proposed model, which is based on the specification of a joint distribution for the random variables  $Z_{1,i}$  e  $Z_{2,i}$ , conditioning on the filtration. Let's generically indicate this distribution in the following way:

$$f_{Z_{1,i}, Z_{2,i}}(z_{1,i}, z_{2,i} | \mathcal{F}_{i-1}; \theta) \quad (4)$$

where  $\theta$  is a set of parameters to be estimated. In order to define the likelihood function, let's introduce the following expressions:

$$\begin{aligned} c_{1,i}(k; \theta) &= \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \Pr(Z_{1,i} \in [k, k + \delta), Z_{2,i} \in (k, \infty) | \mathcal{F}_{i-1}; \theta) \quad (5) \\ &= \int_k^\infty f_{Z_{1,i}, Z_{2,i}}(k, z_2 | \mathcal{F}_{i-1}; \theta) dz_2 \end{aligned}$$

$$\begin{aligned} c_{2,i}(k; \theta) &= \lim_{\delta \rightarrow 0^+} \frac{1}{\delta} \Pr(Z_{1,i} \in (k, \infty), Z_{2,i} \in [k, k + \delta) | \mathcal{F}_{i-1}; \theta) \quad (6) \\ &= \int_k^\infty f_{Z_{1,i}, Z_{2,i}}(z_1, k | \mathcal{F}_{i-1}; \theta) dz_1 \end{aligned}$$



Expression (5) is the contribution to the likelihood function when one event of the first type is observed ( $Y_i = 1$ ), while (6) is the contribution to the likelihood function when one event of the second type is observed ( $Y_i = 0$ ). Therefore, the likelihood function for a sample of  $N(T)$  observations on  $(X_i, Y_i)$  is given by:

$$L(\theta) = \prod_{i=1}^{N(T)} c_{1,i}(x_i; \theta)^{y_i} c_{2,i}(x_i; \theta)^{(1-y_i)} \quad (7)$$

Parameter estimates may be obtained by maximizing (7) with respect to  $\theta$ .

### 3.1 Basic specification: EBACD

Within the framework illustrated above, we will now introduce our basic specification for (4), namely the Exponential Bivariate ACD, or EBACD. One key feature of the proposed model is that non-causality may be easily defined and tested for. Moreover, unlike other specifications proposed in the literature, in the absence of causality in either direction, the model collapses into two univariate ACD models. Finally, our approach allows for a natural generalization to the  $k$ -variate setting ( $k > 2$ ). The basic model is based on two simplifying assumptions, which will be relaxed in the next Section:

- independence of the Latent Residual Durations conditional on the history of the process
- constancy of the hazard function (i.e. exponential distribution for the LRD's)

More specifically  $Z_{1,i}$  and  $Z_{2,i}$  are assumed to be independent, conditionally on the filtration:

$$f_{Z_{1,i}, Z_{2,i}}(z_{1,i}, z_{2,i} | \mathcal{F}_{i-1}; \theta) = f_{Z_{1,i}}(z_{1,i} | \mathcal{F}_{i-1}; \theta_1) f_{Z_{2,i}}(z_{2,i} | \mathcal{F}_{i-1}; \theta_2) \quad (8)$$

where if  $\theta_1 \in \Theta_1$  and  $\theta_2 \in \Theta_2$  than  $\theta \in \Theta_1 \times \Theta_2$ . The independence assumption is rather unrealistic, and will be removed later.

Since our purpose is to generalize the univariate specification, it seems natural to make the same steps as in Engle and Russell (1998). Therefore, at first we propose the Exponential distribution for the LRD's:

$$\begin{aligned} Z_{1,i} | \mathcal{F}_{i-1} &\sim \text{Exp}\left(\frac{1}{\psi_{1,i}}\right) \\ Z_{2,i} | \mathcal{F}_{i-1} &\sim \text{Exp}\left(\frac{1}{\psi_{2,i}}\right) \end{aligned} \quad (9a)$$

where:

$$\begin{aligned} \psi_{1,i} &= E(Z_{1,i} | \mathcal{F}_{i-1}) = \psi_{1,i}(\mathcal{F}_{i-1}; \theta_1) \\ \psi_{2,i} &= E(Z_{2,i} | \mathcal{F}_{i-1}) = \psi_{2,i}(\mathcal{F}_{i-1}; \theta_2) \end{aligned} \quad (10)$$

As in the univariate case, a simple way to parametrize the expressions (10) is by specifying them as a linear function of past durations of both processes. For notational convenience, let us define the following variables:

- $x_{1,i}^{*(j)} = x_{1,N_1(T_{i-1})+1-j}$ :  $j$ -th last complete duration of process 1 when the  $i$ -th interval of the pooled process begins.
- $x_{2,i}^{*(j)} = x_{2,N_2(T_{i-1})+1-j}$ :  $j$ -th last complete durations of process 2 when the  $i$ -th interval of the pooled process begins.

Using this notation, an Auto Regressive like specification for the bivariate conditional expectation is given by:

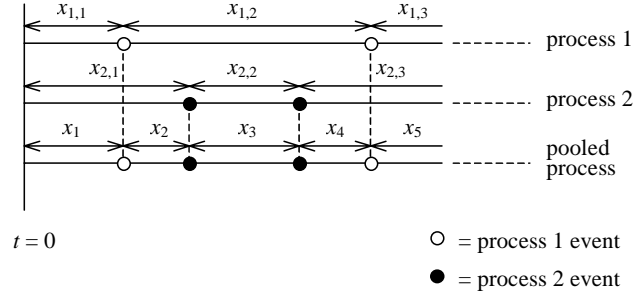
$$\begin{aligned}\psi_{1,i} &= \omega_1 + \sum_{j=1}^{m_{1,1}} \alpha_{1,1,j} x_{1,i}^{*(j)} + \sum_{j=1}^{m_{1,2}} \alpha_{1,2,j} x_{2,i}^{*(j)} \\ \psi_{2,i} &= \omega_2 + \sum_{j=1}^{m_{2,1}} \alpha_{2,1,j} x_{1,i}^{*(j)} + \sum_{j=1}^{m_{2,2}} \alpha_{2,2,j} x_{2,i}^{*(j)}\end{aligned}\quad (11)$$

where  $m_{a,b}$  is the number of autoregressive terms of process  $b$  into the conditional mean equation of process  $a$ . Notice that model (11) is not exactly an Auto Regressive model. In fact, the conditional expectation of the  $i$ -th LRD's are not defined in terms of past LRD's, since these are not observable, but rather in terms of past observed durations. However, this seems a natural way to use the information in  $\mathcal{F}_{i-1}$  in order to make predictions on the future. As in the univariate ACD models, a more parsimonious representation of the process might be obtained by including past conditional expectations into expressions (11), corresponding to past durations. One possible specification is the following

$$\begin{aligned}\psi_{1,i} &= \omega_1 + \sum_{j=1}^{m_{1,1}} \alpha_{1,1,j} x_{1,i}^{*(j)} + \sum_{j=1}^{m_{1,2}} \alpha_{1,2,j} x_{2,i}^{*(j)} + \sum_{j=1}^{q_{1,1}} \beta_{1,1,j} \psi_{1,i}^{*(j)} + \sum_{j=1}^{q_{1,2}} \beta_{1,2,j} \psi_{2,i}^{*(j)} \\ \psi_{2,i} &= \omega_2 + \sum_{j=1}^{m_{2,1}} \alpha_{2,1,j} x_{1,i}^{*(j)} + \sum_{j=1}^{m_{2,2}} \alpha_{2,2,j} x_{2,i}^{*(j)} + \sum_{j=1}^{q_{2,1}} \beta_{2,1,j} \psi_{1,i}^{*(j)} + \sum_{j=1}^{q_{2,2}} \beta_{2,2,j} \psi_{2,i}^{*(j)}\end{aligned}\quad (12)$$

where:

- $\psi_{1,i}^{*(j)} = E\left(Z_{1,i-N(T_{1,N_1(T_{i-1})-j}, T_{i-1})} \mid \mathcal{F}_{i-N(T_{1,N_1(T_{i-1})-j}, T_{i-1})-1}\right)$ : conditional expectation of the LRD of type 1 starting when the complete duration  $x_{1,i}^{*(j)} = x_{1,N_1(T_{i-1})+1-j}$  starts
- $\psi_{2,i}^{*(j)} = E\left(Z_{2,i-N(T_{2,N_2(T_{i-1})-j}, T_{i-1})} \mid \mathcal{F}_{2,i-N(T_{2,N_2(T_{i-1})-j}, T_{i-1})-1}\right)$ : conditional expectation of the LRD of type 2 starting when the complete duration  $x_{2,i}^{*(j)} = x_{2,N_2(T_{i-1})+1-j}$  starts;



$i$ -th period of the pooled process	$x_{1,i}^{*(1)}$	$x_{2,i}^{*(1)}$	$\psi_{1,i}^{*(1)}$	$\psi_{2,i}^{*(1)}$
1	$x_{1,0}$	$x_{2,0}$	$\psi_{1,0}$	$\psi_{2,0}$
2	$x_{1,1}$	$x_{2,0}$	$\psi_{1,1}$	$\psi_{2,0}$
3	$x_{1,1}$	$x_{2,1}$	$\psi_{1,1}$	$\psi_{2,1}$
4	$x_{1,1}$	$x_{2,2}$	$\psi_{1,1}$	$\psi_{2,3}$
5	$x_{1,2}$	$x_{2,2}$	$\psi_{1,2}$	$\psi_{2,3}$

Figure 2: Illustration of the notation

The notation is illustrated in Figure 2. Notice that, as discussed above, model (12) is not exactly an ARMA model, since the unobserved past LRD's are replaced by the corresponding past complete durations. Expressions (12), added to the Exponential and independence assumptions give the structure of this model which, in analogy with the univariate specification, we call Exponential Bivariate ACD, or EBACD( $m, q$ ), where  $m = \max(m_{i,j})$  and  $q = \max(q_{i,j})$ .

These structure allows to test for causality very easily; for instance, if we want to test for absence of causality of process 1 on process 2 and vice versa, we just have to test (with traditional techniques) the following hypothesis:

$$\alpha_{a,b,k} = \beta_{a,b,k} = 0 \quad \forall a, b, k; \quad a \neq b \quad (13)$$

When condition (13) holds, neither process causes the other, and the BACD collapses into two independent univariate ACD processes.

As in the univariate model, the main problem with the linear specification of (10) is that the conditional mean might turn negative if at least one coefficient is negative. In applications where this drawback is not irrelevant, we can use

other functional forms as, for instance:

$$\begin{aligned}\psi_{1,i} &= \exp \left( \begin{aligned} &\omega_1 + \sum_{j=1}^{m_{1,1}} \alpha_{1,1,j} \log x_{1,i}^{*(j)} + \sum_{j=1}^{m_{1,2}} \alpha_{1,2,j} \log x_{2,i}^{*(j)} + \\ &+ \sum_{j=1}^{q_{1,1}} \beta_{1,1,j} \log \psi_{1,i}^{*(j)} + \sum_{j=1}^{q_{1,2}} \beta_{1,2,j} \log \psi_{2,i}^{*(j)} \end{aligned} \right) \\ \psi_{2,i} &= \exp \left( \begin{aligned} &\omega_2 + \sum_{j=1}^{m_{2,1}} \alpha_{2,1,j} \log x_{1,i}^{*(j)} + \sum_{j=1}^{m_{2,2}} \alpha_{2,2,j} \log x_{2,i}^{*(j)} + \\ &+ \sum_{j=1}^{q_{2,1}} \beta_{2,1,j} \log \psi_{1,i}^{*(j)} + \sum_{j=1}^{q_{2,2}} \beta_{2,2,j} \log \psi_{2,i}^{*(j)} \end{aligned} \right) \end{aligned} \quad (14)$$

Whatever the functional form defined for the dynamic equations, the likelihood function is easily obtained by using the well known property of lack of memory of the Exponential distribution, which implies that if complete durations are Exponential distributed, residual durations are Exponential distributed as well. Therefore (7) becomes:

$$\begin{aligned}L &= \prod_{i=1}^{N(T)} f_{Z_{1,i}}(x_i)^{y_i} S_{Z_{1,i}}(x_i)^{(1-y_i)} f_{Z_{2,i}}(x_i)^{(1-y_i)} S_{Z_{2,i}}(x_i)^{y_i} = \\ &= \prod_{i=1}^{N(T)} \left( \frac{1}{\psi_{1,i}} \right)^{y_i} \left( \frac{1}{\psi_{2,i}} \right)^{1-y_i} \exp \left( -x_i \frac{\psi_{1,i} + \psi_{2,i}}{\psi_{1,i}\psi_{2,i}} \right)\end{aligned}$$

## 4 Extending the model

In this Sections we propose two alternative specifications for (4), relaxing the restrictive assumptions of the EBACD model in two directions. Both models share with the EBACD model the properties that non-causality may be easily defined and tested for and that in the absence of causality in either direction, the model collapses into two univariate ACD models.

### 4.1 Relaxing constancy of the hazard: WBACD

The exponential distribution, although is often used in applications for its analytical simplicity, is not always appropriate for empirical data, since it implies constancy of the hazard function. Following Engle and Russell (1998), our approach is generalized to the Weibull distribution. Weibull distribution does not share the property of lack of memory of the Exponential distribution; therefore, in order to develop a bivariate Weibull model, we need the probability function for the LRD's in the Weibull case. Let us assume the complete duration  $X$  is Weibull distributed:

$$f_X(x) = \kappa\gamma (\kappa x)^{\gamma-1} \exp[-(\kappa x)^\gamma]$$

then the residual duration  $Z = X - \bar{x}$  is distributed as follows:

$$f_Z(z) = \frac{f_X(x)}{S_X(\bar{x})} = \kappa\gamma [\kappa(z + \bar{x})]^{\gamma-1} \exp\{-\kappa^\gamma [(z + \bar{x})^\gamma - \bar{x}^\gamma]\} \quad (15)$$

We will refer to (15) as the Generalized Weibull distribution. Obviously, this distribution is Weibull only for  $\bar{x} = 0$ . The survival function for the Generalized Weibull is:

$$S_Z(z) = \exp\{-\kappa^\gamma [(z + \bar{x})^\gamma - \bar{x}^\gamma]\}$$

Expected residual duration can be easily obtained using the following relation, see Johnson, Kotz and Balakrishnan (1994), p. 630:

$$E(Z) = E(X - \bar{x} | X \geq \bar{x}) = \int_{\bar{x}}^{\infty} \frac{S_X(u)}{S_X(\bar{x})} du$$

which in our case gives:

$$E(Z) = \int_{\bar{x}}^{\infty} \frac{\exp[-(\kappa u)^\gamma]}{\exp[-(\kappa \bar{x})^\gamma]} du = \exp[(\kappa \bar{x})^\gamma] \int_{\bar{x}}^{\infty} \exp[-(\kappa u)^\gamma] du \quad (16)$$

Raja Rao and Talwalker (1989) derived upper and lower bound for this function. Formula (16) shows that the relation between expected residual duration and Weibull parameters is much more complex than for Exponential distribution. This makes the derivation of the bivariate Weibull ACD model rather complicated, even if there is no major conceptual difference with respect to the Exponential case. We will now show how is it possible to proceed operatively.

What follows is based on process 1, but obviously analogous remarks are valid for process 2 as well. This specification of the BACD model raises from the idea that complete durations are Weibull distributed. Let us assume that the distribution of the  $j$ -th complete duration of the process 1 is a Weibull of parameters  $\kappa_{1,j}$  and  $\gamma_{1,j}$ . In the spirit of the approach of Engle and Russell (1998), we assume the shape parameter  $\gamma_{1,j}$  (which makes the hazard function increasing or decreasing) to be constant for all the complete durations. Moreover, since the model is based on the recursive structure of the expected duration, the scale parameter  $\kappa_{1,j}$  is expressed as a function of the expected duration  $\psi_{1,j}$  and of the shape parameter:

$$\kappa_{1,j} = \frac{\Gamma(1 + 1/\gamma_1)}{\psi_{1,j}}$$

in order to have:

$$E(X_{1,j}) = \frac{\Gamma(1 + 1/\gamma_1)}{\kappa_{1,j}} = \psi_{1,j}$$

As we previously discussed, if complete durations are Weibull distributed, residual durations are Generalized Weibull distributed; so we have:

$$Z_{1,i} | F_{i-1} \sim GW(\kappa_{1,i}, \gamma_1, \bar{x}_{1,i})$$

where:

- $\gamma_1 = \text{constant}$  (shape parameter)
- $\kappa_{1,i} = \Gamma(1 + 1/\gamma_1) / \psi_{1,i}$  (scale parameter)

- $\bar{x}_{1,i} = T_{i-1} - T_{1,N_1(T_{i-1})}$  (portion of duration already completed)

We want to remark the role of  $\psi_{1,i}$ : in this case it is not the expected latent residual duration (as it is for the EBACD model), since it represents the expected value the duration would have if a portion of it should not be already completed (i.e. if  $\bar{x}_{1,i}$  were 0). In order to get the expected residual duration, the only way is to compute with numerical techniques the integral (16).

Let us complete the specification of this model, which we refer to as the WBACD model. Assuming

$$\begin{aligned} Z_{1,i}|F_{i-1} &\sim GW(\kappa_{1,i}, \gamma_1, \bar{x}_{1,i}) \\ Z_{2,i}|F_{i-1} &\sim GW(\kappa_{2,i}, \gamma_2, \bar{x}_{2,i}) \end{aligned}$$

and  $Z_{1,i}, Z_{2,i}$  independent (conditionally to past history), (4) becomes:

$$\begin{aligned} f_{Z_{1,i}, Z_{2,i}}(z_{1,i}, z_{2,i}|F_{i-1}; \theta) &= \kappa_{1,i} \gamma_1 \kappa_{2,i} \gamma_2 \\ &\quad [\kappa_{1,i} (z_{1,i} + \bar{x}_{1,i})]^{\gamma_1 - 1} [\kappa_{2,i} (z_{2,i} + \bar{x}_{2,i})]^{\gamma_2 - 1} \\ &\quad \exp\{-\kappa_{1,i}^{\gamma_1} [(z_{1,i} + \bar{x}_{1,i})^{\gamma_1} - \bar{x}_{1,i}^{\gamma_1}] \\ &\quad - \kappa_{2,i}^{\gamma_2} [(z_{2,i} + \bar{x}_{2,i})^{\gamma_2} - \bar{x}_{2,i}^{\gamma_2}]\} \end{aligned}$$

If  $\gamma_1 = \gamma_2 = 1$  we get back the EBACD model.  $\psi_{1,i}$  and  $\psi_{2,i}$  can be specified in terms of past history as discussed above for the EBACD model, i.e. by expressions (12) or (14). This allows correct testing for causality.

Given the independence assumption, the likelihood function becomes:

$$\begin{aligned} L &= \prod_{i=1}^{N(T)} f_{Z_{1,i}}(x_i)^{y_i} S_{Z_{1,i}}(x_i)^{(1-y_i)} f_{Z_{2,i}}(x_i)^{(1-y_i)} S_{Z_{2,i}}(x_i)^{y_i} = \\ &= \prod_{i=1}^{N(T)} \left\{ \kappa_{1,i} \gamma_1 [\kappa_{1,i} (x_i + \bar{x}_{1,i})]^{\gamma_1 - 1} \right\}^{y_i} \left\{ \kappa_{2,i} \gamma_2 [\kappa_{2,i} (x_i + \bar{x}_{2,i})]^{\gamma_2 - 1} \right\}^{(1-y_i)} \\ &\quad \exp\{-\kappa_{1,i}^{\gamma_1} [(x_i + \bar{x}_{1,i})^{\gamma_1} - \bar{x}_{1,i}^{\gamma_1}] - \kappa_{2,i}^{\gamma_2} [(x_i + \bar{x}_{2,i})^{\gamma_2} - \bar{x}_{2,i}^{\gamma_2}]\} \end{aligned}$$

## 4.2 Relaxing independence of LRD's: GBACD

The EBACD and WBACD specifications share the condition of independence of residual durations. This condition in some applications may result too restrictive, so we have developed a third formulation of the BACD model with potential correlation between residual durations. This was done generalizing the EBACD formulation by the second bivariate Exponential distribution of Gumbel (1960):

$$\begin{aligned} f_{Z_{1,i}, Z_{2,i}}(z_{1,i}, z_{2,i}|F_{i-1}; \theta) &= \frac{1}{\psi_{1,i} \psi_{2,i}} \exp\left[-\left(\frac{z_{1,i}}{\psi_{1,i}} + \frac{z_{2,i}}{\psi_{2,i}}\right)\right] \\ &\quad \left[1 + \alpha \left(2e^{-\frac{z_{1,i}}{\psi_{1,i}}} - 1\right) \left(2e^{-\frac{z_{2,i}}{\psi_{2,i}}} - 1\right)\right] \end{aligned}$$

with  $-1 \leq \alpha \leq 1$ . The parameter  $\alpha$  captures the potential correlation between residual duration and is related to the coefficient of linear correlation  $\rho$  simply by  $\rho = \alpha/4$ . So correlation is negative for  $-1 \leq \alpha \leq 0$  and positive for  $0 \leq \alpha \leq 1$ ; moreover, since  $|\alpha| \leq 1$ ,  $\rho$  can not exceed 0.25 and can not be lower than  $-0.25$ .

Conditional expected residual duration can be specified in terms of past history by equations (12) or (14), in order to complete the specification of the Gumbel BACD model (GBACD).

Contributes to likelihood function are:

$$\begin{aligned} c_{1,i}(x_i) &= \frac{1}{\psi_{1,i}} \exp \left[ - \left( \frac{\psi_{1,i} + \psi_{2,i}}{\psi_{1,i}\psi_{2,i}} \right) x_i \right] \left[ 1 + \alpha \left( 2e^{-\frac{x_i}{\psi_{1,i}}} - 1 \right) \left( e^{-\frac{x_i}{\psi_{2,i}}} - 1 \right) \right] \\ c_{2,i}(x_i) &= \frac{1}{\psi_{2,i}} \exp \left[ - \left( \frac{\psi_{1,i} + \psi_{2,i}}{\psi_{1,i}\psi_{2,i}} \right) x_i \right] \left[ 1 + \alpha \left( 2e^{-\frac{x_i}{\psi_{2,i}}} - 1 \right) \left( e^{-\frac{x_i}{\psi_{1,i}}} - 1 \right) \right] \end{aligned}$$

The resulting likelihood function is:

$$\begin{aligned} L &= \prod_{i=1}^{N(T)} \exp \left[ - \left( \frac{\psi_{1,i} + \psi_{2,i}}{\psi_{1,i}\psi_{2,i}} \right) x_i \right] \\ &\quad \left\{ \frac{1}{\psi_{1,i}} \left[ 1 + \alpha \left( 2e^{-\frac{x_i}{\psi_{1,i}}} - 1 \right) \left( e^{-\frac{x_i}{\psi_{2,i}}} - 1 \right) \right] \right\}^{y_i} \\ &\quad \left\{ \frac{1}{\psi_{2,i}} \left[ 1 + \alpha \left( 2e^{-\frac{x_i}{\psi_{2,i}}} - 1 \right) \left( e^{-\frac{x_i}{\psi_{1,i}}} - 1 \right) \right] \right\}^{(1-y_i)} \end{aligned}$$

## 5 Modelling the effect of calendar time in the bivariate framework

In this section we extend to the bivariate case the approach adopted by Engle and Russell (1998) in modelling the impact of calendar time on durations. Let us decompose the expected  $i$ -th residual duration of process  $a$  in a stochastic component  $\tilde{\psi}_{a,i}$  (explained by the BACD model) and a deterministic component  $\phi_a$  (function of the calendar time):

$$\begin{aligned} E(Z_{a,i}|F_{i-1}) &= \psi_{a,i} = \phi_a(t_{a,N_a(T_{i-1})}; \theta_{a,\phi}) \\ &\quad \tilde{\psi}_{a,i} \left( \tilde{x}_{1,N_1(T_{i-1})}, \dots, \tilde{x}_{1,1}, \tilde{x}_{2,N_2(T_{i-1})}, \dots, \tilde{x}_{2,1}; \theta_{a,\tilde{\psi}} \right) \end{aligned}$$

where  $\tilde{x}_{a,i_a}$  represents the  $i_a$ -th complete duration diurnally adjusted of process  $a$ :

$$\tilde{x}_{a,i_a} = x_{a,i_a} / \phi_a(t_{a,i_a-1}; \theta_{a,\phi})$$

We remark that the argument of  $\phi_a$  indicates when the  $i_a$ -th complete duration of process  $a$  begins: the diurnal factor, as in the univariate case, is assumed to

depend on the starting time of the duration and to be constant until the end of the duration itself. In the bivariate framework, this means that when an event of process  $b$  realizes, the diurnal factor for the residual duration of process  $a$  does not change. In this way, we can properly test for causality: infact, if causal relations are absent between processes, when an event of type  $b$  happens, the probability structure of the next event of type  $a$  does not change, not only for the autoregressive component, but for the deterministic one as well.

We will show now how to implement this method for a EBACD(1,1) model. The parameters in expressions (9a) become:

$$\begin{aligned}\psi_{1,i} &= \phi_1(t_{1,N_1(T_{i-1})}) \tilde{\psi}_{1,i} \\ \psi_{2,i} &= \phi_2(t_{2,N_2(T_{i-1})}) \tilde{\psi}_{2,i}\end{aligned}$$

Including diurnal factors, the model is even more complex than before, so we advise to estimate the two components separately, starting from the deterministic one. As in the univariate case, diurnal factors may be specified by a cubic spline function, subjected to continuity and derivability restrictions at each node. Cubic spline functions are:

$$\begin{aligned}\phi_1(t_{1,i_1-1}) &= \sum_{j=1}^k I_{j,i_1}^{(1)} \left[ \begin{array}{l} b_{0,j}^{(1)} + b_{1,j}^{(1)}(t_{1,i_1} - k_j) \\ + b_{2,j}^{(1)}(t_{1,i_1} - k_j)^2 + b_{3,j}^{(1)}(t_{1,i_1} - k_j)^3 \end{array} \right] \\ \phi_2(t_{2,i_2-1}) &= \sum_{j=1}^k I_{j,i_2}^{(2)} \left[ \begin{array}{l} b_{0,j}^{(2)} + b_{1,j}^{(2)}(t_{2,i_2} - k_j) \\ + b_{2,j}^{(2)}(t_{2,i_2} - k_j)^2 + b_{3,j}^{(2)}(t_{2,i_2} - k_j)^3 \end{array} \right]\end{aligned}\quad (17)$$

where  $k_j$  is the  $j$ -th node ( $j$ -th trading hour),  $I_{j,i_a}^{(a)}$  is a dummy which is 1 if the  $i_a$ -th complete duration of process  $a$  started between  $k_j$  and  $k_{j+1}$  (i.e.  $k_j \leq t_{a,i_a-1} \leq k_{j+1}$ ) and 0 otherwise.

Once estimated diurnal factors, we can specify properly the autoregressive components:

$$\begin{aligned}\tilde{\psi}_{1,i} &= \omega_1 + \alpha_{1,1} \tilde{x}_{1,i}^{*(1)} + \alpha_{1,2} \tilde{x}_{2,i}^{*(1)} + \beta_{1,1} \tilde{\psi}_{1,i}^{*(1)} + \beta_{1,2} \tilde{\psi}_{2,i}^{*(1)} \\ \tilde{\psi}_{2,i} &= \omega_2 + \alpha_{2,1} \tilde{x}_{1,i}^{*(1)} + \alpha_{2,2} \tilde{x}_{2,i}^{*(1)} + \beta_{2,1} \tilde{\psi}_{1,i}^{*(1)} + \beta_{2,2} \tilde{\psi}_{2,i}^{*(1)}\end{aligned}$$

where  $\tilde{x}_{a,i}^{*(1)}$  is last diurnally adjusted complete duration of process  $a$  when the  $i$ -th interval of the pooled process begins, i.e.  $x_{a,i}^{*(1)}/\phi_a(t_{a,N_a(T_{i-1})-1})$ , and  $\tilde{\psi}_{a,i}^{*(1)}$  the expected value of that duration when it started.

## 6 Application

The data used in the application are extracted from the database *Trade and Quote (TAQ)*, delivered by the *New York Stock Exchange*, which includes data



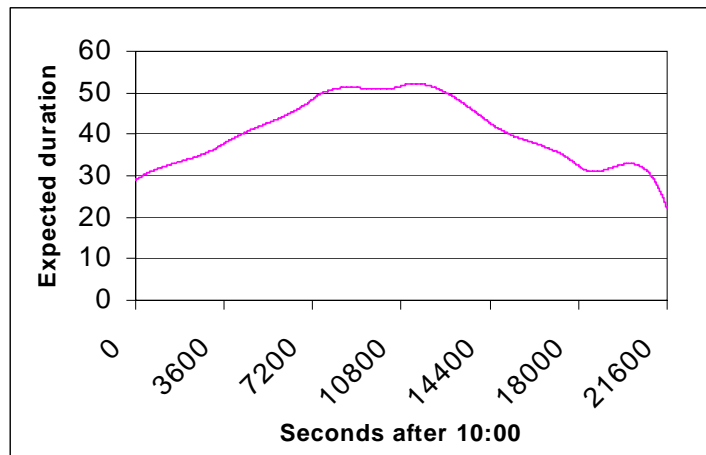


Figure 3: Intra-day pattern for transactions process

on single transaction and quote. Data refer to the asset *Federal National Mortgage (FNM)*, one of the highest capitalization security traded at *NYSE*. The sample is formed by the 42 trading days between 1st August 1997 and 30th September 1997. Trading procedures operating at *NYSE* are well discussed in Schwartz (1993) and Hasbrouck, Sofianos and Sosebee (1993).

Transaction data arising from initial batch auction were deleted: since sometimes the first transaction in continuous auction takes place after 9.45 (despite batch auction officially ends at 9.30), all data between 9.30 and 9.50 were deleted, while data between 9.50 and 10.00 were used to initialize both processes every day. Data after 16.00 (unusual transaction) were deleted as well.

Both stochastic processes present significative diurnal effects, so we need to diurnally adjust the data in order to properly apply the bivariate models specified above. Nodes were fixed at exact hours and expressed in terms of seconds after 10.00. OLS estimates of calendar time coefficients are given in appendix, while Figure 3 and Figure 4 show the intra-day pattern for transactions process and quotes process respectively. Both processes present more events at opening and closing time; patterns are similar, anyway they also show individual peculiarities which are properly caught by the modelling proposed above. Every

trading day has been initialized by restricting past durations in the autoregressive equations to be equal to the mean of the durations (for each process) realized from 9.50 to 10.00. Past conditional expected values in the same equations have been fixed to 1, that is the non conditional expected value of the autoregressive component.

Transactions and quotes in the *TAQ* database are recorded with approximation to the nearest second, which is fine enough to apply our model efficiently, but cannot exclude contemporaneous events. Two or more events of the same

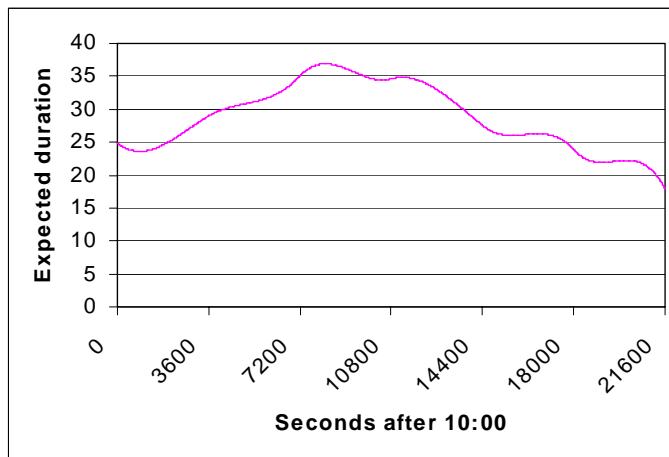


Figure 4: Intra-day pattern for quotes process

	Transactions	Quotes
Observations	22975	32114
Sample Mean	39.4	28.2
Standard Deviation	46.8	34.5
Skewness	2.88	3.61
Kurtosis	13.9	26.1

Table 1: Descriptive statistics for transactions and quotes.

kind must be treated carefully, since they are theoretically excluded in ACD modelling, see Engle and Russell (1998) for details. In our sample we have 18 simultaneous transactions (out of almost 23000) and about 300 simultaneous quotes (out of over 32000). Joint events of different types occur approximately 400 times, and could be handled by making the bivariate model more complicated. However, we preferred to handle multiple events of all kinds by adding a random draw from a  $\text{Uniform}(-0.5, 0.5)$  to the time of all simultaneous events. This gives a random ranking and sets to zero the probability of ties.<sup>4</sup> The number of multiple events is so small that such approximation does not have any impact on results arising from our models. Descriptive statistics are shown in Table 1, where we can notice that new quotes are on average more frequent than transactions.

Our bivariate model is applied to test for causality relations between transactions and quotes processes. Estimates were obtained using the numerical

<sup>4</sup>An alternative way to handle ties could be to aggregate multiple transactions since they often represents *block trade*. Simultaneous quote revisions could be aggregated as well: in fact we noticed that they usually consisted in two separated revision, one for prices and one for quantities, so aggregation should not cause any loss of information.

COEFF.	ESTIMATE	STD DEV.	T-TEST	p-value
$\omega_1$	0.00771	0.00132	5.83	0.000
$\omega_2$	0.0255	0.00304	8.37	0.000
$\alpha_{1,1}$	0.0195	0.00148	13.11	0.000
$\alpha_{1,2}$	0.00201	0.000946	2.12	0.034
$\alpha_{2,1}$	0.00110	0.00117	0.94	0.345
$\alpha_{2,2}$	0.0624	0.00240	26.06	0.000
$\beta_{1,1}$	0.975	0.00238	409.19	0.000
$\beta_{1,2}$	-0.00392	0.00213	-1.84	0.066
$\beta_{2,1}$	0.0141	0.00380	3.72	0.000
$\beta_{2,2}$	0.895	0.00477	187.7	0.000

Table 2: Estimates of EBACD(1,1) model.

optimization algorithm of Berndt, Hall, Hall, Hausmann (1974) implemented in RATS. In order to reduce complexity, each bivariate specification was taken with just one autoregressive element of every kind, i.e. only (1,1) models were estimated. EBACD(1,1) ML estimates, based on 52600 observations on the pooled process, are shown in Table 2.

Two of the coefficients ( $\alpha_{1,2}$  and  $\beta_{2,1}$ ) which incorporate the potential causality relation between this processes are significative and positive, so when durations of one process are shorter, realizations of the other process tend to be more frequent as well. Moreover all  $\alpha_{k,k}$  and  $\beta_{k,k}$  are positive and highly significative, which means that both processes are subjected to clustering of events. In particular,  $\beta_{1,1}$  and  $\beta_{2,2}$  are close to 1, so that high persistence is found, as it was in Engle and Russell (1998).

Let us discuss causality test results. First we have tested for the absence of causality in both directions ( $H_0 : \alpha_{1,2} = \beta_{1,2} = \alpha_{2,1} = \beta_{2,1} = 0$ ). Maximized log-likelihood under the null is  $-245291.308$ , so that the  $LR$  test rejects null hypothesis at usual significance level (p-value: 0.00138). We have also tested for unidirectional non-causality, i.e. "transactions do not cause quotes" ( $\alpha_{2,1} = \beta_{2,1} = 0$ ) and "quotes do not cause transactions" ( $\alpha_{1,2} = \beta_{1,2} = 0$ ). Maximized log-likelihood under the null are  $-245288.403$  and  $-245284.298$  respectively, so that the first  $LR$  test rejects (p-value: 0.00255), while the second does not (p-value: 0.15493).

This results suggest that the transactions process Granger causes the quotes process, but not vice versa. The first result seems to confirm information-based microstructural theories, since it seems that quote revision frequency depends on the observed transaction density.<sup>5</sup> The second result also seems consistent with information-based theories, since in this paradigm informed traders trade only when they have private information at their disposal (supposed exogenous to the events observed in the market), while trade times of liquidity traders are

<sup>5</sup>Notice that this is not the same as Easley and O'Hara (1992) hypothesis, which implies that transaction density has an impact on the Bid-Ask spread. This theory can be empirically tested by an extension of the BACD model, which is presented in Mosconi-Olivetti (2000).

COEFF.	ESTIMATE	STD DEV.	T-TEST	p-value
$\omega_1$	0.00769	0.00138	5.56	0.000
$\omega_2$	0.0255	0.00316	8.07	0.000
$\alpha_{1,1}$	0.0194	0.00154	12.55	0.000
$\alpha_{1,2}$	0.00194	0.000988	1.97	0.049
$\alpha_{2,1}$	0.00100	0.00122	0.83	0.409
$\alpha_{2,2}$	0.0624	0.00250	24.00	0.000
$\beta_{1,1}$	0.975	0.00250	390.62	0.000
$\beta_{1,2}$	-0.00379	0.00223	-1.70	0.090
$\beta_{2,1}$	0.0143	0.00340	3.63	0.000
$\beta_{2,2}$	0.895	0.00497	180.03	0.000
$\gamma_1$	0.972	0.00569	170.73	0.000
$\gamma_2$	0.974	0.00421	231.57	0.000

Table 3: Estimates of WBACD(1,1) model.

supposed totally random. Given these assumption, it seems natural that the quotes process does not have any effect on the transactions process.

The same topics were analyzed with the WBACD(1,1) specification, whose estimates are given in Table 3.  $\gamma_1$  and  $\gamma_2$  are close to 1, but the hypotheses  $H_0 : \gamma_1 = \gamma_2 = 1$ ,  $H_0 : \gamma_1 = 1$  and  $H_0 : \gamma_2 = 1$  are all rejected, so that neither duration process appears to be exponentially distributed. Causality tests yielded the same results of the EBACD model: absence of causality in both directions is rejected (p-value: 0.00232), "transactions do not cause quotes" is rejected (p-value: 0.00349) and "quotes do not cause transactions" is not rejected (p-value: 0.19124).

GBACD(1,1) yielded slightly different results. First of all, remind that  $\alpha$  must be between  $-1$  and  $1$ ; therefore we rewrite the model in terms of the unrestricted parameter  $\alpha'$  which is related to  $\alpha$  by:

$$\alpha = \frac{2}{1 + e^{\alpha'}} - 1$$

Notice that when  $\alpha'$  is 0,  $\alpha$  is 0 as well. Maximum likelihood estimates are given in Table 4. The correlation parameter  $\alpha$  tends to his upper limit and the null  $\alpha = 0$  is strongly rejected. Positive correlation between transactions and quotes implies that, even conditioning on past information  $F_{i-1}$ , shorter transaction durations are associated with more frequent quotes revisions. An economic interpretation of this result may be found if we accept that the transaction density is related to informative factors. In this case, a positive correlation between transactions and quotes indicates that the specialist is an informed agent, whose behaviour partly reflects the same information set which drives the traders' behaviour. In other words, when new information is available, informed traders exploit their informative advantage by trading more frequently; on the other hand the specialist, who shares at least part of the private information, reviews

COEFF.	ESTIMATE	STD DEV.	T-TEST	p-value
$\omega_1$	0.00981	0.00130	7.540	0.000
$\omega_2$	0.0221	0.00215	10.270	0.000
$\alpha_{1,1}$	0.0194	0.00133	14.618	0.000
$\alpha_{1,2}$	0.00278	0.000835	3.327	0.001
$\alpha_{2,1}$	0.00208	0.000952	2.189	0.029
$\alpha_{2,2}$	0.0501	0.00188	26.573	0.000
$\beta_{1,1}$	0.967	0.00266	363.260	0.000
$\beta_{1,2}$	-0.00449	0.00203	-2.217	0.027
$\beta_{2,1}$	0.00803	0.00324	2.475	0.013
$\beta_{2,2}$	0.912	0.00390	233.474	0.000
$\alpha'$	-31.936	0.000	0.000	0.000
$\alpha$	1.000			

Table 4: Estimates of GBACD(1,1) model.

his quotes more frequently. Notice that this interpretation is in contrast with the assumption of non-informed specialist, which is standard in information-based theory.

Let us now discuss whether the GBACD model confirms the causality relations discussed above. Absence of causality, "transactions do not cause quotes" and "quotes do not cause transactions" null hypotheses are all rejected (p-values are 0.00022, 0.00415, 0.00732, respectively). This means that in this case there is also a significant impact of quotes process on transactions process. We can interpret this result as follows: if the specialist is an informed agent (as suggested by the correlation result), it seems reasonable that traders may infer from quotes revision frequency whether the specialist holds private information or not, and behave consequently. This is the same logic used by the specialist inferring private information from transaction density. According to this interpretation, the usual microstructural assumptions of non-informed specialist and independence of trading activity on specialist's behavior does not seem to be empirically confirmed.

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