Scalar BEKK and Indirect DCC

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Abstract: The paper derives the scalar special case of the BEKK model of Engle and Kroner (1995) using a multivariate extension of the Random Coefficient Autoregressive (RCA) model of Tsay (1987). This representation establishes the relevant structural and asymptotic properties of the scalar BEKK model using the theoretical results in Comte and Lieberman (2003) and Ling and McAleer (2003). Sufficient conditions for the direct DCC model of Engle (2002) to be consistent with a scalar BEKK representation are established. Moreover, an indirect DCC model that is consistent with the scalar BEKK representation is obtained, and is compared with the direct DCC model using an empirical example. It is found that the scalar BEKK model, and hence indirect DCC, generally provides less volatile estimates of the conditional variances and correlations than does the direct DCC model.

Keywords: Asymptotic properties, BEKK, Direct DCC, Dynamic conditional correlations, Dynamic conditional covariances, Indirect DCC, Risk management, Structural properties.

JEL codes: C32, C51, C52

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1. Introduction

Over the last decade, the multivariate GARCH literature has expanded significantly, with many new models and empirical applications (see McAleer (2005) and Bauwens, Laurent and Rombouts (2005) for recent discussions). The modelling of conditional correlations has attracted particular interest, given their relevance for portfolio allocation, risk measurement and management, and the forecasting of Value-at-Risk (VaR) thresholds according to the principles of the Basel Accord. In the area of time-varying conditional correlations, the simplest and possibly the most popular specification is the Dynamic Conditional Correlation (DCC) model of Engle (2002).

Several recent multivariate conditional volatility models have attracted considerable interest, especially given the smaller number of parameters as compared with the highly overparameterized multivariate BEKK and Vech models of Engle and Kroner (1995). However, they do not represent preferred solutions if the primary interest is to evaluate and measure risk rather than selecting an optimal portfolio. Regardless of whether dynamic conditional correlation models are useful for portfolio allocation and management, they are not as useful for risk evaluation and measurement, where conditional variances play a prominent role. The exclusion of any spillover effect among conditional variances and covariances, which is standard in DCC-type models, may lead to serious biases in the estimates and outcomes.

Within the risk measurement framework, the relationship between the scalar BEKK and direct DCC models are established. This representation establishes the relevant structural and asymptotic properties of scalar BEKK using the theoretical results in Comte and Lieberman (2003) and Ling and McAleer (2003). Sufficient conditions for the direct DCC model of Engle (2002) to be consistent with a scalar BEKK representation are established, and an indirect DCC model that is implied by the scalar BEKK representation is obtained. Finally, the direct and indirect DCC models are compared empirically.

The plan of the remainder of the paper is as follows. Section 2 presents a derivation of the scalar BEKK model from a multivariate extension of the Random Coefficient Autoregressive (RCA) model of Tsay (1987). In Section 3, the scalar BEKK (and hence indirect DCC) and direct DCC models are compared using an empirical example based on the DAX, CAC40 and FTSE100 stock market indexes.

2. Scalar BEKK and Dynamic Correlations

Let y_t represent an $m \times 1$ vector of asset returns, $E(y_t \mid I_{t-1})$ the conditional mean, ε_t the random shock to returns, and I_{t-1} the information set at time $t-1$. The typical model of multivariate returns and risk can be represented as follows:

$$y_t = E(y_t \mid I_{t-1}) + \varepsilon_t \tag{1a}$$

$$\varepsilon_t = D_t \eta_t \tag{1b}$$

$$Q_t = D_t \Gamma_t D_t \tag{1c}$$

where $D_t = diag(h_{1t}, h_{2t}, ..., h_{mt})$, $h_{it} = E(\varepsilon_{it}^2 \mid I_{t-1})$, $Q_t = E(\varepsilon_t \varepsilon_t^{'} \mid I_{t-1})$ is the matrix of conditional covariances, $\eta_t = D_t^{-1/2} \varepsilon_t$ and $\Gamma_t = E(\eta_t \eta_t^{'} \mid I_{t-1})$ is the matrix of conditional correlations.

The BEKK model assumes that the dynamic positive definite conditional covariance matrix is given as follows:

$$Q_t = QQ^{'} + A\varepsilon_{t-1}\varepsilon_{t-1}^{'}A^{'} + BQ_{t-1}B^{'} \tag{2}$$

where A and B are square coefficient matrices, and Q is a triangular coefficient matrix. On the other hand, the DCC model estimates the univariate GARCH(1,1) model in the first step. In the second step, DCC is estimated using the GARCH(1,1) standardized residuals. However, in order to interpret the dynamic components as valid conditional correlations, an appropriate standardization is required. The standardized conditional correlations are given as:

$$\Gamma_t = (1 - \alpha - \beta)\Gamma + \alpha \eta_{t-1}\eta_{t-1}^{'} + \beta \Gamma_{t-1} \tag{3}$$

where α and β are scalar parameters, and Γ is the constant conditional correlation matrix in the absence of dynamics. The dynamic conditional correlations are finally given by:

$$\tilde{\Gamma}_t = \left(\Gamma_t^*\right)^{-1} \Gamma_t \left(\Gamma_t^*\right)^{-1},$$

where

$$\Gamma_t^* = diag\left(\sqrt{\Gamma_{11,t}}, \sqrt{\Gamma_{22,t}}, \ldots \sqrt{\Gamma_{kk,t}}\right).$$

McAleer et al. (2005) show that the multivariate BEKK model can be obtained from a Vector Random Coefficient Autoregressive (VRCA) process, namely:

$$\varepsilon_t = A_t \varepsilon_{t-1} + \xi_t, \quad A_t \sim IID(0, A), \quad \xi_t \sim IID(0, QQ'). \tag{4}$$

Given the VRCA process for the unconditional shocks in equation (1a), the moment conditions and asymptotic theory follow directly from the theoretical results in Comte and Lieberman (2003) and Ling and McAleer (2003). In particular, for the scalar BEKK model considered in this paper, it can be shown that:

$$A = \theta_1^{1/2} I_m, \quad B = \theta_2^{1/2} I_m,$$
$$Q_t = QQ' + \theta_1 \varepsilon_{t-1} \varepsilon_{t-1}{}' + \theta_2 Q_{t-1},$$

which follows directly from the scalar VRCA model, which is given for the AR(1) process as:

$$\varepsilon_t = \alpha_t \varepsilon_{t-1} + v_t,$$
$$\alpha_t \sim IID\left(0, \alpha\right),$$
$$v_t \sim IID\left(0, V\right),$$

it follows that

$$Var\left(\varepsilon_t \mid I_{t-1}\right) = Q_t = \alpha \varepsilon_{t-1} \varepsilon_{t-1}{}' + V.$$

Replacing the AR(1) with an AR(∞) VRCA process yields the scalar BEKK model, with a transformation of V as QQ'.

From (1c), it follows that Γ_t can be obtained indirectly from Q_t as:

$$\Gamma_t = D_t^{-1} Q_t D_t^{-1}. \tag{5}$$

Substitution of (2) into (5) gives the indirect DCC model, as follows:

$$\Gamma_t = (D_t^{-1}Q)(D_t^{-1}Q)' + (D_t^{-1}A\varepsilon_{t-1})(D_t^{-1}A\varepsilon_{t-1})' + (D_t^{-1}BQ_{t-1}^{1/2})(D_t^{-1}BQ_{t-1}^{1/2})', \qquad (6)$$

where $Q_{t-1}^{1/2}Q_{t-1}^{1/2} = Q_t$. The conditional correlation matrix in (6) is structurally valid and directly interpretable as it has been derived from a VRCA process. Thus, Γ_t in (6) has explicit regularity conditions and asymptotic properties for the Quasi-Maximum Likelihood Estimates (QMLE) in the absence of multivariate normality of the vector of standardized residuals. It is worth comparing Γ_t in (3) and (6), namely the direct and indirect DCC specifications, as follows:

$$(1-\alpha-\beta)\Gamma = (D_t^{-1}Q)(D_t^{-1}Q)', \qquad (7a)$$

$$\alpha\eta_{t-1}\eta_{t-1}' = (D_t^{-1}A\varepsilon_{t-1})(D_t^{-1}A\varepsilon_{t-1})', \qquad (7b)$$

$$\beta\Gamma_{t-1} = (D_t^{-1}BQ_{t-1}^{1/2})(D_t^{-1}BQ_{t-1}^{1/2})'. \qquad (7c)$$

When $\alpha = \beta = 0$ in (7a), D_t^{-1} is also a matrix of constants. However, the restrictions on the parameters and variables inherent in (7b)-(7c) to obtain the direct DCC model from scalar BEKK are infeasible as:

(i) except when $\alpha = 0$ and $A = 0$, (7b) is not consistent with the definition of the vector of standardized residuals, η_t;

(ii) except when $\beta = 0$ and $B = 0$, (7c) is not consistent with the definition of the vector of standardized residuals, η_t.

Therefore, the direct DCC model cannot be derived from a Vector Random Coefficient Autoregressive process with valid restrictions on the parameters and variables, which distinguished it from the indirect DCC model. Thus, it would seem that the direct DCC model is unlikely to have any valid moment conditions or asymptotic properties.

3. Empirical Comparison

Three stock market indexes are selected to compare the dynamic conditional correlations derived from the direct and indirect DCC models, namely DAX, CAC40 and FTSE100. The sample is restricted to the daily closing market indexes from January 1999 to September 2005, giving a total of 1733 days. The daily logarithmic returns are calculated, and missing observations (due to different holidays) are replaced with zero returns.

In order to filter out the mean dynamics, a VAR(1) model was fitted to the returns to compute the mean residuals, which are the object of the empirical exercise. Two models were estimated for the conditional variances and correlations, namely the scalar BEKK (or indirect DCC) model for the conditional covariances and the direct DCC model with GARCH(1,1) for the conditional variances. With the conditional first and second moments given as $E\left[\varepsilon_t \mid I_{t-1}\right] = 0$ and $E\left[\varepsilon_t \varepsilon_t' \mid I_{t-1}\right] = \Sigma_t$, respectively, the direct and indirect DCC models are given as follows:

Scalar BEKK (Indirect DCC)

$$\Sigma_t = \left(1-\alpha-\beta\right)\Sigma + \alpha\varepsilon_{t-1}\varepsilon_{t-1}' + \beta\Sigma_{t-1},$$
$$\Sigma = T^{-1}\sum_{t=1}^{T}\varepsilon_t\varepsilon_t'.$$

Direct DCC

$$\Sigma_t = D_t\Gamma_t D_t, \quad D_t = diag\left(\sigma_{1,t}, \sigma_{2,t}, ..., \sigma_{k,t}\right),$$
$$\sigma_{j,t}^2 = \omega_j + \alpha_j\varepsilon_{j,t-1}^2 + \beta_j\sigma_{j,t-1}^2, \quad \eta_t = D_t^{-1}\varepsilon_t,$$
$$Q_t = \left(1-\theta_1-\theta_2\right)\Gamma + \theta_1\eta_{t-1}\eta_{t-1}' + \theta_2 Q_{t-1},$$
$$\Gamma = T^{-1}\sum_{j=1}^{T}\eta_t\eta_t', \quad \Gamma_t = \bar{Q}_t^{-1}Q_t\bar{Q}_t^{-1}, \quad \bar{Q}_t = diag\left(\sqrt{q_{11,t}}, \sqrt{q_{22,t}}, ..., \sqrt{q_{kk,t}}\right).$$

Furthermore, the dynamic conditional correlations from the scalar BEKK model are as follows:

$$\tilde{\Gamma}_t = \bar{\Sigma}_t^{-1}\Sigma_t\bar{\Sigma}_t^{-1},$$
$$\bar{\Sigma}_t = diag\left(\sigma_{11,t}, \sigma_{22,t}, ..., \sigma_{kk,t}\right).$$

Tables 1 and 2 report the estimated parameters for the direct DCC and scalar BEKK (or indirect DCC) models, while Tables 3 and 4 report the sample moments of the estimated conditional variances and correlations of the two models. Furthermore, Figures 1-3 report the conditional variances produced by the direct and indirect DCC models, while Figures 4-6 report the conditional correlations from the two models.

Note that both models provide highly significant estimates. Furthermore, the GARCH(1,1) models lead to very similar parameter estimates. This is to be expected result as markets are closely integrated, so they are likely to share common patterns and to react in a similar manner to shocks. The DCC model does not provide a particularly high estimate of $\hat{\theta}_2 = 0.819,$ while the scalar BEKK model produces a highly persistent estimate of $\hat{\alpha} + \hat{\beta} = 0.972,$ which is in line with typical GARCH(1,1) estimates.

Tables 1 and 2 report the full system likelihoods, that is, the maximized likelihood for the scalar BEKK model, while this is equivalent to the maximized likelihood including the effects of the GARCH(1,1) models (that is, the likelihood for ε_t) for DCC. Note that even if scalar BEKK (and hence indirect DCC) is a simplistic solution compared with the direct DCC model and GARCH(1,1), it provides a significantly higher likelihood value (specifically, at more than 10% higher). This is an interesting result, and may arise because BEKK estimates the conditional variances and covariances simultaneously.

It is clear from the graphs that the scalar BEKK model provides smoother conditional variances and correlation estimates as compared with the direct DCC estimates. In general, the scalar BEKK (and hence indirect DCC) estimates are less volatile than are their direct DCC counterparts.

Finally, Table 4 reports the sample correlation between the conditional variances, covariances and correlations estimated by the Scalar BEKK and the DCC model. The conditional variance and covariances series are very close while some differences emerge for the conditional correlations. In fact, the correlation coefficients decrease in particular when we include the FTSE index. These relevant discrepancies are due to the different approaches of both models and strengthen the preference for the Scalar-BEKK.

4. Concluding Remarks

The paper derives the scalar special case of the BEKK model of Engle and Kroner (1995) using a multivariate extension of the Random Coefficient Autoregressive (RCA) model of Tsay (1987). This representation establishes the relevant structural and asymptotic properties of the scalar BEKK model using the theoretical results in Comte and Lieberman (2003) and Ling and McAleer (2003). Sufficient conditions for the direct DCC model of Engle (2002) to be consistent with a scalar BEKK representation are established. Moreover, an indirect DCC model that is consistent with the scalar BEKK representation is obtained, and is compared with the direct DCC model using an empirical example. It is found that the scalar BEKK model, and hence indirect DCC, generally provides less volatile estimates of the conditional variances and correlations than does the direct DCC model.

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Table 1: Direct DCC with GARCH(1,1) Conditional Variances

GARCH(1,1)				
Data	Estimates	ω	α	β
CAC40	Coeff.	0.002	0.073	0.922
	St.dev.	$2.6\text{x}10^{-5}$	$3.1\text{x}10^{-4}$	$3.1\text{x}10^{-4}$
	T-stat.	87.186	233.971	2971.770
DAX	Coeff.	0.003	0.077	0.917
	St.dev.	$3.3\text{x}10^{-5}$	$2.6\text{x}10^{-4}$	$2.7\text{x}10^{-4}$
	T-stat.	95.206	291.343	3377.455
FTSE	Coeff.	0.002	0.086	0.908
	St.dev.	$1.9\text{x}10^{-5}$	$3.8\text{x}10^{-4}$	$4.1\text{x}10^{-4}$
	T-stat.	87.946	225.064	2214.906
Direct DCC				
Direct DCC	Estimates	θ_1	θ_2	
	Coeff.	0.034	0.819	
	St.dev.	$2.3\text{x}10^{-4}$	$1.9\text{x}10^{-3}$	
	T-stat.	160.050	429.078	
Full system likelihood				-2574.375

Table 2: Scalar BEKK (Indirect DCC) Estimates

Estimates	α	β
Coeff.	0.021	0.951
St.dev.	$1.6\text{x}10^{-4}$	$4.4\text{x}10^{-4}$
T-stat.	134.255	2147.518
Full system likelihood		-2279.620

Table 3: Sample Moments of Dynamic Conditional Variances and Covariances

Moments	CAC	CAC-DAX	DAX	CAC-FTSE	DAX-FTSE	FTSE
Direct DCC with GARCH(1,1) Conditional Variances						
Mean	0.409	0.385	0.526	0.257	0.266	0.263
Stdev	0.387	0.367	0.510	0.253	0.260	0.260
Min	0.075	0.068	0.084	0.047	0.046	0.045
Max	2.167	2.157	3.153	1.657	1.596	1.850
Scalar BEKK (Indirect DCC)						
Mean	0.407	0.388	0.527	0.268	0.276	0.262
Stdev	0.169	0.167	0.224	0.124	0.123	0.108
Min	0.253	0.240	0.321	0.165	0.170	0.163
Max	1.110	1.119	1.471	0.831	0.782	0.784

Table 4: Sample Moments of Dynamic Conditional Correlations

Moments	CAC-DAX	CAC-FTSE	DAX-FTSE
Direct DCC			
Mean	0.836	0.784	0.718
Stdev	0.027	0.033	0.036
Min	0.544	0.527	0.514
Max	0.897	0.851	0.819
Scalar BEKK (Indirect DCC)			
Mean	0.836	0.814	0.738
Stdev	0.032	0.042	0.041
Min	0.596	0.591	0.590
Max	0.913	0.922	0.869

Table 5: Correlation between Scalar BEKK and DCC conditional variances, covariances and correlations

	Series	Correlation
Variances and covariances	CAC	0.983
	CAC-DAX	0.979
	DAX	0.982
	CAC-FTSE	0.972
	DAX-FTSE	0.970
	FTSE	0.970
Correl ations	CAC-DAX	0.822
	CAC-FTSE	0.746
	DAX-FTSE	0.750

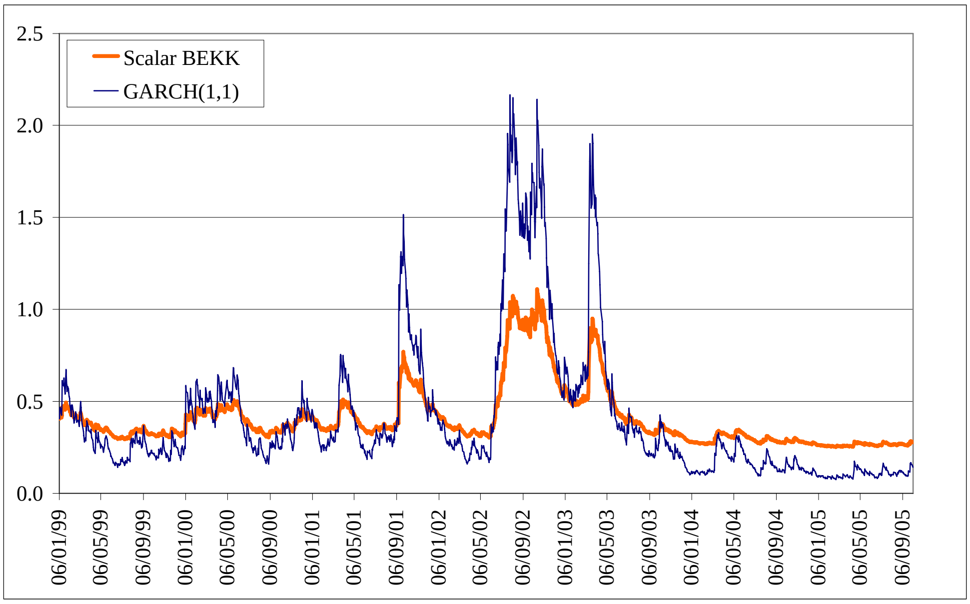

Figure 1: CAC40 conditional variances

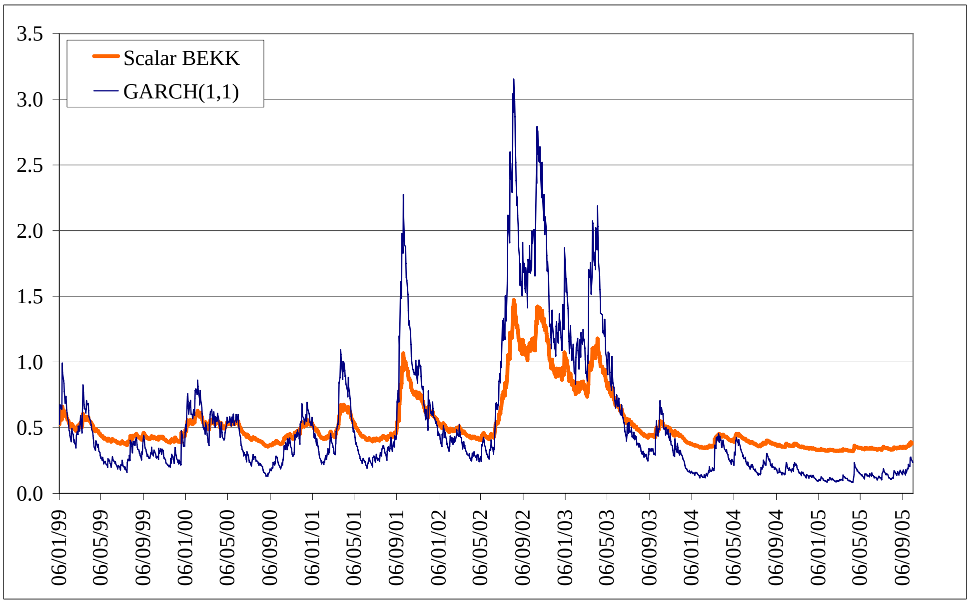

Figure 2: DAX conditional variances

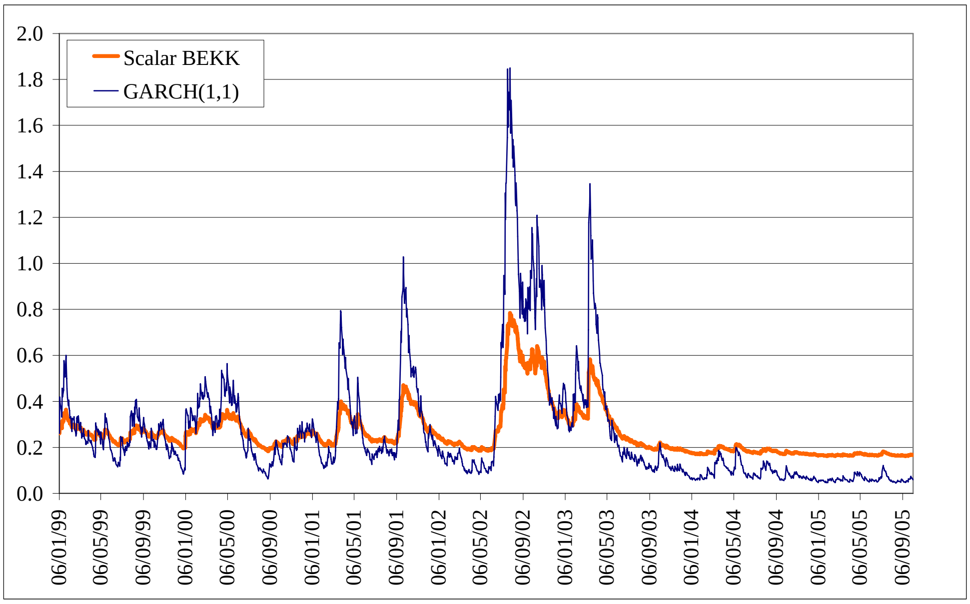

Figure 3: FTSE 100 conditional variances

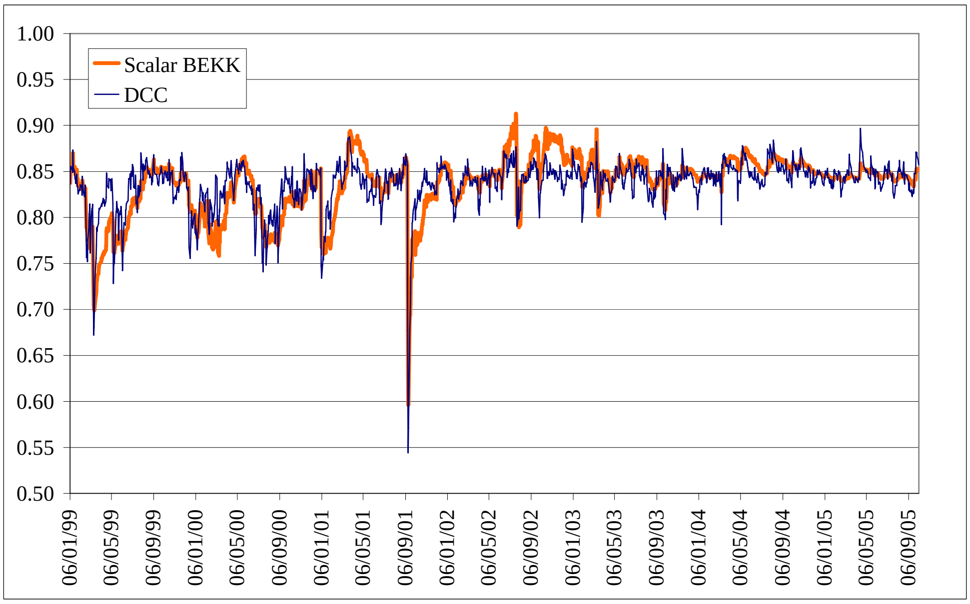

Figure 4: CAC40-DAX conditional correlations

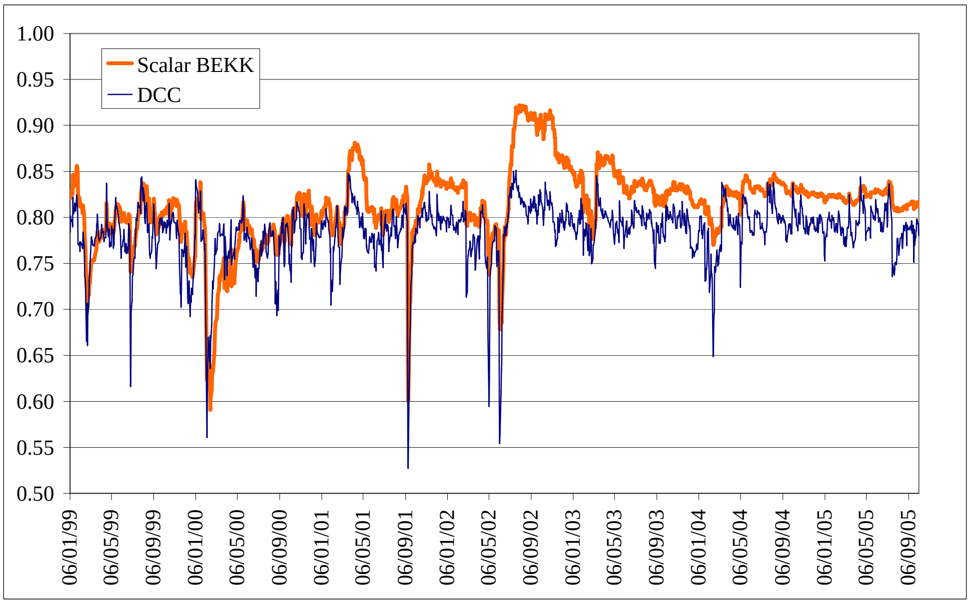

Figure 5: CAC40-FTSE100 conditional correlations

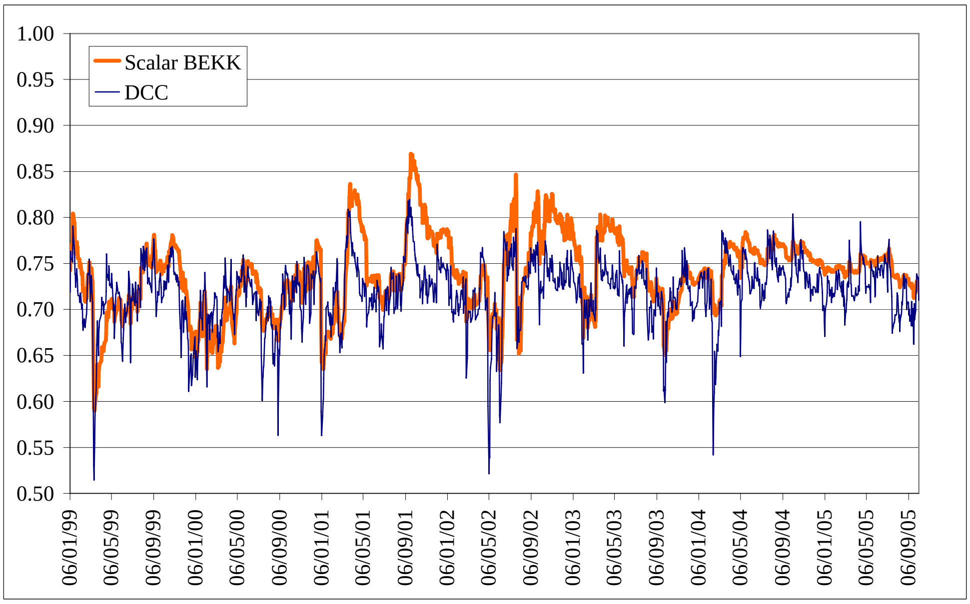

Figure 6: DAX-FTSE100 conditional correlations